Abstract. In this paper, by the use of Zhuang’s inequalities, we establish some reverse inequalities for the celebrated refinement of the Cauchy-Bunyakovsky-Schwarz inequality that was obtained by Callebaut in 1965. A numerical comparison is also provided.

1. Introduction

The following inequality

\[ x^{1-\nu} y^\nu \leq (1 - \nu) x + \nu y \]  \hspace{1cm} (1.1)

is well known in literature as either weighted Arithmetic mean-Geometric mean inequality or as Young’s inequality.

In 1991, Y.-D. Zhuang [14] established the following inequality for \(0 < m \leq x \leq M, 0 < k \leq y \leq K,\) and \(\nu \in [0,1]\)

\[ \nu x + (1 - \nu) y \leq \max \left\{ \frac{\nu M + (1 - \nu) k}{M^\nu k^{1-\nu}}, \frac{\nu m + (1 - \nu) K}{m^\nu K^{1-\nu}} \right\} x^\nu y^{1-\nu} \]  \hspace{1cm} (1.2)

or

\[ x + y \leq \max \left\{ \frac{M + k}{M^\nu k^{1-\nu}}, \frac{m + K}{m^\nu K^{1-\nu}} \right\} x^\nu y^{1-\nu}. \]  \hspace{1cm} (1.3)

The sign of equality in (1.2) and (1.3) holds if and only if either \((x, y) = (m, K)\) or \((x, y) = (M, k)\).

Moreover, if \(m \geq K\), then

\[ \frac{\nu m + (1 - \nu) K}{m^\nu K^{1-\nu}} x^\nu y^{1-\nu} \leq \nu x + (1 - \nu) y \leq \frac{\nu M + (1 - \nu) k}{M^\nu k^{1-\nu}} x^\nu y^{1-\nu}. \]  \hspace{1cm} (1.4)

The sign of inequality on the right-hand side of (1.4) holds if and only if \((x, y) = (M, k)\) and the sign of equality on the left-hand side of (1.4) holds if and only if \((x, y) = (m, K)\).

The sign of inequality in (1.4) is reversed if \(k \geq M\).

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Now, if we take \( y = 1 \), then we have from the above inequalities for \( x \in [m, M] \subset (0, \infty) \) and \( \nu \in [0, 1] \) that
\[
\nu x + 1 - \nu \leq \max \left\{ \frac{\nu M + 1 - \nu}{M^\nu}, \frac{\nu m + 1 - \nu}{m^\nu} \right\} x^\nu \tag{1.5}
\]
and
\[
x + 1 \leq \max \left\{ \frac{M + 1}{M^\nu}, \frac{m + 1}{m^\nu} \right\} x^\nu. \tag{1.6}
\]
If \( m \geq 1 \), then we have
\[
\frac{\nu m + 1 - \nu}{m^\nu} x^\nu \leq \nu x + 1 - \nu \leq \frac{\nu M + 1 - \nu}{M^\nu} x^\nu \tag{1.7}
\]
for \( x \in [m, M] \) and \( \nu \in (0, 1) \).
If \( M \leq 1 \), then we have
\[
\frac{\nu M + 1 - \nu}{M^\nu} x^\nu \leq \nu x + 1 - \nu \leq \frac{\nu m + 1 - \nu}{m^\nu} x^\nu, \tag{1.8}
\]
for \( x \in [m, M] \) and \( \nu \in (0, 1) \).

The inequalities (1.5), (1.7) and (1.8) can be put together as
\[
\begin{cases}
\frac{\nu M + 1 - \nu}{M^\nu} & \text{if } M < 1, \\
\nu m + 1 - \nu & \text{if } m \leq 1 \leq M, \\
\frac{\nu M + 1 - \nu}{M^\nu} & \text{if } 1 < m,
\end{cases}
\]
\[
1 \leq \frac{\nu x + 1 - \nu}{x^\nu}, \tag{1.9}
\]
for \( x \in [m, M] \subset (0, \infty) \) and \( \nu \in [0, 1] \).

The inequality
\[
1 \leq \frac{\nu x + 1 - \nu}{x^\nu}
\]
is the AG-inequality for \( y = 1 \).

We notice that the inequality (1.9) has been also obtained in [5] by a direct approach in studying the margins of the function \( g(x) := \frac{\nu x + 1 - \nu}{x^\nu} \) with \( x \in [m, M] \subset (0, \infty) \) and \( \nu \in [0, 1] \).

