Turkish J. Ineq., 1 (1) (2017), Pages 53 –77 .

Turkish Journal of
INEQUALITIES

Available online at www.tjinequality.com

OSTROWSKI'S TYPE INEQUALITIES FOR FUNCTIONS WHOSE FIRST DERIVATIVES IN ABSOLUTE VALUE ARE MN-CONVEX

M.W. ALOMARI[1] AND M.M. ALMAHAMEED[1]

Abstract. For absolutely continuous functions whose first derivatives in absolute value are MN-convex several inequalities of Ostrowski's type are introduced. Other related results by applying Hölder integral inequality are also provided.

1. Introduction

1.1. MN**-Convexity.** Let I be a real interval. A function $f : I \to \mathbb{R}$ is called convex iff

$$f\left(t\alpha + (1-t)\beta\right) \le tf\left(\alpha\right) + (1-t)f\left(\beta\right), \tag{1.1}$$

for all points $\alpha, \beta \in I$ and all $t \in [0,1]$. If -f is convex then we say that f is concave. Moreover, if f is both convex and concave, then f is said to be affine.

In general, it is not easy to check whether a given function is convex or not. Because of that, there are several criteria were known in literature. The celebrated known criterion is that of mid-convex (or Jensen convex) functions, which deal with the arithmetic mean. They are precisely the functions $f : I \to \mathbb{R}$ such that

$$f\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{2}.$$

This fact was discussed by Jensen in ([21], p. 10). Namely, he proved his famous criterion:

Theorem 1.1. *Let $f : I \to \mathbb{R}$ be a continuous function. Then f is convex iff f is mid-convex.*

Definition 1.1. A function $\mathrm{M} : (0,\infty) \to (0,\infty)$ is called a Mean function if

(a) Symmetry: $\mathrm{M}(x,y) = \mathrm{M}(y,x)$.
(b) Reflexivity: $\mathrm{M}(x,x) = x$.
(c) Monotonicity: $\min\{x,y\} \le \mathrm{M}(x,y) \le \max\{x,y\}$.
(d) Homogeneity: $\mathrm{M}(\lambda x, \lambda y) = \lambda \mathrm{M}(x,y)$, for any positive scalar λ.

Key words and phrases. Ostrowski's inequality, MN-Convex function, Means.
2010 *Mathematics Subject Classification.* 26A15, 26A51, 26D10.
Received:31.07.2017
Accepted:18.12.2017.

The most famous and old known mathematical means are listed as follows:

(a) The arithmetic mean :

$$A := A(\alpha,\beta) = \frac{\alpha+\beta}{2}, \quad \alpha,\beta \in \mathbb{R}_+.$$

(b) The geometric mean :

$$G := G(\alpha,\beta) = \sqrt{\alpha\beta}, \quad \alpha,\beta \in \mathbb{R}_+.$$

(c) The harmonic mean :

$$H := H(\alpha,\beta) = \frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}}, \quad \alpha,\beta \in \mathbb{R}_+ - \{0\}.$$

(d) The power mean :

$$M_r(\alpha,\beta) = \left(\frac{\alpha^r+\beta^r}{2}\right)^{\frac{1}{r}}, \quad r \geq 1,\ \alpha,\beta \in \mathbb{R}_+.$$

(e) The identric mean:

$$I(\alpha,\beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right)^{\frac{1}{\beta-\alpha}}, & \alpha \neq \beta \\ \alpha, & \alpha = \beta \end{cases}, \quad \alpha,\beta > 0$$

(f) The logarithmic mean :

$$L := L(\alpha,\beta) = \frac{\alpha-\beta}{\ln|\alpha| - \ln|\beta|}, \ |\alpha| \neq |\beta|,\ \alpha,\beta \neq 0,\ \alpha,\beta \in \mathbb{R}_+.$$

(g) The generalized log-mean:

$$L_p := L_p(\alpha,\beta) = \left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right]^{\frac{1}{p}}, \ p \in \mathbb{R}\backslash\{-1,0\},\ \alpha,\beta > 0.$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$, with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality

$$H \leq G \leq L \leq I \leq A.$$

1.2. **Ostrowski's Inequality.** In 1938, Ostrowski established his celebrated inequality for differentiable mappings with bounded derivatives, which reads [18]:

Theorem 1.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'(x)| \leq M$, then the following inequality,*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right| \leq M(b-a)\left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right] \tag{1.2}$$

holds for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

The proof of (1.2) can be done using Montgomery identity, which known as:

$$f(x)=\frac{1}{b-a}\int_a^b f(t)\,dt+\int_a^b K(x,t)\,f'(t)\,dt \tag{1.3}$$

where $K(x,t)$ is the Peano kernel,

$$K(x,t)=\begin{cases}\frac{t-a}{b-a}, & a\le t\le x,\\ \frac{t-b}{b-a}, & x<t\le b,\end{cases}$$

for any $x\in[a,b]$.

An Ostrowski type inequality for convex functions was pointed out by Barnett et al. in [8], as follows:

Theorem 1.3. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$ such that $|f'|$ is convex on $[a,b]$. Then for any $x\in[a,b]$ we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\le\begin{cases}\frac{1}{2}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left[f'(x)+\|f'\|_\infty\right], & f'\in L_\infty[a,b];\\[2ex] \frac{1}{2}\cdot\frac{1}{(q+1)^{1/q}}\left[\left(\frac{b-x}{b-a}\right)^{q+1}+\left(\frac{x-a}{b-a}\right)^{q+1}\right]^{1/q}(b-a)^{1/q}\left\||f'(x)|+|f'|\right\|_\infty \\ \qquad\qquad f'\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1\\[2ex] \frac{1}{2}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[(b-a)f'(x)+\|f'\|_1\right].\end{cases} \tag{1.4}$$

The constant $\frac{1}{2}$ in the first and second inequalities is sharp as is the first $\frac{1}{2}$ in the final.

Another result concerning convexity was proved by Alomari in ([1], p. 79)

Theorem 1.4. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\le\frac{b-a}{6}\left[\left(4\left(\frac{b-x}{b-a}\right)^3-3\left(\frac{b-x}{b-a}\right)^2+1\right)|f'(a)|\right.$$
$$\left.+\left(9\left(\frac{b-x}{b-a}\right)^2-4\left(\frac{b-x}{b-a}\right)^3-6\left(\frac{b-x}{b-a}\right)+2\right)|f'(b)|\right], \tag{1.5}$$

for each $x\in[a,b]$. The constant $\frac{1}{6}$ is best possible in the sense that it cannot be replaced by a smaller value.

For more result concerning Ostrowski's inequality, see [2], [3]–[4], [7]– [16].

For two given means M and N, Anderson et al. [5] studied $MN-$convexity of Maclaurin series in terms of the coefficients. In fact, the authors investigated the mid-MN-convexity of a continuous function according to a given mean between the endpoints of a definite interval.

Definition 1.2. Let $f : I\text{ÂĄ} \to \left(0, \text{ÂĄ}\infty\right)$ be continuous, where I is a subinterval of $(0, \text{ÂĄ}\infty)$. Let M and N be any two Mean functions as understood above. We say f is MN-convex (concave) if

$$f\left(\mathrm{M}\left(x,y\right)\right) \le (\ge)\, \mathrm{N}\left(f(x), f(y)\right),$$

for all $x, y \in I$ and $t \in [0,1]$.

Recently, some authors studied inequalities of Ostrowski's inequality under various MN-convexity assumption, a sample of these works can be found in [6],[15],[17],[19]–[23].

One of the most important applications of Ostrowski's inequality is to provide several bounds for mathematical means. So that, in this work, several inequalities of Ostrowski's type for absolutely continuous functions whose first derivatives in absolute value are MN-convex are introduced. Other related results by applying Hölder integral inequality are also provided.

2. Preliminaries

Let $0 < a < b$. Define the function $\widehat{\mathrm{M}} : [0,1] \to [a,b]$ given by $\widehat{\mathrm{M}}\left(t\right) = \widehat{\mathrm{M}}\left(t; a, b\right)$; where by $\widehat{\mathrm{M}}\left(t; a, b\right)$ we mean one of the following functions:

(a) $A_t\left(a,b\right) := \left(1-t\right)a + tb$; The generalized Arithmetic Mean.

(b) $G_t\left(a,b\right) = a^{1-t}b^t$; The generalized Geometric Mean.

(c) $H_t\left(t;a,b\right) := \frac{ab}{ta+(1-t)b}$; The generalized Harmonic Mean.

Note that $\widehat{\mathrm{M}}\left(0; a, b\right) = a$ and $M\left(1; a, b\right) = b$.

It is well-know that the above means are related with celebrated inequality

$$H_t\left(a,b\right) \le G_t\left(a,b\right) \le A_t\left(a,b\right), \qquad \forall\ t \in [0,1].$$

In viewing of the above notions of Mean we may generalize Definition 1.2 of MN-convexity (concavity) excluding the continuity assumption, as follows:

Definition 2.1. Let $f : [a,b]\,\text{ÂĄ} \to \left(0, \text{ÂĄ}\infty\right)$ be any function. Let $\widehat{M}$ and $\widehat{N}$ be any two Mean functions. We say f is $\widehat{\mathrm{M}}\widehat{\mathrm{N}}$-convex (concave) if

$$f\left(\widehat{M}\left(t; x, y\right)\right) \le (\ge)\, \widehat{N}\left(t; f(x), f(y)\right),$$

for all $x, y \in I$ and $t \in [0,1]$.

