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SOLUTION OF HARMONIC VARIATIONAL INEQUALITIES BY TWO-STEP ITERATIVE SCHEME

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Abstract. This manuscript provide two step iterative scheme for finding the solution of the harmonic variational inequalities (HVIs). We use the auxiliary principle technique to propose this iterative scheme.The convergence of the proposed iterative scheme needs partially relaxed harmonic strongly monotonicity of the operator.

1. Introduction

A significant extension and generalization of the variational principles is the variational inequalities. Historically variational inequality was introduced by Stampachhia [20] in 1964. After then variational inequalities become a subject of interest due to its important applications in pure and applied sciences. A number of techniques for finding the solution of variational inequalities and the convergence criteria of methods have also been invented. Numerous applications of the theory of variational inequalities have been observed in various social and applied sciences such as in economics, ergodic theory, regional and engineering sciences, probability, harmonic analysis, optimization theory and convex analysis. For the various classes of variational inequalities, their formulation techniques, algorithms and other aspects see [3–6, 8, 10, 12, 19, 20] and the references therein.

Recently, a number of mathematicians have been given a considerable attention to the classical convexity as well as its several extensions and generalizations. Optimum problems in convexity has many applications in a number of disciplines: e.g., statistics (optimal design), electronic circuit design and other engineering sciences. It is well mentioning that the harmonic mean (H.M) have applications in electrical circuits. More precisely, set of parallel resistors has total resistance equal to the sum of the reciprocals of the individual resistance values, and then taking the reciprocal of their total, see [2]. The H.M has been applied to develop the parallel algorithms for solving a variety of problems. By using

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the H.M, Noor [14] proposed some iterative methods for finding the roots of nonlinear equations. With the help of the weighted H.Ms, one usually states the harmonic convex set. Harmonic convex set can be regarded as another extension of the convex set. This approach is used to introduced the idea of the harmonic convex functions. Harmonic convex function is an important generalization of the convex functions, see [9]. Various features of the harmonic convex functions have been examined in Anderson et al [1]. Some integral inequalities namely Hermite-Hadamard, Simpson, Trapezoid and Ostrowski type for the harmonic convex functions have been obtained by Iscan [7] and Noor et al [17]. For current improvements see [17,18] and the references therein. Variational inequalities theory and the convex functions are closely related. The optimality of the differentiable convex function on a convex set in a norm space can be categorized via the variational inequalities. For more about this connection one can see [3,6,8,10–14] and the references there in. Noor and Noor [15] explored the characterizations of the differentiable harmonic convex functions. They have revealed that the optimality conditions of the differentiable harmonic convex functions can be characterized by a new class of variational inequalities, namely harmonic variational inequalities. They have utilized the auxiliary principal technique [5] to propose an implicit method for solving the harmonic variational inequalities. Inspired by the recent research in this area, we use such auxiliary principal technique to suggest a two step iterative scheme for solving the harmonic variational inequalities. Convergence of the proposed iterative scheme is considered under the partially relaxed harmonic strongly monotonicity of the operator. The manuscript is arranged as: some preliminaries and useful definitions are given in the next section, in the last section main results with their proofs and convergence is discussed. We expect that many avenues, the ideas and techniques for further research in this direction might be fruitfully explored. Finding its applications in various fields of pure and applied sciences is an open problem.

2. Preliminaries

Consider that H be a real Hilbert space. Let $K_h \subseteq H - \{0\}$ be a non empty closed and harmonic convex set. We symbolize the inner product and norm by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. If T is given to be a nonlinear operator, consider the problem of finding $u \in K_h$, such that

$$\left\langle Tu, \frac{uv}{u-v} \right\rangle \geq 0, \qquad \forall\, v \in K_h. \tag{2.1}$$

The inequality (2.1) is called the HVIs. It has been shown [15,16] that the minimum of a differentiable harmonic convex function can be characterized by the HVIs of type (2.1). For the sake of inclusiveness and to deliver the key concept, we give the related details.

Definition 2.1. [19]The set $K_h \subset H - \{0\}$ is defined to be a harmonic convex set, if

$$\frac{uv}{v + \lambda (u - v)} \in K_h, \qquad \forall u, v \in K_h, \qquad \lambda \in [0, 1] .$$

Definition 2.2. [7]A function $f : K_h \subset H - \{0\} \to K_h$ on the harmonic convex set K_h is defined to be harmonic convex function, if

$$f\left(\frac{uv}{v+\lambda(u-v)}\right) \leq (1-\lambda) f(u) + \lambda f(v), \qquad \forall u, v \in K_h, \qquad \lambda \in [0,1].$$

The function f will be harmonic concave, if and only if its negative is harmonically convex.

It has been shown that the minimum of a differentiable harmonic convex function on the harmonic convex set K_h can be categorized by the HVIs (2.1). This conclusion has been shown by Noor and Noor [15].