For other similar results, see [1] and [3]-[13].

The following refinement of the Cauchy-Bunyakovsky-Schwarz inequality was obtained by Callebaut [2] in 1965:
\[
\left( \sum_{i=1}^{n} p_i a_i b_i \right)^2 \leq \sum_{i=1}^{n} p_i a_i^{2(1-\nu)} b_i^{2\nu} \sum_{i=1}^{n} p_i a_i^{2\nu} b_i^{2(1-\nu)} \leq \sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} p_i b_i^2. \tag{1.10}
\]

In this paper, by the use of Zhuang’s inequalities (1.2) and (1.3) we establish some upper bounds for the quotient
\[
\frac{\sum_{i \in I} p_i a_i^2}{\sum_{i \in I} p_i a_i^{2(1-\nu)} b_i^{2\nu}} \frac{\sum_{i \in I} p_i a_i^{2\nu} b_i^{2(1-\nu)}}{\sum_{i \in I} p_i a_i^{2(1-\nu)} b_i^{2\nu}} \sum_{i \in I} p_i a_i^2 \sum_{i \in I} p_i b_i^2
\]
under suitable conditions for the sequences \( a_k, b_k > 0 \) and \( p_k \geq 0, k \in \mathbb{N} \).
These results can be applied for operator inequalities as in [1], [5]-[7] and [9].

2. DISCRETE INEQUALITIES

We start with the following result:

**Theorem 2.1.** Let $a_k, b_k > 0, k \in \mathbb{N}$ and $I, J$ be finite sets of indices such that

$$m \leq \frac{b_i}{a_i} \leq M \quad \text{and} \quad k \leq \frac{b_j}{a_j} \leq K$$

(2.1)

for some constants $0 < m < M, 0 < k < K$, for any $i \in I$ and $j \in J$. If $p_i \geq 0$ for $i \in I$, $q_j \geq 0$ for $j \in J$ and $\nu \in [0, 1]$, then we have the inequality

$$\nu \sum_{i \in I} p_i b_i^2 \sum_{j \in J} q_j a_j^2 + (1 - \nu) \sum_{i \in I} p_i a_i^2 \sum_{j \in J} q_j b_j^2$$

$$\leq \max \left\{ \frac{\nu M^2 + (1 - \nu) k^2}{M^{2\nu} k^{2(1 - \nu)}}, \frac{\nu m^2 + (1 - \nu) K^2}{m^{2\nu} K^{2(1 - \nu)}} \right\}$$

$$\times \sum_{i \in I} p_i a_i^{2(1 - \nu)} b_i^{2\nu} \sum_{j \in J} q_j a_j^{2\nu} b_j^{2(1 - \nu)}$$

and the inequality

$$\sum_{i \in I} p_i b_i^2 \sum_{j \in J} q_j a_j^2 + \sum_{i \in I} p_i a_i^2 \sum_{j \in J} q_j b_j^2$$

$$\leq \max \left\{ \frac{M^2 + k^2}{M^{2\nu} k^{2(1 - \nu)}}, \frac{m^2 + K^2}{m^{2\nu} K^{2(1 - \nu)}} \right\} \sum_{i \in I} p_i a_i^{2(1 - \nu)} b_i^{2\nu} \sum_{j \in J} q_j a_j^{2\nu} b_j^{2(1 - \nu)}.$$ 

(2.3)

**Proof.** If we write the inequality (1.2) for $x = \left( \frac{b_i}{a_i} \right)^2$ and $y = \left( \frac{b_j}{a_j} \right)^2$, then we get

$$\nu \left( \frac{b_i}{a_i} \right)^2 + (1 - \nu) \left( \frac{b_j}{a_j} \right)^2$$

$$\leq \max \left\{ \frac{\nu M^2 + (1 - \nu) k^2}{M^{2\nu} k^{2(1 - \nu)}}, \frac{\nu m^2 + (1 - \nu) K^2}{m^{2\nu} K^{2(1 - \nu)}} \right\} \left( \frac{b_i}{a_i} \right)^{2(1 - \nu)} \left( \frac{b_j}{a_j} \right)^{2\nu}$$

(2.4)

for any $i \in I$ and $j \in J$.