Clearly, if one chooses $t = \frac{1}{2}$ in the Definition 2.1, then we reduces to mid-MN-convexity which is intended in Definition 1.2. Accordingly, we can state that:

(a) f is A_tA_t-convex iff (1.1) holds or equivalently by setting $s = t\alpha + (1-t)\beta$, we write

$$f(s) \leq \frac{s-\beta}{\alpha-\beta} f(\alpha) + \frac{\alpha-s}{\alpha-\beta} f(\beta), \qquad \alpha \leq s \leq \beta. \tag{2.1}$$

(b) f is A_tG_t-convex iff

$$f(t\alpha + (1-t)\beta) \leq [f(\alpha)]^t [f(\beta)]^t, \qquad 0 \leq t \leq 1,$$

or equivalently, we write

$$f(s) \leq [f(\alpha)]^{\frac{\beta-s}{\beta-\alpha}} [f(\beta)]^{\frac{s-\alpha}{\beta-\alpha}}, \qquad \alpha \leq s \leq \beta. \tag{2.2}$$

(c) f is A_tH_t-convex iff

$$f((1-t)\alpha + t\beta) \leq \frac{f(\alpha) f(\beta)}{tf(\alpha) + (1-t) f(\beta)}, \qquad 0 \leq t \leq 1,$$

or equivalently by setting $s = (1-t)\alpha + t\beta$, we write

$$f(s) \leq \frac{(\beta-\alpha) f(\alpha) f(\beta)}{(\beta-s) f(\alpha) + (s-\alpha) f(\beta)}, \qquad \alpha \leq s \leq \beta. \tag{2.3}$$

(d) f is G_tA_t-convex iff

$$f\left(\alpha^t \beta^{1-t}\right) \leq tf(\alpha) + (1-t) f(\beta), \qquad 0 \leq t \leq 1,$$

or equivalently by setting $s = \alpha^t \beta^{1-t}$, we write

$$f(s) \leq \frac{\ln(s) - \ln(\beta)}{\ln(\alpha) - \ln(\beta)} \cdot f(\alpha) + \frac{\ln(\alpha) - \ln(s)}{\ln(\alpha) - \ln(\beta)} \cdot f(\beta), \qquad \alpha \leq s \leq \beta. \tag{2.4}$$

(e) f is G_tG_t-convex iff

$$f\left(\alpha^t \beta^{1-t}\right) \leq [f(\alpha)]^t [f(\beta)]^{1-t}, \qquad 0 \leq t \leq 1,$$

or equivalently, we write

$$f(s) \leq [f(\alpha)]^{\frac{\ln(s)-\ln(\beta)}{\ln(\alpha)-\ln(\beta)}} [f(\beta)]^{\frac{\ln(\alpha)-\ln(s)}{\ln(\alpha)-\ln(\beta)}}, \qquad \alpha \leq s \leq \beta. \tag{2.5}$$

(f) f is G_tH_t-convex iff

$$f\left(\alpha^{1-t} \beta^{t}\right) \leq \frac{f(\alpha) f(\beta)}{tf(\alpha) + (1-t) f(\beta)}, \qquad 0 \leq t \leq 1,$$

or equivalently by setting $s = \alpha^{1-t}\beta^t$, we write

$$f(s) \leq \frac{f(\alpha) f(\beta) [\ln(\alpha) - \ln(\beta)]}{[\ln(\alpha) - \ln(s)] f(\alpha) + [\ln(s) - \ln(\beta)] f(\beta)}, \qquad \alpha \leq s \leq \beta. \tag{2.6}$$

(g) f is H_tA_t-convex iff

$$f\left(\frac{\alpha\beta}{(1-t)\alpha + t\beta}\right) \leq tf(\alpha) + (1-t) f(\beta), \qquad 0 \leq t \leq 1,$$

or equivalently by setting $s = \frac{\alpha\beta}{(1-t)\alpha+t\beta}$, we write

$$f(s) \le \frac{\beta(s-\alpha)}{s(\beta-\alpha)} f(\alpha) + \frac{\alpha(\beta-s)}{s(\beta-\alpha)} f(\beta), \qquad \alpha \le s \le \beta. \tag{2.7}$$

(h) f is H_tG_t-convex iff

$$f\left(\frac{\alpha\beta}{(1-t)\alpha+t\beta}\right) \le [f(\alpha)]^t [f(\beta)]^{1-t}, \qquad 0 \le t \le 1,$$

or equivalently, we write

$$f(s) \le [f(\alpha)]^{\frac{\beta(s-\alpha)}{s(\beta-\alpha)}} [f(\beta)]^{\frac{\alpha(\beta-s)}{s(\beta-\alpha)}}, \qquad \alpha \le s \le \beta. \tag{2.8}$$

(i) f is H_tH_t-convex iff

$$f\left(\frac{\alpha\beta}{(1-t)\alpha+t\beta}\right) \le \frac{f(\alpha) f(\beta)}{(1-t) f(\alpha) + t f(\beta)}, \qquad 0 \le t \le 1,$$

or equivalently, we write

$$f(s) \le \frac{s(\beta-\alpha) f(\alpha) f(\beta)}{\beta(s-\alpha) f(\alpha) + \alpha(\beta-s) f(\beta)}, \qquad \alpha \le s \le \beta. \tag{2.9}$$

Remark 2.1. Theorem 2.4 and Corollary 2.5 in [5] are still hold for the previous $\widehat{\mathrm{M}}\widehat{\mathrm{N}}$-convexities.

3. When $|f'|$ is A$\widehat{\mathrm{N}}$-convex

In this section, inequalities of Ostrowski's type for absolutely continuous functions whose first derivatives in absolute value are A$\widehat{\mathrm{N}}$-convex are proved, where $A := A_t(a,b)$ and $\widehat{\mathrm{N}} := A_t(a,b), G_t(a,b), H_t(a,b)$.

3.1. When $|f'|$ is AG-convex.

Theorem 3.1. *Let I be a real interval, $a, b \in \mathbb{R}$ $(a < b)$ with a, b in I° (the interior of I). Let $f : I \to \mathbb{R}$ be an absolutely continuous function on I. If $|f'|$ is AG-convex, then*

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| \\ &\le \frac{2|f'(a)|^{\frac{b-x}{b-a}} |f'(b)|^{\frac{x-a}{b-a}}}{(\ln|f'(b)| - \ln|f'(a)|)^2} \cdot \left[\left(x - \frac{a+b}{2}\right)\ln\left(\frac{|f'(b)|}{|f'(a)|}\right) - (b-a)\right] \\ &\quad + \frac{(b-a)(|f'(a)| + |f'(b)|)}{(\ln|f'(b)| - \ln|f'(a)|)^2}, \end{aligned} \tag{3.1}$$

for all $a \le x \le b$. In particular, for $x = \frac{a+b}{2}$ we have

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| \le \frac{(b-a)}{4} \cdot L^2\left(\sqrt{|f'(a)|}, \sqrt{|f'(b)|}\right),$$

where $L(\cdot,\cdot)$ is the Logarithmic mean.

Proof. Rewrite (1.3) and then taking the absolute value in, by employing the triangle inequality, since $|f'|$ is AG-convex (i.e., (2.2) holds) we have

$$\begin{aligned}
\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| &= \left|\int_a^b K(x,s)\,f'(s)\,ds\right| \\
&\le \int_a^b |K(x,s)|\,|f'(s)|\,ds \\
&\le \int_a^b |K(x,s)|\,[|f'(a)|]^{\frac{b-s}{b-a}}\,[|f'(b)|]^{\frac{s-a}{b-a}}\,ds \\
&= \frac{1}{b-a}\int_a^x (s-a)\,[|f'(a)|]^{\frac{b-s}{b-a}}\,[|f'(b)|]^{\frac{s-a}{b-a}}\,ds \\
&\quad + \frac{1}{b-a}\int_x^b (b-s)\,[|f'(a)|]^{\frac{b-s}{b-a}}\,[|f'(b)|]^{\frac{s-a}{b-a}}\,ds \\
&= \frac{2\,|f'(a)|^{\frac{b-x}{b-a}}\,|f'(b)|^{\frac{x-a}{b-a}}}{\left(\ln|f'(b)|-\ln|f'(a)|\right)^2}\cdot\left[\left(x-\frac{a+b}{2}\right)\ln\left(\frac{|f'(b)|}{|f'(a)|}\right)-(b-a)\right] \\
&\quad + \frac{(b-a)\left(|f'(a)|+|f'(b)|\right)}{\left(\ln|f'(b)|-\ln|f'(a)|\right)^2},
\end{aligned}$$

the last equality follows from integration by parts and basic simplification, and this proves the desired result in (3.1). □

Theorem 3.2. *Let I be a real interval, $a,b\in\mathbb{R}$ $(a<b)$ with a,b in I° (the interior of I). Let $f:I\to\mathbb{R}$ be an absolutely continuous function on I. If $|f'|^p$ $(p>1)$ is AG-convex, then*

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(s)\,ds\right| \\
&\qquad\le (b-a)^{-\frac{1}{q}}\left[\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}}\cdot L^{\frac{1}{p}}\left(|f'(a)|^p,|f'(b)|^p\right), \quad (3.2)
\end{aligned}$$

for all $a\le x\le b$ and $p>1$ with $\frac{1}{p}+\frac{1}{q}=1$. In particular, for $x=\frac{a+b}{2}$ we have

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|\le\frac{(b-a)}{2(q+1)^{\frac{1}{q}}}\cdot L^{\frac{1}{p}}\left(|f'(a)|^p,|f'(b)|^p\right),$$

where $L(\cdot,\cdot)$ is the Logarithmic mean.