Theorem 2.1. [14, 15]*Suppose that f be a differentiable harmonic convex function on the harmonic convex set K_h. Then $u \in K_h$ is a minimum of f, if and only if $u \in K_h$ is the root of the inequality*

$$\left\langle f'(u), \frac{uv}{u-v}\right\rangle \geq 0, \qquad \forall v \in K_h, \tag{2.2}$$

which is called the HVIs.

We would like to remark that Theorem 2.1 provides that harmonic convex programming can be studied via the HVIs (2.1).

Theorem 2.2. [14, 15] *Suppose that K_h be a harmonic convex set. Let f be a differentiable harmonic convex function on K_h. Then*

(a) $f(v) - f(u) \geq \left\langle f'(u), \frac{uv}{u-v}\right\rangle, \qquad \forall u, v \in K_h.$

(b) $\left\langle f'(u) - f'(v), \frac{uv}{v-u}\right\rangle \geq 0, \qquad \forall u, v \in K_h,$

where $f'(u)$ is the differential of f at u in the direction $\frac{uv}{v-u}$.

3. Main Results

The present section is our main goal. We will propose here an iterative scheme with the help of the auxiliary principal technique. The origin of auxiliary principal technique is Glowinski et al [5]. We will use our proposed scheme to solve the HVIs (2.1). We will give the conditions under which convergence of our suggested technique hold.

For a given $u \in K_h$ and $\rho > 0$ satisfying (2.1), consider the problem of finding $w \in K_h$, such that

$$\left\langle \rho Tu, \frac{vw}{w-v}\right\rangle + \langle w-u, v-w\rangle \geq 0, \qquad \forall v \in K_h, \tag{3.1}$$

which is called the auxiliary HVIs. We remark that, if $w = u$, then w is a solution of the HVIs (2.1). We used this opinion to suggest and investigate a two-step method for solving the problem (2.1).

Before proceeding to our main results, we define partially relaxed harmonic strongly monotone operator. This definition will play valuable role to get the main results of this manuscript.

Definition 3.1. An operator T is called partially relaxed harmonic strongly monotone if there exists a constant $\alpha > 0$ such that

$$\left\langle Tu - Tv, \frac{vz}{v-z} \right\rangle \geq -\alpha \left\| z - u \right\|^2, \qquad \forall u, v, z \in H.$$

Algorithm 3.1. For a given $u_0 \in H$, find the estimated solution u_{n+1} by the iterative scheme

$$\left\langle \beta T u_n, \frac{v y_n}{y_n - v} \right\rangle + \langle y_n - u_n, v - y_n \rangle \geq 0, \qquad \forall v \in K_h, \tag{3.2}$$

$$\left\langle \rho T y_n, \frac{v u_{n+1}}{u_{n+1} - v} \right\rangle + \langle u_{n+1} - y_n, v - u_{n+1} \rangle \geq 0, \qquad \forall v \in K_h, \tag{3.3}$$

where $\beta > 0, \rho > 0$ are constants. The above scheme is called, the two step iterative scheme.

The convergence behavior of the proposed Algorithm 3.1 is discussed in next theorem.

Theorem 3.2. *Suppose that $u \in K_h$ be a root of* (2.1)*. Let u_{n+1} be the estimated solution acquired from Algorithm* 3.1*. If the operator T is partially relaxed harmonic strongly monotone with constant $\alpha > 0$, then*

$$\left\| y_n - u \right\|^2 \leq \left\| u_n - u \right\|^2 - (1 - 2\alpha\beta) \left\| y_n - u_n \right\|^2, \tag{3.4}$$

$$\left\| u_{n+1} - u \right\|^2 \leq \left\| y_n - u \right\|^2 - (1 - 2\alpha\rho) \left\| u_{n+1} - y_n \right\|^2, \tag{3.5}$$

where $\beta > 0, \rho > 0$ are constants.

Proof. Let $u \in K_h$ be a solution of (2.1). Then

$$\left\langle \beta T u, \frac{uv}{u - v} \right\rangle \geq 0, \qquad \forall v \in K_h \tag{3.6}$$

and

$$\left\langle \rho T u, \frac{uv}{u - v} \right\rangle \geq 0, \qquad \forall v \in K_h. \tag{3.7}$$

Taking $v = y_n$ in (3.6), and $v = u$ in (3.2), we get

$$\left\langle \beta T u, \frac{u y_n}{u - y_n} \right\rangle \geq 0, \tag{3.8}$$

and

$$\left\langle \beta T u_n, \frac{u y_n}{y_n - u} \right\rangle + \langle y_n - u_n, u - y_n \rangle \geq 0, \tag{3.9}$$

From (3.8) and (3.9)

$$\langle y_n - u_n, u - y_n \rangle \geq \beta \left\langle T u_n - T u, \frac{u y_n}{u - y_n} \right\rangle \tag{3.10}$$

$$\geq -\beta\alpha \left\| y_n - u_n \right\|^2, \tag{3.11}$$

since T is partially relaxed harmonic strongly monotone with constant $\alpha > 0$.