By multiplying (2.4) with $a_i^2 a_j^2 \geq 0$ we get

$$\nu b_i^2 a_j^2 + (1 - \nu) a_i^2 b_j^2$$

$$\leq \max \left\{ \frac{\nu M^2 + (1 - \nu) k^2}{M^{2\nu} k^{2(1 - \nu)}}, \frac{\nu m^2 + (1 - \nu) K^2}{m^{2\nu} K^{2(1 - \nu)}} \right\} a_i^{2(1 - \nu)} b_i^{2\nu} a_j^{2\nu} b_j^{2(1 - \nu)}$$

(2.5)

for any $i \in I$ and $j \in J$.

Multiply the inequality (2.5) by $q_j \geq 0$ and sum over $j \in J$ to get

$$\nu b_i^2 \sum_{j \in J} q_j a_j^2 + (1 - \nu) a_i^2 \sum_{j \in J} q_j b_j^2$$

$$\leq \max \left\{ \frac{\nu M^2 + (1 - \nu) k^2}{M^{2\nu} k^{2(1 - \nu)}}, \frac{\nu m^2 + (1 - \nu) K^2}{m^{2\nu} K^{2(1 - \nu)}} \right\} a_i^{2(1 - \nu)} b_i^{2\nu} \sum_{j \in J} q_j a_j^{2\nu} b_j^{2(1 - \nu)}$$

(2.6)
for any $i \in I$. If we multiply (2.6) by $p_i \geq 0$ and sum over $i \in I$, we get the desired inequality (2.2).

By the inequality (1.3) for $x = \left(\frac{b_i}{a_i}\right)^2$ and $y = \left(\frac{b_j}{a_j}\right)^2$ we have

$$\left(\frac{b_i}{a_i}\right)^2 + \left(\frac{b_j}{a_j}\right)^2 \leq \max \left\{ \frac{M^2 + k^2}{M^{2\nu}k^{2(1-\nu)}}, \frac{m^2 + K^2}{m^{2\nu}K^{2(1-\nu)}} \right\} \left(\frac{b_i}{a_i}\right)^{2\nu} \left(\frac{b_j}{a_j}\right)^{2(1-\nu)} \tag{2.7}$$

for any $i \in I$ and $j \in J$. On making use of a similar argument as above, we deduce (2.3). \hfill $\square$

**Corollary 2.1.** Let $a_k, b_k > 0, k \in \mathbb{N}$ and $I$ be a finite set of indices such that

$$m \leq \frac{b_i}{a_i} \leq M \tag{2.8}$$

for some constants $0 < m < M$ and any $i \in I$. If $p_i \geq 0$ for $i \in I$ and $\nu \in [0,1]$, then we have the inequality

$$\frac{\sum_{i \in I} p_i b_i^2}{\sum_{i \in I} p_i a_i^2} \sum_{i \in I} p_i a_i^2 \leq \max \left\{ \frac{\nu M^2 + (1 - \nu) m^2}{M^{2\nu}m^{2(1-\nu)}}, \frac{\nu m^2 + (1 - \nu) M^2}{m^{2\nu}M^{2(1-\nu)}} \right\} \tag{2.9}$$

and the inequality

$$\frac{\sum_{i \in I} p_i b_i^2}{\sum_{i \in I} p_i a_i^2} \sum_{i \in I} p_i a_i^2 \leq \frac{M^2 + m^2}{2} \max \left\{ \frac{1}{M^{2\nu}m^{2(1-\nu)}}, \frac{1}{m^{2\nu}M^{2(1-\nu)}} \right\} \tag{2.10}$$

The inequalities (2.9) and (2.10) therefore provide multiplicative reverses of the second Callebaut inequality (1.10).

The following result also holds:

**Theorem 2.2.** Let $a_k, b_k > 0, k \in \mathbb{N}$ and $I$ be a finite set of indices such that

$$a \leq a_i \leq A, b \leq b_i \leq B \tag{2.11}$$

for some constants $0 < a < A, 0 < b < B$ and any $i \in I$. If $w_i \geq 0$ for $i \in I$ with $\sum_{i \in I} w_i = 1$ and $\nu \in [0,1]$, then we have the inequality