Proof. since $|f'|^p$ is AG-convex by employing the Hölder integral inequality on the right-hand side of the second inequality in the proof of Theorem 3.1, we get

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\
&\le \left(\int_a^b |K(x,s)|^q\,ds\right)^{\frac{1}{q}}\left(\int_a^b |f'(s)|^p\,ds\right)^{\frac{1}{p}} \\
&\le \frac{1}{b-a}\left(\int_a^x (s-a)^q\,ds+\int_x^b (b-s)^q\,ds\right)^{\frac{1}{q}}\left(\int_a^b [|f'(a)|]^{p\cdot\frac{b-s}{b-a}}\,[|f'(b)|]^{p\cdot\frac{s-a}{b-a}}\,ds\right)^{\frac{1}{p}} \\
&= (b-a)^{-\frac{1}{q}}\left[\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}}\cdot L^{\frac{1}{p}}\left(|f'(a)|^p,|f'(b)|^p\right),
\end{aligned}$$

which proves the desired result. □

Remark 3.1. As $p\to 1$ and $q\to\infty$ in (3.2), we have

$$\begin{aligned}
\lim_{q\to\infty}\left[\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}} &\le \lim_{q\to\infty}\sup_{a\le x\le b}\left[\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}} \\
&= \lim_{q\to\infty}\left[\frac{(b-a)^{q+1}}{q+1}\right]^{\frac{1}{q}} = (b-a)\,, \qquad (3.3)
\end{aligned}$$

and so that (3.2) becomes

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(s)\,ds\right| \le (b-a)\cdot L\left(|f'(a)|,|f'(b)|\right),$$

for all $a\le x\le b$.

3.2. When $|f'|$ is AH-convex.

Theorem 3.3. *Let I be a real interval, $a,b\in\mathbb{R}$ $(a<b)$ with a,b in I° (the interior of I). Let $f:I\to\mathbb{R}$ be an absolutely continuous function on I. If $|f'|$ is AH-convex, then*

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(s)\,ds\right| \\
&\le \frac{|f'(a)|\,|f'(b)|}{|f'(b)|-|f'(a)|}\cdot\Big\{(a+b-2x)+(b-a)\ln\left[(x-a)\,|f'(a)|+(b-x)\,|f'(b)|\right] \\
&\qquad +\frac{(b-a)}{|f'(b)|-|f'(a)|}\cdot\ln\left[(b-a)^{(|f'(a)|+|f'(b)|)}\cdot|f'(a)|^{|f'(a)|}\cdot|f'(b)|^{|f'(b)|}\right]\Big\}, \qquad (3.4)
\end{aligned}$$

for all $a \leq x \leq b$*. In particular, if* $x = \frac{a+b}{2}$ *then we have*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(s)\,ds \right|$$
$$\leq \frac{(b-a)\left|f'(a)\right|\left|f'(b)\right|}{\left|f'(b)\right| - \left|f'(a)\right|}\left\{ \ln\left(\tfrac{|f'(a)|+|f'(b)|}{2}\right) + \frac{\ln\left[(b-a)^{(|f'(a)|+|f'(b)|)} \cdot \left|f'(a)\right|^{|f'(a)|} \cdot \left|f'(b)\right|^{|f'(b)|}\right]}{\left|f'(b)\right| - \left|f'(a)\right|} \right\}.$$

Proof. As in the proof of Theorem 3.1, taking into account that $|f'|$ is AH-convex (i.e., (2.3) holds), we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\
&\leq \int_a^b |K(x,s)| \left|f'(s)\right| ds \\
&\leq \int_a^b |K(x,s)| \cdot \frac{(b-a)\left|f'(a)\right|\left|f'(b)\right|}{(b-s)\left|f'(a)\right| + (s-a)\left|f'(b)\right|} ds \\
&= \left|f'(a)\right|\left|f'(b)\right| \int_a^x \frac{s-a}{(b-s)\left|f'(a)\right| + (s-a)\left|f'(b)\right|} ds \\
&\quad + \left|f'(a)\right|\left|f'(b)\right| \int_x^b \frac{b-s}{(b-s)\left|f'(a)\right| + (s-a)\left|f'(b)\right|} ds \\
&= \left|f'(a)\right|\left|f'(b)\right| \cdot \left[\frac{(a+b-2x) + (b-a)\ln\left[(x-a)\left|f'(a)\right| + (b-x)\left|f'(b)\right|\right]}{\left|f'(b)\right| - \left|f'(a)\right|} \right. \\
&\quad \left. + \frac{(b-a)\left|f'(a)\right|\ln\left[(b-a)\left|f'(a)\right|\right] + (b-a)\left|f'(b)\right|\ln\left[(b-a)\left|f'(b)\right|\right]}{\left(\left|f'(b)\right| - \left|f'(a)\right|\right)^2} \right],
\end{aligned}$$

which proves the required result in (3.4). □

Theorem 3.4. *Let* I *be a real interval,* $a, b \in \mathbb{R}$ $(a < b)$ *with* a, b *in* I° *(the interior of* I*). Let* $f : I \to \mathbb{R}$ *be an absolutely continuous function on* I*. If* $|f'|^p$ $(p > 1)$ *is* AH*-convex, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(s)\,ds \right|$$
$$\leq (b-a)^{-\frac{1}{q}} \left|f'(a)\right|\left|f'(b)\right| \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{1}{q}} \cdot L^{-\frac{1}{p}}\left(\left|f'(a)\right|^p, \left|f'(b)\right|^p\right), \quad (3.5)$$

for all $a \leq x \leq b$ *and* $p > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*. In particular, if* $x = \frac{a+b}{2}$ *then we have*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| \leq \frac{(b-a)}{2(q+1)^{\frac{1}{q}}} \left|f'(a)\right|\left|f'(b)\right| \cdot L^{-\frac{1}{p}}\left(\left|f'(a)\right|^p, \left|f'(b)\right|^p\right).$$

Proof. As in the proof of Theorem 3.2, and since $|f'|$ is AH-convex we have

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\
&\leq \left(\int_a^b |K(x,s)|^q\,ds\right)^{\frac{1}{q}}\left(\int_a^b |f'(s)|^p\,ds\right)^{\frac{1}{p}} \\
&\leq \frac{1}{b-a}\left(\int_a^x (s-a)^q\,ds+\int_x^b (b-s)^q\,ds\right)^{\frac{1}{q}}\left(\int_a^b \frac{(b-a)\,|f'(a)|^p\,|f'(b)|^p}{(b-s)\,|f'(a)|^p+(s-a)\,|f'(b)|^p}ds\right)^{\frac{1}{p}} \\
&= (b-a)^{-\frac{1}{q}}\,|f'(a)|\,|f'(b)|\left[\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}}\cdot\left[\frac{\ln|f'(b)|^p-\ln|f'(a)|^p}{|f'(b)|^p-|f'(a)|^p}\right]^{\frac{1}{p}} \\
&= (b-a)^{-\frac{1}{q}}\,|f'(a)|\,|f'(b)|\left[\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}}\cdot L^{-\frac{1}{p}}\left(|f'(a)|^p,|f'(b)|^p\right),
\end{aligned}$$

and this ends the proof. □

Remark 3.2. As $p\to 1$ and $q\to\infty$ in (3.5), then by (3.3), we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|\leq (b-a)\,|f'(a)|\,|f'(b)|\cdot L^{-1}\left(|f'(a)|,|f'(b)|\right),$$

for all $a\leq x\leq b$.

4. When $|f'|$ is G$\widehat{\text{N}}$-convex

In this section, inequalities of Ostrowski's type for absolutely continuous functions whose first derivatives in absolute value are G$\widehat{\text{N}}$-convex are proved, where $G:=G_t(a,b)$ and $\widehat{\text{N}}:=A_t(a,b),G_t(a,b),H_t(a,b)$.

4.1. When $|f'|$ is GA-convex.

Theorem 4.1. *Let I be a real interval, $a,b\in\mathbb{R}$ $(a<b)$ with a,b in I° (the interior of I). Let $f:I\to\mathbb{R}$ be an absolutely continuous function on I. If $|f'|$ is GA-convex, then*

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(s)\,ds\right| \\
&\leq \frac{1}{2(b-a)\ln\left(\frac{b}{a}\right)}\left\{\left[\frac{1}{2}\left(a^2+b^2-2x^2\right)-2(b-a)(a+b-x)\right]\left(|f'(b)|-|f'(a)|\right)\right. \qquad (4.1)\\
&\qquad\left.+\left[a^2|f'(a)|+b^2|f'(b)|\right]\ln\left(\frac{b}{a}\right)+2x(a+b-x)\ln\left(\frac{x^{|f'(a)|}a^{|f'(b)|}}{b^{|f'(a)|}x^{|f'(b)|}}\right)\right\},
\end{aligned}$$

for all $a \le x \le b$. In particular, if we choose $x = \frac{a+b}{2}$, then

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| \\
&\le \frac{1}{2(b-a)\ln\left(\frac{b}{a}\right)} \left\{ \frac{1}{4}\left[(b-a)^2 - 4\left(b^2-a^2\right)\right] \left(|f'(b)| - |f'(a)|\right) \right. \\
&\quad \left. + a^2|f'(a)| + b^2|f'(b)| \ln\left(\frac{b}{a}\right) + \frac{(a+b)^2}{2} \cdot \ln\left(\frac{\left(\frac{a+b}{2}\right)^{|f'(a)|} a^{|f'(b)|}}{b^{|f'(a)|}\left(\frac{a+b}{2}\right)^{|f'(b)|}} \right) \right\}.
\end{aligned}$$