From (3.11) and using the relation

$$2 \langle u, v \rangle = \left\| u + v \right\|^2 - \left\| u \right\|^2 - \left\| v \right\|^2 \qquad \forall u, v \in H, \tag{3.12}$$

we obtain

$$\left\| y_n - u \right\|^2 \leq \left\| u_n - u \right\|^2 - (1 - 2\alpha\beta) \left\| y_n - u_n \right\|^2,$$

the required.

Now taking $v = u_{n+1}$ in (3.7) and $v = u$ in (3.3), we have

$$\rho \left\langle Tu, \frac{uu_{n+1}}{u - u_{n+1}} \right\rangle \geq 0, \tag{3.13}$$

and

$$\left\langle \rho T y_n, \frac{uu_{n+1}}{u_{n+1} - u} \right\rangle + \langle u_{n+1} - y_n, u - u_{n+1} \rangle \geq 0. \tag{3.14}$$

From (3.13) and (3.14), we get

$$\begin{aligned} \langle u_{n+1} - y_n, u - u_{n+1} \rangle &\geq \rho \left\langle T y_n - Tu, \frac{uu_{n+1}}{u - u_{n+1}} \right\rangle \\ &\geq -\rho\alpha \left\| u_{n+1} - y_n \right\|^2, \end{aligned} \tag{3.15}$$

because T is partially relaxed harmonic strongly monotone with constant $\alpha > 0$.

From (3.15) and using the relation (3.12), we have

$$\left\| u_{n+1} - u \right\|^2 \leq \left\| y_n - u \right\|^2 - (1 - 2\alpha\rho) \left\| u_{n+1} - y_n \right\|^2,$$

which complete the proof. □

Theorem 3.3. *Consider that H be a finite dimensional Hilbert space and a partially relaxed harmonic strongly monotone operator T with constant $\alpha > 0$. Let $u \in K_h$ be a root of the problem* (2.1) *and u_{n+1} be the estimated root acquired from Algorithm* 3.1*. If $0 < \rho < \frac{1}{2\alpha}$ and $0 < \beta < \frac{1}{2\alpha}$ then $\lim_{n\to\infty} u_n = u$.*

Proof. Suppose that $u \in K_h$ be a root of (2.1). Then from (3.4) and (3.5), it can be observed that the sequences $\{\|y_n - u\|\}$ and $\{\|u_n - u\|\}$ are nonincreasing. Therefore the sequences $\{y_n\}$ and $\{u_n\}$ are bounded. Also from (3.4) and (3.5), we obtain

$$\sum_{n=0}^{\infty} (1 - 2\alpha\beta) \left\| y_n - u_n \right\|^2 \leq \left\| u_0 - u \right\|^2$$

and

$$\sum_{n=0}^{\infty} (1 - 2\rho\alpha) \left\| u_{n+1} - y_n \right\|^2 \leq \left\| y_0 - u \right\|^2,$$

since $0 < \rho < \frac{1}{2\alpha}$ and $0 < \beta < \frac{1}{2\alpha}$, we have

$$\lim_{n\to\infty} \|y_n - u_n\| = 0,$$

and

$$\lim_{n\to\infty} \|u_{n+1} - y_n\| = 0.$$

Thus

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| \leq \lim_{n\to\infty} \|u_{n+1} - y_n\| + \lim_{n\to\infty} \|y_n - u_n\| = 0 \tag{3.16}$$

Let $\widehat{u}$ be the cluster point of $\{u_n\}$, there exists a subsequence $\{u_{n_i}\}$ such that $\{u_{n_i}\}$ converges to $\widehat{u} \in K_h$. Putting u_{n_i} in place of y_n in (3.2) and (3.3), we have

$$\left\langle \beta T u_n, \frac{vu_{n_i}}{u_{n_i} - v} \right\rangle + \langle u_{n_i} - u_n, v - u_{n_i} \rangle \geq 0, \qquad \forall v \in K_h,$$

and

$$\left\langle \rho T u_{n_i}, \frac{v u_{n+1}}{u_{n+1} - v} \right\rangle + \langle u_{n+1} - u_{n_i}, v - u_{n+1} \rangle \geq 0, \qquad \forall v \in K_h.$$

Taking the limit as $n_i \to \infty$ and using (3.16), we obtain

$$\left\langle T\widehat{u}, \frac{\widehat{u}v}{\widehat{u} - v} \right\rangle \geq 0, \qquad \forall v \in K_h,$$

which shows that $\widehat{u}$ is a solution of the HVIs (2.1) and consequently

$$\|u_{n+1} - \widehat{u}\|^2 \leq \|u_n - \widehat{u}\|^2 .$$

Using the above inequality, one can easily show that the sequence $\{u_n\}$ has exactly one cluster point and $\lim_{n\to\infty} u_n = \widehat{u}$. The required result. □

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