$$\frac{\left(\sum_{i \in I} w_i a_i^2\right)^{1-\nu} \left(\sum_{i \in I} w_i b_i^2\right)^{\nu}}{\sum_{j \in I} w_j a_j^{2\nu} b_j^{2(1-\nu)}} \leq \max \left\{ \frac{\nu A^2B^2 + (1 - \nu) a^2b^2}{A^{2\nu}a^{2(1-\nu)}B^{2\nu}b^{2(1-\nu)}}, \frac{\nu a^2b^2 + (1 - \nu) A^2B^2}{A^{2(1-\nu)}a^{2\nu}B^{2(1-\nu)}b^{2\nu}} \right\} \tag{2.12}$$
and the inequality
\[
\frac{\left(\sum_{i \in I} w_i a_i^2\right)^\nu \left(\sum_{i \in I} w_i b_i^2\right)^{1-\nu}}{\sum_{j \in I} w_j a_j^{2\nu} b_j^{1-\nu}} \leq \frac{A^2 B^2 + a^2 b^2}{2} \times \max \left\{ \frac{1}{A^{2\nu} a^{2(1-\nu)} B^{2\nu} b^{2(1-\nu)}}, \frac{1}{A^{2(1-\nu)} a^{2\nu} B^{2(1-\nu)} b^{2\nu}} \right\}.
\]  
(2.13)

**Proof.** Let \( x = \frac{a^2}{\sum_{i \in I} w_i a_i^2} \) and \( y = \frac{b^2}{\sum_{i \in I} w_i b_i^2} \) for \( j \in I \), then we get
\[
a^2 \leq x \leq \frac{A^2}{a^2}, \quad j \in I
\]
and
\[
b^2 \leq y \leq \frac{B^2}{b^2}, \quad j \in I.
\]

If we write the inequality (1.2) for \( x \) and \( y \) as above, then we get
\[
\nu \frac{a_j^2}{\sum_{i \in I} w_i a_i^2} + (1 - \nu) \frac{b_j^2}{\sum_{i \in I} w_i b_i^2} \leq \max \left\{ \frac{\nu A^2 + (1 - \nu) \frac{b_j^2}{b^2}}{\left(\frac{A^2}{a^2}\right)^\nu \left(\frac{b^2}{b^2}\right)^{1-\nu}}, \frac{\nu a^2 + (1 - \nu) \frac{b_j^2}{a^2}}{\left(\frac{a^2}{a^2}\right)^\nu \left(\frac{b^2}{b^2}\right)^{1-\nu}} \right\}
\times \frac{a_j^{2\nu}}{\left(\sum_{i \in I} w_i a_i^2\right)^\nu} \left(\sum_{i \in I} w_i b_i^2\right)^{1-\nu}
\]
for any \( j \in I \).

Since
\[
\frac{\nu A^2 + (1 - \nu) \frac{b_j^2}{b^2}}{\left(\frac{A^2}{a^2}\right)^\nu \left(\frac{b^2}{b^2}\right)^{1-\nu}} = \frac{\nu A^2 B^2 + (1 - \nu) a^2 b^2}{A^{2\nu} a^{2(1-\nu)} B^{2\nu} b^{2(1-\nu)}}
\]
and
\[
\frac{\nu a^2 + (1 - \nu) \frac{b_j^2}{a^2}}{\left(\frac{a^2}{a^2}\right)^\nu \left(\frac{b^2}{b^2}\right)^{1-\nu}} = \frac{\nu a^2 b^2 + (1 - \nu) A^2 B^2}{A^{2(1-\nu)} a^{2\nu} B^{2(1-\nu)} b^{2\nu}},
\]
then by (2.14) we have
\[
\nu \frac{a_j^2}{\sum_{i \in I} w_i a_i^2} + (1 - \nu) \frac{b_j^2}{\sum_{i \in I} w_i b_i^2} \leq \max \left\{ \frac{\nu A^2 B^2 + (1 - \nu) a^2 b^2}{A^{2\nu} a^{2(1-\nu)} B^{2\nu} b^{2(1-\nu)}}, \frac{\nu a^2 b^2 + (1 - \nu) A^2 B^2}{A^{2(1-\nu)} a^{2\nu} B^{2(1-\nu)} b^{2\nu}} \right\}
\times \frac{a_j^{2\nu}}{\left(\sum_{i \in I} w_i a_i^2\right)^\nu} \left(\sum_{i \in I} w_i b_i^2\right)^{1-\nu}
\]
for any \( j \in I \).
If we multiply (2.15) by $w_j$ and sum, then we get
\[
\sum_{j \in I} \frac{w_j a_j^2}{\sum_{i \in I} w_i a_i^2} + (1 - \nu) \sum_{j \in I} \frac{w_j b_j^2}{\sum_{i \in I} w_i b_i^2}
\leq \max \left\{ \frac{\nu A^2 B^2 + (1 - \nu) a^2 b^2}{A^{2 \nu} a^{2(1 - \nu)} B^{2 \nu} b^{2(1 - \nu)}}, \frac{\nu a^2 b^2 + (1 - \nu) A^2 B^2}{A^{2(1 - \nu)} a^{2 \nu} B^{2(1 - \nu)} b^{2 \nu}} \right\}
\times \frac{\sum_{j \in I} w_j a_j^{2\nu} b_j^{2(1 - \nu)}}{(\sum_{i \in I} w_i a_i^2) \nu (\sum_{i \in I} w_i b_i^2) \nu}
\]
that is equivalent to (2.12).