Proof. As in the proof of Theorem 3.1, taking into account that $|f'|$ is GA-convex (i.e., (2.4) holds), we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\
&\le \int_a^b |K(x,s)|\,|f'(s)|\,ds \\
&\le \int_a^b |K(x,s)| \left[\frac{\ln(b)-\ln(s)}{\ln(b)-\ln(a)} \cdot |f'(a)| + \frac{\ln(s)-\ln(a)}{\ln(b)-\ln(a)} \cdot |f'(b)| \right] ds \\
&= \frac{1}{b-a}\int_a^x (s-a) \left[\frac{\ln(b)-\ln(s)}{\ln(b)-\ln(a)} \cdot |f'(a)| + \frac{\ln(s)-\ln(a)}{\ln(b)-\ln(a)} \cdot |f'(b)| \right] ds \\
&\quad + \frac{1}{b-a}\int_x^b (b-s) \left[\frac{\ln(b)-\ln(s)}{\ln(b)-\ln(a)} \cdot |f'(a)| + \frac{\ln(s)-\ln(a)}{\ln(b)-\ln(a)} \cdot |f'(b)| \right] ds \\
&= \frac{1}{(b-a)\ln\left(\frac{b}{a}\right)} \left\{ \left[\frac{1}{4}\left(a^2+b^2-2x^2\right) - (b-a)(a+b-x) \right] \left(|f'(b)| - |f'(a)|\right) \right. \\
&\quad \left. + \frac{1}{2}\left[a^2|f'(a)| + b^2|f'(b)|\right] \ln\left(\frac{b}{a}\right) + x(a+b-x)\ln\left(\frac{x^{|f'(a)|}a^{|f'(b)|}}{b^{|f'(a)|}x^{|f'(b)|}} \right) \right\},
\end{aligned}$$

the last equality follows from integration by parts and basic simplification, and this proves the desired result in (4.1). □

Theorem 4.2. *Let I be a real interval, $a, b \in \mathbb{R}$ $(a < b)$ with a, b in I° (the interior of I). Let $f : I \to \mathbb{R}$ be an absolutely continuous function on I. If $|f'|^p$ $(p > 1)$ is GA-convex, then*

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| \\
&\le \frac{1}{b-a}\left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{1}{q}} \qquad (4.2) \\
&\quad \times \left[\frac{\left(b|f'(b)|^p - a|f'(a)|^p\right)\ln\left(\frac{b}{a}\right) - (b-a)\left(|f'(b)|^p - |f'(a)|^p\right)}{\ln b - \ln a} \right]^{\frac{1}{p}},
\end{aligned}$$

for all $a \le x \le b$ and $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. In particular, if we choose $x = \frac{a+b}{2}$, then

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(s)\,ds \right|$$
$$\le \frac{(b-a)^{\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}} \left[\frac{\left(b\left|f'(b)\right|^p - a\left|f'(a)\right|^p\right)\ln\left(\frac{b}{a}\right) - (b-a)\left(\left|f'(b)\right|^p - \left|f'(a)\right|^p\right)}{\ln b - \ln a} \right]^{\frac{1}{p}}.$$

Proof. As in the proof of Theorem 3.2 and since $|f'|^p$ is $(p > 1)$ is GA-convex, then we have

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right|$$
$$\le \left(\int_a^b |K(x,s)|^q\,ds \right)^{\frac{1}{q}} \left(\int_a^b \left|f'(s)\right|^p ds \right)^{\frac{1}{p}}$$
$$\le \frac{1}{b-a} \left(\int_a^x (s-a)^q\,ds + \int_x^b (b-s)^q\,ds \right)^{\frac{1}{q}}$$
$$\times \left(\int_a^b \left[\frac{\ln(b) - \ln(s)}{\ln(b) - \ln(a)} \cdot \left|f'(a)\right|^p + \frac{\ln(s) - \ln(a)}{\ln(b) - \ln(a)} \cdot \left|f'(b)\right|^p \right] ds \right)^{\frac{1}{p}}$$
$$= \frac{1}{b-a} \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{1}{q}}$$
$$\times \left[\frac{\left(b\left|f'(b)\right|^p - a\left|f'(a)\right|^p\right)\ln\left(\frac{b}{a}\right) - (b-a)\left(\left|f'(b)\right|^p - \left|f'(a)\right|^p\right)}{\ln b - \ln a} \right]^{\frac{1}{p}},$$

which proves the required result. □

Remark 4.1. As $p \to 1$ and $q \to \infty$ in (4.2), then by (3.3) we have

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(s)\,ds \right|$$
$$\le \left[\frac{\left(b\left|f'(b)\right| - a\left|f'(a)\right|\right)\ln\left(\frac{b}{a}\right) - (b-a)\left(\left|f'(b)\right| - \left|f'(a)\right|\right)}{\ln b - \ln a} \right],$$

for all $a \le x \le b$.

4.2. **When $|f'|$ is GG-convex.**

Theorem 4.3. *Let I be a real interval, $a,b\in\mathbb{R}$ $(a<b)$ with a,b in I° (the interior of I). Let $f:I\to\mathbb{R}$ be an absolutely continuous function on I. If $|f'|$ is GG-convex, then*

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(s)\,ds\right| \\
&\leq\left[\frac{2x^2\ln\left(\frac{b}{a}\right)}{\ln\left(\frac{b}{a}\right)^2+\ln\left(\frac{|f'(b)|}{|f'(a)|}\right)}+\frac{x(b-a)\ln\left(\frac{b}{a}\right)}{\ln\left(\frac{b}{a}\right)+\ln\left(\frac{|f'(b)|}{|f'(a)|}\right)}\right]\left[|f'(a)|\right]^{\frac{\ln(b)-\ln(x)}{\ln(b)-\ln(a)}}\left[|f'(b)|\right]^{\frac{\ln(x)-\ln(a)}{\ln(b)-\ln(a)}} \quad (4.3)\\
&\quad+\left[\frac{a^2\ln\left(\frac{b}{a}\right)}{\ln\left(\frac{b}{a}\right)^2+\ln\left(\frac{|f'(b)|}{|f'(a)|}\right)}-\frac{a^2\ln\left(\frac{b}{a}\right)}{\ln\left(\frac{b}{a}\right)+\ln\left(\frac{|f'(b)|}{|f'(a)|}\right)}\right]|f'(a)| \\
&\quad+\left[\frac{b^2\ln\left(\frac{b}{a}\right)}{\ln\left(\frac{b}{a}\right)^2+\ln\left(\frac{|f'(b)|}{|f'(a)|}\right)}-\frac{b^2\ln\left(\frac{b}{a}\right)}{\ln\left(\frac{b}{a}\right)+\ln\left(\frac{|f'(b)|}{|f'(a)|}\right)}\right]|f'(b)|,
\end{aligned}$$

for all $a\leq x\leq b$.

Proof. As in the proof of Theorem 4.2, and since $|f'|$ is *GG*-convex (i.e., (2.5) holds), we have

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\
&\leq\int_a^b|K(x,s)|\,|f'(s)|\,ds \\
&\leq\int_a^b|K(x,s)|\cdot\left[|f'(a)|\right]^{\frac{\ln(b)-\ln(s)}{\ln(b)-\ln(a)}}\left[|f'(b)|\right]^{\frac{\ln(s)-\ln(a)}{\ln(b)-\ln(a)}}ds \\
&=\frac{1}{b-a}\int_a^x(s-a)\left[|f'(a)|\right]^{\frac{\ln(b)-\ln(s)}{\ln(b)-\ln(a)}}\left[|f'(b)|\right]^{\frac{\ln(s)-\ln(a)}{\ln(b)-\ln(a)}}ds \\
&\quad+\frac{1}{b-a}\int_x^b(b-s)\left[|f'(a)|\right]^{\frac{\ln(b)-\ln(s)}{\ln(b)-\ln(a)}}\left[|f'(b)|\right]^{\frac{\ln(s)-\ln(a)}{\ln(b)-\ln(a)}}ds \\
&=\left[\frac{2x^2\ln\left(\frac{b}{a}\right)}{\ln\left(\frac{b}{a}\right)^2+\ln\left(\frac{|f'(b)|}{|f'(a)|}\right)}+\frac{x(b-a)\ln\left(\frac{b}{a}\right)}{\ln\left(\frac{b}{a}\right)+\ln\left(\frac{|f'(b)|}{|f'(a)|}\right)}\right]\left[|f'(a)|\right]^{\frac{\ln(b)-\ln(x)}{\ln(b)-\ln(a)}}\left[|f'(b)|\right]^{\frac{\ln(x)-\ln(a)}{\ln(b)-\ln(a)}} \\
&\quad+\left[\frac{a^2\ln\left(\frac{b}{a}\right)}{\ln\left(\frac{b}{a}\right)^2+\ln\left(\frac{|f'(b)|}{|f'(a)|}\right)}-\frac{a^2\ln\left(\frac{b}{a}\right)}{\ln\left(\frac{b}{a}\right)+\ln\left(\frac{|f'(b)|}{|f'(a)|}\right)}\right]|f'(a)| \\
&\quad+\left[\frac{b^2\ln\left(\frac{b}{a}\right)}{\ln\left(\frac{b}{a}\right)^2+\ln\left(\frac{|f'(b)|}{|f'(a)|}\right)}-\frac{b^2\ln\left(\frac{b}{a}\right)}{\ln\left(\frac{b}{a}\right)+\ln\left(\frac{|f'(b)|}{|f'(a)|}\right)}\right]|f'(b)|,
\end{aligned}$$

where the last equality holds using integration by parts, and this ends the proof. □