By the inequality (1.3) we also have
\[
\sum_{i \in I} w_i a_i^2 + \sum_{i \in I} w_i b_i^2 \leq \max \left\{ \frac{A^2}{\alpha^2} + \frac{\nu^2}{\beta^2}, \frac{A^2}{\alpha^2} \nu \left( \frac{\nu^2}{\beta^2} \right)^{1 - \nu}, \frac{A^2}{\alpha^2} \nu \left( \frac{\nu^2}{\beta^2} \right)^{1 - \nu} \right\}
\times \frac{a_j^{2\nu} b_j^{2(1 - \nu)}}{(\sum_{i \in I} w_i a_i^2) \nu (\sum_{i \in I} w_i b_i^2) \nu}
\]
for any $j \in I$ and since
\[
\max \left\{ \frac{A^2}{\alpha^2} + \frac{\nu^2}{\beta^2}, \frac{A^2}{\alpha^2} \nu \left( \frac{\nu^2}{\beta^2} \right)^{1 - \nu}, \frac{A^2}{\alpha^2} \nu \left( \frac{\nu^2}{\beta^2} \right)^{1 - \nu} \right\}
= \left( A^2 B^2 + a^2 b^2 \right)
\times \max \left\{ \frac{1}{A^{2 \nu} a^{2(1 - \nu)} B^{2 \nu} b^{2(1 - \nu)}}, \frac{1}{A^{2(1 - \nu)} a^{2 \nu} B^{2(1 - \nu)} b^{2 \nu}} \right\},
\]
then by (2.16) we get
\[
\sum_{j \in I} \frac{w_j a_j^2}{\sum_{i \in I} w_i a_i^2} + \sum_{j \in I} \frac{w_j b_j^2}{\sum_{i \in I} w_i b_i^2} \leq \left( A^2 B^2 + a^2 b^2 \right)
\times \max \left\{ \frac{1}{A^{2 \nu} a^{2(1 - \nu)} B^{2 \nu} b^{2(1 - \nu)}}, \frac{1}{A^{2(1 - \nu)} a^{2 \nu} B^{2(1 - \nu)} b^{2 \nu}} \right\}
\times \frac{a_j^{2\nu} b_j^{2(1 - \nu)}}{(\sum_{i \in I} w_i a_i^2) \nu (\sum_{i \in I} w_i b_i^2) \nu}
\]
for any $j \in I$.

If we multiply (2.17) by $w_j$ and sum, then we get the desired result (2.13).
Remark 2.1. With the assumptions of Theorem 2.2 we have the Callebaut reverse inequalities
\[
\sum_{i \in I} w_i a_i^2 \sum_{i \in I} w_i b_i^2 \\
\sum_{j \in I} w_j a_j^{2(1-\nu)} \sum_{j \in I} w_j a_j^{2(1-\nu)} b_j^{2\nu}
\leq \max \left\{ \frac{\nu A^2 B^2 + (1-\nu) a^2 b^2}{A^{4\nu} a^{4(1-\nu)} B^{4\nu} b^{4(1-\nu)}}, \frac{\nu a^2 b^2 + (1-\nu) A^2 B^2}{A^{4(1-\nu)} a^{4\nu} B^{4(1-\nu)} b^{4\nu}} \right\}
\]
(2.18)

and
\[
\sum_{i \in I} w_i a_i^2 \sum_{i \in I} w_i b_i^2 \\
\sum_{j \in I} w_j a_j^{2(1-\nu)} \sum_{j \in I} w_j a_j^{2(1-\nu)} b_j^{2\nu}
\leq \frac{A^2 B^2 + a^2 b^2}{2} \times \max \left\{ \frac{1}{A^{4\nu} a^{4(1-\nu)} B^{4\nu} b^{4(1-\nu)}}, \frac{1}{A^{4(1-\nu)} a^{4\nu} B^{4(1-\nu)} b^{4\nu}} \right\}.
\]
(2.19)

Indeed, by the inequality (2.12) for \(1-\nu\) instead of \(\nu\) we have
\[
\left(\sum_{i \in I} w_i a_i^2\right)^{1-\nu} \left(\sum_{i \in I} w_i b_i^2\right)^{\nu}
\sum_{j \in I} w_j a_j^{2(1-\nu)} b_j^{2\nu}
\leq \max \left\{ \frac{(1-\nu) A^2 B^2 + \nu a^2 b^2}{A^2 b^2 a^2(1-\nu) b^{2(1-\nu)}}, \frac{(1-\nu) a^2 b^2 + \nu A^2 B^2}{A^2 a^2(1-\nu) B^{2(1-\nu)} b^2} \right\}.
\]
(2.20)

If we multiply (2.12) with (2.20) we obtain (2.18).

The inequality (2.19) follows in a similar way by (2.13) and the details are omitted.

The inequalities from (2.19) and (2.20) can be however improved as follows:

**Theorem 2.3.** Let \(a_k, b_k > 0, k \in \mathbb{N}\) and \(I\) a finite set of indices such that the inequality (2.11) is valid for some constants \(0 < a < A, 0 < b < B\) for any \(i \in I\). If \(w_i \geq 0\) for \(i \in I\) with \(\sum_{i \in I} w_i = 1\) and \(\nu \in [0, 1]\), then we have the inequalities
\[
\sum_{i \in I} w_i a_i^2 \sum_{i \in I} w_i b_i^2 \\
\sum_{j \in I} w_j a_j^{2(1-\nu)} \sum_{j \in I} w_j a_j^{2(1-\nu)} b_j^{2\nu}
\leq \max \left\{ \frac{\nu A^2 B^2 + (1-\nu) a^2 b^2}{A^{2\nu} B^{2\nu} a^{2(1-\nu)} b^{2(1-\nu)}}, \frac{\nu a^2 b^2 + (1-\nu) A^2 B^2}{a^{2\nu} b^{2\nu} A^{2(1-\nu)} B^{2(1-\nu)}} \right\}
\]
(2.21)

and
\[
\sum_{i \in I} w_i a_i^2 \sum_{i \in I} w_i b_i^2 \\
\sum_{j \in I} w_j a_j^{2(1-\nu)} \sum_{j \in I} w_j a_j^{2(1-\nu)} b_j^{2\nu}
\leq \frac{A^2 B^2 + a^2 b^2}{2} \times \max \left\{ \frac{1}{A^{2\nu} B^{2\nu} a^{2(1-\nu)} b^{2(1-\nu)}}, \frac{1}{a^{2\nu} b^{2\nu} A^{2(1-\nu)} B^{2(1-\nu)}} \right\}.
\]
(2.22)
Proof. Let \( x = a_i^2 b_j^2 \) and \( y = a_j^2 b_i^2 \) for \( i, j \in I \). Then by the condition (2.11) we have
\[
a^2 b^2 \leq x \leq A^2 B^2 \text{ and } a^2 b^2 \leq y \leq A^2 B^2.
\]
By the inequalities (1.2) and (1.3) we have
\[
\nu a_i b_j^2 + (1 - \nu) a_j^2 b_i^2 \leq \max \left\{ \frac{\nu A^2 B^2 + (1 - \nu) a^2 b^2}{(A^2 B^2)^\nu (a^2 b^2)^{1-\nu}}, \frac{\nu a^2 b^2 + (1 - \nu) A^2 B^2}{(a^2 b^2)^\nu (A^2 B^2)^{1-\nu}} \right\} \times (a_i^2 b_j^2)^\nu (a_j^2 b_i^2)^{1-\nu} \times a_i^2 b_i^2 \leq \left( A^2 B^2 + a^2 b^2 \right)
\]
and
\[
a_j^2 b_j^2 + a_i^2 b_i^2 \leq \left( A^2 B^2 + a^2 b^2 \right)
\]
for \( i, j \in I \).