Theorem 4.4. *Let I be a real interval, $a,b\in\mathbb{R}$ $(a<b)$ with a,b in I° (the interior of I). Let $f:I\to\mathbb{R}$ be an absolutely continuous function on I. If $|f'|^p$ $(p>1)$ is GG-convex,* *then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|$$
$$\leq \frac{1}{b-a}\left[\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}}\times\left[\frac{\left(b\left|f'(b)\right|^p-a\left|f'(a)\right|^p\right)\ln\left(\frac{b}{a}\right)}{\ln\left(\frac{b}{a}\right)+p\ln\left(\frac{|f'(b)|}{|f'(a)|}\right)}\right]^{\frac{1}{p}}, \quad (4.4)$$

for all $a\leq x\leq b$ *and* $p>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$. *In particular for* $x=\frac{a+b}{2}$ *we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|\leq\frac{(b-a)^{\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}\cdot\left[\frac{\left(b\left|f'(b)\right|^p-a\left|f'(a)\right|^p\right)\ln\left(\frac{b}{a}\right)}{\ln\left(\frac{b}{a}\right)+p\ln\left(\frac{|f'(b)|}{|f'(a)|}\right)}\right]^{\frac{1}{p}}.$$

Proof. As in the proof of Theorem 4.2 talking into account that $|f'|^p$ $(p>1)$ is GG-convex, then

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq\left(\int_a^b |K(x,s)|^q\,ds\right)^{\frac{1}{q}}\left(\int_a^b |f'(s)|^p\,ds\right)^{\frac{1}{p}}$$
$$\leq\frac{1}{b-a}\left(\int_a^x (s-a)^q\,ds+\int_x^b (b-s)^q\,ds\right)^{\frac{1}{q}}$$
$$\times\left(\int_a^b \left[|f'(a)|\right]^{p\cdot\frac{\ln(b)-\ln(s)}{\ln(b)-\ln(a)}}\left[|f'(b)|\right]^{p\cdot\frac{\ln(s)-\ln(a)}{\ln(b)-\ln(a)}}\,ds\right)^{\frac{1}{p}}$$
$$=\frac{1}{b-a}\left[\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}}\times\left[\frac{\left(b\left|f'(b)\right|^p-a\left|f'(a)\right|^p\right)\ln\left(\frac{b}{a}\right)}{\ln\left(\frac{b}{a}\right)+p\ln\left(\frac{|f'(b)|}{|f'(a)|}\right)}\right]^{\frac{1}{p}},$$

which proves the required result. □

Remark 4.2. As $p\to 1$ and $q\to\infty$ in (4.4), then by (3.3) we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|\leq\frac{\left(b\left|f'(b)\right|-a\left|f'(a)\right|\right)\ln\left(\frac{b}{a}\right)}{\ln\left(\frac{b}{a}\right)+\ln\left(\frac{|f'(b)|}{|f'(a)|}\right)},$$

for all $a\leq x\leq b$.

4.3. **When $|f'|$ is GH-convex.**

Theorem 4.5. *Let I be a real interval, $a, b \in \mathbb{R}$ $(a < b)$ with a, b in I° (the interior of I). Let $f : I \to \mathbb{R}$ be an absolutely continuous function on I. If $|f'|$ is GH-convex, then*

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| \\
&\le \frac{|f'(a)|\,|f'(b)| \ln\left(\frac{b}{a}\right)}{b-a} \cdot \frac{a^{\frac{|f'(b)|}{|f'(b)|-|f'(a)|}} \cdot b^{\frac{|f'(a)|}{|f'(a)|-|f'(b)|}}}{|f'(b)| - |f'(a)|} \\
&\quad \times \left\{ (x-a)\left[Ei\left(\frac{|f'(a)|\ln\left(\frac{a}{b}\right)}{|f'(b)|-|f'(a)|} \right) - Ei\left(\frac{|f'(b)|\ln\left(\frac{a}{x}\right) + |f'(a)|\ln\left(\frac{x}{b}\right)}{|f'(b)|-|f'(a)|} \right) \right] \right. \\
&\quad\quad \left. + (b-x)\left[Ei\left(\frac{|f'(b)|\ln\left(\frac{a}{b}\right)}{|f'(b)|-|f'(a)|} \right) - Ei\left(\frac{|f'(a)|\ln\left(\frac{x}{b}\right) + |f'(b)|\ln\left(\frac{a}{x}\right)}{|f'(b)|-|f'(a)|} \right) \right] \right\},
\end{aligned} \tag{4.5}$$

for all $a \le x \le b$, where

$$Ei(x) = V.P. \int_{-x}^{\infty} \frac{e^{-t}}{t}\,dt$$

is the Exponential Integral function.

Proof. Repeating the proof of Theorem 4.3 and using the fact the $|f'|$ is GH-convex (i.e., (2.6) holds), so that we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\
&\le \int_a^b |K(x,s)|\,|f'(s)|\,ds \\
&\le \int_a^b |K(x,s)| \cdot \left[\frac{|f'(a)|\,|f'(b)|\,[\ln(b) - \ln(a)]}{[\ln(b) - \ln(s)]\,|f'(a)| + [\ln(s) - \ln(a)]\,|f'(b)|} \right] ds \\
&= \frac{|f'(a)|\,|f'(b)| \ln\left(\frac{b}{a}\right)}{b-a} \cdot \int_a^x \frac{s-a}{[\ln(b) - \ln(s)]\,|f'(a)| + [\ln(s) - \ln(a)]\,|f'(b)|} ds \\
&\quad + \frac{|f'(a)|\,|f'(b)| \ln\left(\frac{b}{a}\right)}{b-a} \cdot \int_x^b \frac{b-s}{[\ln(b) - \ln(s)]\,|f'(a)| + [\ln(s) - \ln(a)]\,|f'(b)|} ds.
\end{aligned}$$

The last two integrations cannot be evaluated directly by using usual methods. So that, we use estimation; since

$$\begin{aligned}
&\int_a^x \frac{s-a}{[\ln(b) - \ln(s)]\,|f'(a)| + [\ln(s) - \ln(\alpha)]\,|f'(b)|} ds \\
&\qquad \le (x-a) \int_a^x \frac{ds}{[\ln(b) - \ln(s)]\,|f'(a)| + [\ln(s) - \ln(a)]\,|f'(b)|},
\end{aligned} \tag{4.6}$$

and

$$\int_x^b \frac{b-s}{[\ln(b)-\ln(s)]|f'(a)|+[\ln(s)-\ln(\alpha)]|f'(b)|}ds$$
$$\leq (b-x)\int_x^b \frac{ds}{[\ln(b)-\ln(s)]|f'(a)|+[\ln(s)-\ln(a)]|f'(b)|}. \tag{4.7}$$

Now, using Maple Software in evaluating the right hand sides of (4.6) and (4.7), we get that

$$\int_a^x \frac{ds}{[\ln(b)-\ln(s)]|f'(a)|+[\ln(s)-\ln(a)]|f'(b)|}$$
$$= \frac{a^{\frac{|f'(b)|}{|f'(b)|-|f'(a)|}}\cdot b^{\frac{|f'(a)|}{|f'(a)|-|f'(b)|}}}{|f'(b)|-|f'(a)|}$$
$$\times\left[Ei\left(\frac{|f'(a)|\ln\left(\frac{a}{b}\right)}{|f'(b)|-|f'(a)|}\right)-Ei\left(\frac{|f'(b)|\ln\left(\frac{a}{x}\right)+|f'(a)|\ln\left(\frac{x}{b}\right)}{|f'(b)|-|f'(a)|}\right)\right],$$

and

$$\int_x^b \frac{ds}{[\ln(b)-\ln(s)]|f'(a)|+[\ln(s)-\ln(a)]|f'(b)|}$$
$$= \frac{a^{\frac{|f'(b)|}{|f'(b)|-|f'(a)|}}\cdot b^{\frac{|f'(a)|}{|f'(a)|-|f'(b)|}}}{|f'(b)|-|f'(a)|}$$
$$\times\left[Ei\left(\frac{|f'(b)|\ln\left(\frac{a}{b}\right)}{|f'(b)|-|f'(a)|}\right)-Ei\left(\frac{|f'(a)|\ln\left(\frac{x}{b}\right)+|f'(b)|\ln\left(\frac{a}{x}\right)}{|f'(b)|-|f'(a)|}\right)\right].$$

Substituting the above two integrals in (4.6) and (4.7) and combining the result with the main inequality we get the desired result in (4.5). □

The bound in (4.5) is complicated and not applicable, so that let us give the following refinement of the inequality (4.5).