If we multiply (2.23) and (2.24) by \( w_i w_j \) and sum over \( i, j \in I \) we get the desired inequalities (2.21) and (2.22).

\[\square\]

3. A Numerical Comparison

We consider the Kantorovich’s ratio defined by
\[
K(h) := \frac{(h + 1)^2}{4h}, \quad h > 0.
\]
The function \( K \) is decreasing on \((0, 1)\) and increasing on \([1, \infty)\), \( K(h) \geq 1 \) for any \( h > 0 \) and \( K(h) = K \left( \frac{1}{h} \right) \) for any \( h > 0 \).

The following multiplicative reverse of Young inequality in terms of Kantorovich’s ratio holds
\[
(1 - \nu) a + \nu b \leq K^R \left( \frac{a}{b} \right) a^{1-\nu} b^\nu,
\]
where \( a, b > 0, \nu \in [0, 1] \) and \( R = \max \{1-\nu, \nu\} \).

This inequality was obtained by Liao et al. [11].

In [8] the first author obtained the following reverse of Callebaut inequality
\[
\frac{\sum_{i \in I} p_i b_i^2}{\sum_{i \in I} p_i a_i^2} \leq K^\text{max\{\nu, 1-\nu\}} \left( \frac{M}{m} \right)^2
\]
where \( a_k, b_k > 0, k \in \mathbb{N} \) and \( I \) a finite set of indices such that the condition (2.8) is valid for some constants \( 0 < m < M \) and any \( i \in I, w_i \geq 0 \) for \( i \in I \) with \( \sum_{i \in I} w_i = 1 \) and \( \nu \in [0, 1] \).
From (2.9), (2.10) and (3.3) we have the following upper bounds for the quotient

\[
\frac{\sum_{i \in I} p_i b_i^2}{\sum_{i \in I} p_i a_i^2} \leq B_1 (m, M, \nu), \ B_2 (m, M, \nu), \ B_3 (m, M, \nu)
\]

where

\[
B_1 (m, M, \nu) := \max \left\{ \frac{\nu M^2 + (1 - \nu) m^2}{M^{2\nu} m^{2(1 - \nu)}}, \frac{\nu m^2 + (1 - \nu) M^2}{m^{2\nu} M^{2(1 - \nu)}} \right\},
\]

\[
B_2 (m, M, \nu) := \frac{M^2 + m^2}{2} \max \left\{ \frac{1}{M^{2\nu} m^{2(1 - \nu)}}, \frac{1}{m^{2\nu} M^{2(1 - \nu)}} \right\},
\]

and

\[
B_3 (m, M, \nu) := K^{\max\{\nu,1-\nu\}} \left( \frac{M}{m} \right)^2.
\]

Here \(0 < m \leq M < \infty\) and \(\nu \in [0,1]\).

For \(m = 1\), we consider the differences

\[
D_1 (M, \nu) : = B_1 (1, M, \nu) - B_2 (1, M, \nu),
\]

\[
D_2 (M, \nu) : = B_3 (1, M, \nu) - B_1 (1, M, \nu),
\]

\[
D_3 (M, \nu) : = B_3 (1, M, \nu) - B_2 (1, M, \nu)
\]

for \(M \geq 1\) and \(\nu \in [0,1]\).

The plots of the differences \(D_1 (M, \nu), \ D_2 (M, \nu)\) and \(D_3 (M, \nu)\) in the box \([1, 3] \times [0, 1]\) are depicted in Figures 1, 2 and 3 below. They show that in (3.4) the bound \(B_1\) is better than \(B_3\) that is better than \(B_2\).

**Problem 1.** Is the following inequality

\[
B_1 (m, M, \nu) \leq B_3 (m, M, \nu) \leq B_2 (m, M, \nu)
\]

valid for any \(0 < m \leq M < \infty\) and \(\nu \in [0,1]\)?
Figure 1. Plot of $D_1(M, \nu)$ in $[1, 3] \times [0, 1]$

Figure 2. Plot of $D_2(M, \nu)$ on $[1, 3] \times [0, 1]$

Figure 3. Plot of $D_3(M, \nu)$ on $[1, 3] \times [0, 1]$
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