Corollary 4.1. *Let I be a real interval, $a,b\in\mathbb{R}$ $(a<b)$ with a,b in I° (the interior of I). Let $f: I\to\mathbb{R}$ be an absolutely continuous function on I. If $|f'|$ is GH-convex, then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|$$
$$\leq \frac{|f'(a)|\,|f'(b)|\ln\left(\frac{b}{a}\right)}{b-a}\cdot\left[\frac{b-a}{4}+\left(x-\frac{a+b}{2}\right)^2\right]\cdot\max\{K_1,K_2\}, \tag{4.8}$$

for all $a\leq x\leq b$. In particular if $x=\frac{a+b}{2}$, then we have

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|\leq\frac{|f'(a)|\,|f'(b)|\ln\left(\frac{b}{a}\right)}{4}\cdot\max\{K_1,K_2\},$$

where

$$K_1=\sup_{a\leq s\leq x}\left\{\frac{1}{[\ln(b)-\ln(s)]|f'(a)|+[\ln(s)-\ln(a)]|f'(b)|}\right\},$$

and

$$K_2 = \sup_{x \le s \le b} \left\{ \frac{1}{[\ln(b) - \ln(s)] |f'(a)| + [\ln(s) - \ln(a)] |f'(b)|} \right\}.$$

Proof. From the proof of Theorem 4.5, we have

$$\begin{aligned}
&\int_a^x \frac{s-a}{[\ln(b) - \ln(s)] |f'(a)| + [\ln(s) - \ln(a)] |f'(b)|} ds \\
&= \sup_{a \le s \le x} \left\{ \frac{1}{[\ln(b) - \ln(s)] |f'(a)| + [\ln(s) - \ln(a)] |f'(b)|} \right\} \int_a^x (s-a) ds \\
&= \frac{(x-a)^2}{2} \sup_{a \le s \le x} \left\{ \frac{1}{[\ln(b) - \ln(s)] |f'(a)| + [\ln(s) - \ln(a)] |f'(b)|} \right\},
\end{aligned}$$

and

$$\begin{aligned}
&\int_x^b \frac{b-s}{[\ln(b) - \ln(s)] |f'(a)| + [\ln(s) - \ln(a)] |f'(b)|} ds \\
&= \sup_{a \le s \le x} \left\{ \frac{1}{[\ln(b) - \ln(s)] |f'(a)| + [\ln(s) - \ln(a)] |f'(b)|} \right\} \int_a^x (b-s) ds \\
&= \frac{(b-x)^2}{2} \sup_{x \le s \le b} \left\{ \frac{1}{[\ln(b) - \ln(s)] |f'(a)| + [\ln(s) - \ln(a)] |f'(b)|} \right\}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a} \int_a^b f(u)\, du \right| \\
&\le \int_a^b |K(x,s)| \left| f'(s) \right| ds \\
&\le \int_a^b |K(x,s)| \cdot \left[\frac{|f'(a)| |f'(b)| [\ln(b) - \ln(a)]}{[\ln(b) - \ln(s)] |f'(a)| + [\ln(s) - \ln(a)] |f'(b)|} \right] ds \\
&\le \frac{|f'(a)| |f'(b)| \ln\left(\frac{b}{a}\right)}{b-a} \cdot \left[\frac{b-a}{4} + \left(x - \frac{a+b}{2} \right)^2 \right] \cdot \max\{K_1, K_2\},
\end{aligned}$$

where K_1 and K_2 are defined above. □

A refinement of Theorem 4.4 is given as:

Theorem 4.6. *Let I be a real interval, $a, b \in \mathbb{R}$ $(a < b)$ with a, b in I° (the interior of I). Let $f : I \to \mathbb{R}$ be an absolutely continuous function on I. If $|f'|^p$ $(p > 1)$ is GH-convex, then*

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a} \int_a^b f(s)\, ds \right| \\
&\qquad \le \left(\frac{\ln b - \ln a}{b-a} \right)^{\frac{1}{q}} \max\left\{ \left| f'(a) \right|, \left| f'(b) \right| \right\} \cdot \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{1}{q}}, \quad (4.9)
\end{aligned}$$

for all $a \le x \le b$ and $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. As in the proof of Theorem 4.4 (and as in Theorem 4.5), we have

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\
&\leq \left(\int_a^b |K(x,s)|^q\,ds\right)^{\frac{1}{q}}\left(\int_a^b |f'(s)|^p\,ds\right)^{\frac{1}{p}} \\
&\leq \frac{|f'(a)|\,|f'(b)|\ln\left(\frac{b}{a}\right)}{b-a}\left(\int_a^x (s-a)^q\,ds+\int_x^b (b-s)^q\,ds\right)^{\frac{1}{q}} \\
&\qquad\times\left(\int_a^b \frac{ds}{[\ln(b)-\ln(s)]\,|f'(a)|^p+[\ln(s)-\ln(a)]\,|f'(b)|^p}\right)^{\frac{1}{p}} \\
&\leq \frac{|f'(a)|\,|f'(b)|\ln\left(\frac{b}{a}\right)}{b-a}\left(\int_a^x (s-a)^q\,ds+\int_x^b (b-s)^q\,ds\right)^{\frac{1}{q}} \\
&\qquad\times\sup_{a\leq s\leq b}\left\{\frac{1}{[\ln(b)-\ln(s)]\,|f'(a)|^p+[\ln(s)-\ln(a)]\,|f'(b)|^p}\right\}^{\frac{1}{p}} \\
&\leq (b-a)^{-\frac{1}{q}}\,|f'(a)|\,|f'(b)|\ln\left(\frac{b}{a}\right)\left[\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}} \\
&\qquad\qquad\times[\ln(b)-\ln(a)]^{-\frac{1}{p}}\max\left\{\frac{1}{|f'(a)|},\frac{1}{|f'(b)|}\right\} \\
&= (b-a)^{-\frac{1}{q}}\,[\ln(b)-\ln(a)]^{\frac{1}{q}}\max\{|f'(a)|,|f'(b)|\}\cdot\left[\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}},
\end{aligned}$$

which proves the required result. □

5. When $|f'|$ is $\mathrm{H\widehat{N}}$-convex

In this section, inequalities of Ostrowski's type for absolutely continuous functions whose first derivatives in absolute value are $\mathrm{H\widehat{N}}$-convex are proved, where $H := H_t(a,b)$ and $\widehat{\mathrm{N}} := A_t(a,b), G_t(a,b), H_t(a,b)$.

5.1. When $|f'|$ is HA-convex.

Theorem 5.1. *Let I be a real interval, $a,b\in\mathbb{R}$ $(a<b)$ with a,b in I° (the interior of I). Let $f: I\to\mathbb{R}$ be an absolutely continuous function on I. If $|f'|$ is HA-convex, then*

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(s)\,ds\right| \\
&\leq \frac{b\cdot|f'(a)|-a\cdot|f'(b)|}{(b-a)^2}\left[\frac{b-a}{4}+\left(x-\frac{a+b}{2}\right)^2\right] \qquad (5.1)\\
&\qquad+\frac{ab}{(b-a)^2}\left[\ln\left(\frac{a^a b^b}{x^{a+b}}\right)-(a+b-2x)\right]\left[|f'(b)|-|f'(a)|\right],
\end{aligned}$$

for all $a \leq x \leq b$. In particular if $x = \frac{a+b}{2}$, then we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a} \int_a^b f(s)\, ds \right| \\
&\leq \frac{b \cdot |f'(a)| - a \cdot |f'(b)|}{4(b-a)} + \frac{ab}{(b-a)^2} \ln \left(\frac{2^{a+b} a^a b^b}{(a+b)^{a+b}} \right) [|f'(b)| - |f'(a)|] .
\end{aligned}$$

Proof. As in the proof of Theorem 3.1, taking into account that $|f'|$ is HA-convex (i.e., (2.7) holds), we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a} \int_a^b f(u)\, du \right| \\
&\leq \int_a^b |K(x,s)| \, |f'(s)| \, ds \\
&\leq \int_a^b |K(x,s)| \cdot \left[\frac{b(s-a)}{s(b-a)} |f'(a)| + \frac{a(b-s)}{s(b-a)} |f'(b)| \right] ds \\
&= \frac{1}{b-a} \int_a^x (s-a) \left[\frac{b(s-a)}{s(b-a)} |f'(a)| + \frac{a(b-s)}{s(b-a)} |f'(b)| \right] ds \\
&\quad + \frac{1}{b-a} \int_x^b (b-s) \left[\frac{b(s-a)}{s(b-a)} |f'(a)| + \frac{a(b-s)}{s(b-a)} |f'(b)| \right] ds \\
&= \frac{b \cdot |f'(a)| - a \cdot |f'(b)|}{(b-a)^2} \left[\frac{b-a}{4} + \left(x - \frac{a+b}{2} \right)^2 \right] \\
&\quad - \frac{ab}{(b-a)^2} \left[\ln \left(\frac{x^{a+b}}{a^a b^b} \right) + (a+b-2x) \right] [|f'(b)| - |f'(a)|] ,
\end{aligned}$$

which proves the required result. □

Theorem 5.2. *Let I be a real interval, $a, b \in \mathbb{R}$ $(a < b)$ with a, b in I° (the interior of I). Let $f : I \to \mathbb{R}$ be an absolutely continuous function on I. If $|f'|^p$ $(p > 1)$ is HA-convex, then*

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a} \int_a^b f(s)\, ds \right| \\
&\leq \frac{1}{(b-a)^2} \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{1}{q}} \\
&\quad \times \left\{ b \left[(x-a) + \ln \left(\frac{a^a}{x^a} \right) \right] |f'(a)|^p + a \left[\ln \left(\frac{b^b}{x^b} \right) - (b-x) \right] |f'(b)|^p \right\}^{\frac{1}{p}} ,
\end{aligned} \tag{5.2}$$

for all $a \leq x \leq b$ and $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. As in the proof of Theorem 3.2, taking into account that $|f'|^p$ is HA-convex, we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\
&\le \left(\int_a^b |K(x,s)|^q\,ds\right)^{\frac{1}{q}} \left(\int_a^b |f'(s)|^p\,ds\right)^{\frac{1}{p}} \\
&\le \frac{1}{b-a}\left(\int_a^x (s-a)^q\,ds + \int_x^b (b-s)^q\,ds\right)^{\frac{1}{q}} \\
&\qquad \times \left(\int_a^b \left[\frac{b(s-a)}{s(b-a)}|f'(a)|^p + \frac{a(b-s)}{s(b-a)}|f'(b)|^p\right]ds\right)^{\frac{1}{p}} \\
&= \frac{1}{(b-a)^2}\left[\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}} \\
&\qquad \times \left\{ b\left[(x-a) + \ln\left(\frac{a^a}{x^a}\right)\right]|f'(a)|^p + a\left[\ln\left(\frac{b^b}{x^b}\right) - (b-x)\right]|f'(b)|^p \right\}^{\frac{1}{p}},
\end{aligned}$$

and this proves the required result in (5.2). ☐

Remark 5.1. As $p \to 1$ and $q \to \infty$ in (5.2), then by (3.3) we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| \\
&\qquad \le \frac{1}{b-a}\left\{ b\left[(x-a) + \ln\left(\frac{a^a}{x^a}\right)\right]|f'(a)| + a\left[\ln\left(\frac{b^b}{x^b}\right) - (b-x)\right]|f'(b)| \right\},
\end{aligned}$$

for all $a \le x \le b$.

5.2. When $|f'|$ is HG-convex.

Theorem 5.3. *Let I be a real interval, $a, b \in \mathbb{R}$ $(a < b)$ with a, b in I° (the interior of I). Let $f : I \to \mathbb{R}$ be an absolutely continuous function on I. If $|f'|$ is HG-convex, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| \le \frac{1}{b-a}\cdot \max\{K_3, K_4\}\left[\frac{b-a}{4} + \left(x - \frac{a+b}{2}\right)^2\right], \qquad (5.3)$$

where

$$K_3 = \max\left\{ |f'(b)|, |f'(a)|^{\frac{b(x-a)}{x(b-a)}} |f'(b)|^{\frac{a(b-x)}{x(b-a)}} \right\}$$

and

$$K_4 = \max\left\{ |f'(a)|, |f'(a)|^{\frac{b(x-a)}{x(b-a)}} |f'(b)|^{\frac{a(b-x)}{x(b-a)}} \right\},$$

for all $a \le x \le b$.

Proof. From (5.1), since $|f'|$ is HG-convex (i.e., (2.8) holds), we have

$$\begin{aligned}\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| &\le \int_a^b |K(x,s)|\,|f'(s)|\,ds \\ &\le \int_a^b |K(x,s)|\cdot|f'(a)|^{\frac{b(s-a)}{s(b-a)}}\,|f'(b)|^{\frac{a(b-s)}{s(b-a)}}\,ds \\ &= \frac{1}{b-a}\int_a^x (s-a)\,|f'(a)|^{\frac{b(s-a)}{s(b-a)}}\,|f'(b)|^{\frac{a(b-s)}{s(b-a)}}\,ds \\ &\quad + \frac{1}{b-a}\int_x^b (b-s)\,|f'(a)|^{\frac{b(s-a)}{s(b-a)}}\,|f'(b)|^{\frac{a(b-s)}{s(b-a)}}\,ds.\end{aligned}$$

The above two integrals can be evaluated in terms of Integral Exponential function, but the resulting bound is useless. Therefore, let us write

$$\begin{aligned}&\int_a^x (s-a)\,|f'(a)|^{\frac{b(s-a)}{s(b-a)}}\,|f'(b)|^{\frac{a(b-s)}{s(b-a)}}\,ds \\ &\le \sup_{a\le s\le x}\left\{|f'(a)|^{\frac{b(s-a)}{s(b-a)}}\,|f'(b)|^{\frac{a(b-s)}{s(b-a)}}\right\}\int_a^x (s-a)\,ds \\ &= \frac{(x-a)^2}{2}\max\left\{|f'(b)|\,,|f'(a)|^{\frac{b(x-a)}{x(b-a)}}\,|f'(b)|^{\frac{a(b-x)}{x(b-a)}}\right\}\end{aligned}$$

and

$$\begin{aligned}&\int_x^b (b-s)\,|f'(a)|^{\frac{b(s-a)}{s(b-a)}}\,|f'(b)|^{\frac{a(b-s)}{s(b-a)}}\,ds \\ &\le \sup_{a\le s\le x}\left\{|f'(a)|^{\frac{b(s-a)}{s(b-a)}}\,|f'(b)|^{\frac{a(b-s)}{s(b-a)}}\right\}\int_x^b (b-s)\,ds \\ &= \frac{(b-x)^2}{2}\max\left\{|f'(a)|\,,|f'(a)|^{\frac{b(x-a)}{x(b-a)}}\,|f'(b)|^{\frac{a(b-x)}{x(b-a)}}\right\}.\end{aligned}$$

Hence,

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \le \frac{1}{b-a}\cdot\max\{K_3,K_4\}\left[\frac{b-a}{4}+\left(x-\frac{a+b}{2}\right)^2\right],$$

and this ends the proof. □

Theorem 5.4. *Let I be a real interval, $a,b\in\mathbb{R}$ $(a<b)$ with a,b in I° (the interior of I). Let $f: I\to\mathbb{R}$ be an absolutely continuous function on I. If $|f'|^p$ $(p>1)$ is HG-convex, then*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(s)\,ds\right| \\ &\qquad \le (b-a)^{-2+\frac{1}{p}}\left[\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}}\cdot\max\{|f'(a)|\,,|f'(b)|\}\,, \quad (5.4)\end{aligned}$$

for all $a\le x\le b$ and $p>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. As in the proof of Theorem 5.2, taking into account that $|f'|^p$ is HG-convex, we have

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\
&\le \left(\int_a^b |K(x,s)|^q\,ds\right)^{\frac{1}{q}}\left(\int_a^b |f'(s)|^p\,ds\right)^{\frac{1}{p}} \\
&\le \frac{1}{b-a}\left(\int_a^x (s-a)^q\,ds+\int_x^b (b-s)^q\,ds\right)^{\frac{1}{q}} \\
&\qquad\times\left(\int_a^b |f'(a)|^{p\cdot\frac{b(s-a)}{s(b-a)}}\,|f'(b)|^{p\cdot\frac{a(b-s)}{s(b-a)}}\,ds\right)^{\frac{1}{p}} \\
&\le \frac{1}{(b-a)^2}\left[\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}}\cdot\max\{|f'(a)|,|f'(b)|\}\,(b-a)^{\frac{1}{p}},
\end{aligned}$$

and this ends the proof. □

5.3. When $|f'|$ is HH-convex.

Theorem 5.5. *Let I be a real interval, $a,b\in\mathbb{R}$ $(a<b)$ with a,b in I° (the interior of I). Let $f:I\to\mathbb{R}$ be an absolutely continuous function on I. If $|f'|$ is HH-convex, then*

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(s)\,ds\right| \qquad (5.5)\\
&\le |f'(a)|\,|f'(b)|\left[\frac{a^2+b^2-2x^2}{2\left[a|f'(b)|-b|f'(a)|\right]}+(b-a)\frac{a(a+x)|f'(b)|+b(b-x)|f'(a)|}{\left[a|f'(b)|-b|f'(a)|\right]^2}\right. \\
&\quad+(b-a)\frac{ab\left[|f'(b)|-|f'(a)|\right]}{\left[a|f'(b)|-b|f'(a)|\right]^3}\times\left\{\ln\left(\frac{\left[b(b-a)|f'(a)|\right]^{b|f'(a)|}}{\left[a|f'(b)|(b-x)+b|f'(a)|(x-a)\right]^{b|f'(a)|}}\right)\right. \\
&\qquad\left.\left.+\ln\left(\frac{\left[a(b-a)|f'(b)|\right]^{a|f'(b)|}}{\left[b(x-a)|f'(a)|+a(b-x)|f'(b)|\right]^{a|f'(b)|}}\right)\right\}\right],
\end{aligned}$$

for all $a\le x\le b$. In particular if $x=\frac{a+b}{2}$, then we have

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(s)\,ds\right| \\
&\le \frac{(b-a)^2}{4\left[a|f'(b)|-b|f'(a)|\right]}+(b-a)\frac{a(3a+b)|f'(b)|+b(b-a)|f'(a)|}{2\left[a|f'(b)|-b|f'(a)|\right]^2} \\
&\quad+(b-a)\frac{ab\left[|f'(b)|-|f'(a)|\right]}{\left[a|f'(b)|-b|f'(a)|\right]^3}\times\left\{\ln\left(\frac{\left[2b|f'(a)|\right]^{b|f'(a)|}}{\left[a|f'(b)|+b|f'(a)|\right]^{b|f'(a)|}}\right)\right. \\
&\qquad\left.+\ln\left(\frac{\left[2a|f'(b)|\right]^{a|f'(b)|}}{\left[b|f'(a)|+a|f'(b)|\right]^{a|f'(b)|}}\right)\right\}.
\end{aligned}$$

Proof. As in the proof of Theorem 3.1, taking into account that $|f'|$ is HH-convex (i.e., (2.9) holds), we have

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\
&\leq \int_a^b |K(x,s)|\,|f'(s)|\,ds \\
&\leq \int_a^b |K(x,s)|\,\frac{s(b-a)|f'(a)||f'(b)|}{b(s-a)|f'(a)|+a(b-s)|f'(b)|}ds \\
&= \frac{1}{b-a}\int_a^x (s-a)\,\frac{s(b-a)|f'(a)||f'(b)|}{b(s-a)|f'(a)|+a(b-s)|f'(b)|}ds \\
&\quad+\frac{1}{b-a}\int_x^b (b-s)\,\frac{s(b-a)|f'(a)||f'(b)|}{b(s-a)|f'(a)|+a(b-s)|f'(b)|}ds \\
&= |f'(a)||f'(b)|\left[\frac{a^2+b^2-2x^2}{2[a|f'(b)|-b|f'(a)|]}+(b-a)\,\frac{a(a+x)|f'(b)|+b(b-x)|f'(a)|}{[a|f'(b)|-b|f'(a)|]^2}\right. \\
&\quad+(b-a)\,\frac{ab[|f'(b)|-|f'(a)|]}{[a|f'(b)|-b|f'(a)|]^3}\times\left\{\ln\left(\frac{[b(b-a)|f'(a)|]^{b|f'(a)|}}{[a|f'(b)|(b-x)+b|f'(a)|(x-a)]^{b|f'(a)|}}\right)\right. \\
&\qquad\left.\left.+\ln\left(\frac{[a(b-a)|f'(b)|]^{a|f'(b)|}}{[b(x-a)|f'(a)|+a(b-x)|f'(b)|]^{a|f'(b)|}}\right)\right\}\right],
\end{aligned}$$

which proves the required result. □

A simplified refinement of (5.5) is given as follows:

Corollary 5.1. *Under the assumptions of Theorem* 5.5*, we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \max\{|f'(a)|,|f'(b)|\}\cdot\left[\frac{b-a}{4}+\left(x-\frac{a+b}{2}\right)^2\right], \tag{5.6}$$

for all $a\leq x\leq b$.

Proof. As in the proof of Theorem 5.5, we have

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\
&\leq \int_a^b |K(x,s)|\,|f'(s)|\,ds \\
&\leq \sup_{a\leq s\leq b}\left\{\frac{s(b-a)|f'(a)||f'(b)|}{b(s-a)|f'(a)|+a(b-s)|f'(b)|}\right\}\int_a^b |K(x,s)|\,ds \\
&= \max\{|f'(a)|,|f'(b)|\}\cdot\left[\frac{b-a}{4}+\left(x-\frac{a+b}{2}\right)^2\right],
\end{aligned}$$

which proves the required result. □

Theorem 5.6. *Let I be a real interval, $a, b \in \mathbb{R}$ $(a < b)$ with a, b in I° (the interior of I). Let $f : I \to \mathbb{R}$ be an absolutely continuous function on I. If $|f'|^p$ $(p > 1)$ is HH-convex, then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(s)\, ds \right| \\ \leq (b-a)^{-\frac{1}{q}} \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{1}{q}} \cdot \max\left\{ \left|f'(a)\right|, \left|f'(b)\right| \right\} \tag{5.7}$$

for all $a \leq x \leq b$ and $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. As in the proof of Theorem 5.4, and since $|f'|^p$ is HH-convex, we have

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a} \int_a^b f(u)\, du \right| \\ &\leq \left(\int_a^b |K(x,s)|^q\, ds \right)^{\frac{1}{q}} \left(\int_a^b |f'(s)|^p\, ds \right)^{\frac{1}{p}} \\ &\leq \frac{1}{b-a} \left(\int_a^x (s-a)^q\, ds + \int_x^b (b-s)^q\, ds \right)^{\frac{1}{q}} \\ &\quad \times \left(\int_a^b \frac{s(b-a)\,|f'(a)|^p\,|f'(b)|^p}{b(s-a)\,|f'(a)|^p + a(b-s)\,|f'(b)|^p} ds \right)^{\frac{1}{p}} \\ &\leq (b-a)^{-\frac{1}{q}} \sup_{a \leq s \leq b} \left\{ \frac{s(b-a)\,|f'(a)|\,|f'(b)|}{b(s-a)\,|f'(a)| + a(b-s)\,|f'(b)|} \right\} \cdot \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{1}{q}} \\ &\leq (b-a)^{-\frac{1}{q}} \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{1}{q}} \cdot \max\left\{ \left|f'(a)\right|, \left|f'(b)\right| \right\}, \end{aligned}$$

and this ends the proof. □

References

[1] M.W. Alomari, Several inequalities of Hermite–Hadamard, Ostrowski and Simpson type for s–convex, quasi–convex and r–convex mappings with some applications, PhD Thesis, Universiti Kebangsaan Malaysia, 2011.

[2] M. Alomari and M. Darus, *Some Ostrowski's type inequalities for convex functions with applications,* Preprint, RGMIA Res. Rep. Coll., **13**(1) (2010), Article 3. [online: http://ajmaa.org/RGMIA/v13n1.php]

[3] M.W. Alomari, M.E. Özdemir and H. Kavurmaci, *On companion of Ostrowski inequality for mappings whose first derivatives absolute value are convex with applications,* Miskolc Math. Notes, **13**(2) (2012), 233-248.

[4] M. Alomari , M. Darus, S.S. Dragomir and P. Cerone, *Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense,* Appl. Math. Lett., **23** (2010), 1071-1076.

[5] G.D. Anderson, M.K. Vamanamurthy and M. Vuorinen, *Generalized convexity and inequalities*, J. Math. Anal. Appl., **335** (2007), 1294-1308.

[6] M.A. Ardiç, A.O. Akdemir and E. Set, New Ostrowski like inequalities for GG-convex and GA-convex functions, *Math. Ineq. Applic.*, **19** (4) (2016), 1159–1168.

[7] P. Cerone, S.S. Dragomir, *Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions,* Demonstratio Math., **37**(2) (2004) 299-308.

[8] Barnett, P. Cerone, S.S. Dragomir, M.R. Pinheiro, A. Sofo, *Ostrowski type inequalities for functions whose modulus of derivatives are convex and applications,* Inequalities Theory & Applications, Vol. 2, Nova Sciences Publishers: 19-32.

[9] S.S. Dragomir, *An Ostrowski like inequality for convex functions and applications,* Revista Math. Complutense, **16**(2) (2003), 373-382.

[10] S.S. Dragomir, A. Sofo, *Ostrowski type inequalities for functions whose derivatives are convex,* in: Proceedings of the 4th International Conference on Modelling and Simulation, November 11-13, 2002. Victoria University, Melbourne, Australia, RGMIA Res. Rep. Coll., **5** (2002) Supplement, Article 30. Online: http://rgmia.vu.edu.au/v5(E).html..

[11] S.S. Dragomir and S. Wang, *A new inequality of Ostrowski's type in L_1-norm and applications to some special means and to some numerical quadrature rule,* Tamkang J. Math., **28** (1997), 239-244.

[12] S.S. Dragomir and S. Wang, *A new inequality of Ostrowski's type in L_p-norm and applications to some special means and to some numerical quadrature rules,* Indian J. of Math., **40**(3) (1998), 245-304.

[13] S.S. Dragomir and S. Wang, *Applications of Ostrowski's inequality to the estimation of error bounds for some special means and for some numerical quadrature rule,* Appl. Math. Lett., **11** (1998), 105-109.

[14] S.S. Dragomir and Th. M. Rassias (Eds), *Ostrowski Type Inequalities and Applications in Numerical Integration,* Kluwer Academic Publisher, 2002.

[15] H.A. Coban, I. Iscan amd M. Kunt, *New Ostrowski type inequalities for GA-convex functions,* New Trends in Mathematical Sciences, NTMSCI **4**(4) (2016), 1-11.

[16] M.A. Latif, M.W. Alomari and S. Hussain, *On Ostrowski-type inequalities for functions whose derivatives are m-convex and (α, m)-convex functions with applications,* Tamkang J. Math., **43**(4) (2012), 521-532.

[17] M.V. Mihai, M.A. Noor, K.I. Noor and M.U. Awan, *Some integral inequalities for harmonic h-convex functions involving hypergeometric functions,* Appl. Math. Comput., **252** (2015), 257-262.

[18] A. Ostrowski, *Über die Absolutabweichung einer differentiebaren Funktion von ihrem Integralmittelwert,* Comment. Math. Helv., **10** (1938), 226-227.

[19] M.E. Ozdemir, C. Yildiz, *New Ostrowski Type Inequalities for Geometrically Convex Functions,* International Journal of Modern Mathematical Sciences, **8** (1) (2013), 27-35.

[20] C.P. Niculescu, *Convexity according to the geometric mean,* Math. Inequal. Appl., **3**(2) (2000), 155-167.

[21] C.P. Niculescu, L.E. Persson, *Convex Functions and Their Applications. A Contemporary Approach,* CMS Books Math., 23, Springer-Verlag, New York, 2006.

[22] E. Set, M.E. Özdemir and M.Z. Sarikaya, *New inequalities of Ostrowski's type for s-convex functions in the second sense with applications,* Facta Univ. Ser. Math. Inform., **27**(1) (2012), 67-82.

[23] M. Tunç, *Ostrowski-type inequalities via h-convex functions with applications to special means,* J. Inequal. Appl., 2013, 2013:326.

[1]Department of Mathematics,
Faculty of Science and Information Technology,
Irbid National University,
2600 Irbid 21110, Jordan
E-mail address: mwomath@gmail.com

E-mail address: almahameed2000@yahoo.com