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MERCER'S INEQUALITY FOR h-CONVEX FUNCTIONS

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Abstract. In this work, we study Mercer inequality which is a variant of Jensen inequality for convex functions to be extended for h-convex functions. As application, a weighted generalization of triangle inequality is given.

1. Introduction

The class of h-convex functions, which generalizes convex, s-convex (denoted by K_s^2, [1]), Godunova-Levin functions (denoted by $Q(I)$, [3]) and P-functions (denoted by $P(I)$, [9]), was introduced by Varošanec in [12]. Namely, for real intervals I and J, the h-convex function is defined as a non-negative function $f : I \to \mathbb{R}$ which satisfies

$$f\left(t\alpha + (1-t)\beta\right) \leq h\left(t\right) f\left(\alpha\right) + h\left(1-t\right) f\left(\beta\right),$$

where $h : J \to \mathbb{R}$ is a non-negative function defined on J, such that $t \in (0,1) \subseteq J \subseteq (0,\infty)$ and $x, y \in I$. Accordingly, some properties of h-convex functions were discussed in the same work of Varošanec. The famous references about these classes are [1–4] and [7–11].

Let $w_1, w_2, \cdots, w_n$ be positive real numbers ($n \geq 2$) and $h : J \to \mathbb{R}$ be a non-negative supermultiplicative function. In [12], Varošanec discussed the case that, if f is a non-negative h-convex on I, then for $x_1, x_2, \cdots, x_n \in I$ the following inequality holds

$$f\left(\frac{1}{W_n}\sum_{k=1}^{n} w_k x_k\right) \leq \sum_{k=1}^{n} h\left(\frac{w_k}{W_n}\right) f\left(x_k\right), \tag{1.1}$$

where $W_n = \sum\limits_{k=1}^{n} w_k$. If h is submultiplicative function and f is an h-concave then inequality is reversed. In case $h(t) = t$ we refer to the classical version of Jensen's inequality [5].

If f is convex on I, then for any finite positive increasing sequence $(x_k)_{k=1}^{n} \in I$, we have

$$f\left(x_1 + x_n - \sum_{k=1}^{n} w_k x_k\right) \leq f\left(x_1\right) + f\left(x_n\right) - \sum_{k=1}^{n} w_k f\left(x_k\right), \tag{1.2}$$

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where $w_1, w_2, \cdots, w_n$ are positive real numbers such that $\sum\limits_{k=1}^{n} w_k = 1$. This inequality was established by Mercer in [6] and it is considered as a variant of Jensen's inequality.

In this work, a generalization of Mercer inequality for h-convex function is presented. As application, a weighted generalization of triangle inequality is given.

2. Mercer analogue inequality for h-convex functions

In order to prove our main result, we need the following Lemma which generalizes Lemma 1.3 in [6].

Lemma 2.1. *Let $h : J \to \mathbb{R}$ be a non-negative supermultiplicative function on J. Let $\alpha, \beta \in [0,1]$ such that $\alpha + \beta = 1$ and $h(\alpha) + h(\beta) \leq 1$. For any h-convex function f defined on a real interval I and finite positive increasing sequence $(x_k)_{k=1}^{n} \in I$, we have*

$$f(x_1 + x_n - x_k) \leq f(x_1) + f(x_n) - f(x_k) \qquad (1 \leq k \leq n). \tag{2.1}$$

If h is submultiplicative function, $h(\alpha) + h(\beta) \geq 1$ for all $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$ and f is an h-concave then inequality (2.1) is reversed.

Proof. Let $0 < x_1 \leq \cdots \leq x_n$ and $\alpha, \beta \in [0,1]$ such that $\alpha + \beta = 1$ with $h(\alpha) + h(\beta) \leq 1$. Following Mercer approach in [6]. Let us write $y_k = x_1 + x_n - x_k$. Then $x_1 + x_n = y_k + x_k$, so that the pairs x_1, x_n and x_k, y_k possess the same midpoint. Since that is the case there exists $\alpha, \beta \in [0,1]$ such that $x_k = \alpha x_1 + \beta x_n$ and $y_k = \beta x_1 + \alpha x_n$, where $\alpha + \beta = 1$ and $1 \leq k \leq n$. Employing the h-convexity of f we get

$$\begin{aligned}
f(y_k) = f(\beta x_1 + \alpha x_n) &\leq h(\beta) f(x_1) + h(\alpha) f(x_n) \\
&\leq (1 - h(\alpha)) f(x_1) + (1 - h(\beta)) f(x_n) \\
&= f(x_1) + f(x_n) - [h(\alpha) f(x_1) + h(\beta) f(x_n)] \\
&\leq f(x_1) + f(x_n) - f(\alpha x_1 + \beta x_n) \\
&= f(x_1) + f(x_n) - f(\alpha x_1 + \beta x_n) \\
&= f(x_1) + f(x_n) - f(x_k),
\end{aligned}$$

and this proves the required result. □

Now, we are ready to state our main result.

Theorem 2.1. *Let $h : J \to \mathbb{R}$ be a non-negative supermultiplicative function on J. Let $w_1, w_2, \cdots, w_n$ be positive real numbers $(n \geq 2)$ such that $W_n = \sum\limits_{k=1}^{n} w_k$ and $\sum\limits_{k=1}^{n} h\left(\frac{w_k}{W_n}\right) \leq 1$. If f is h-convex on I, then for any finite positive increasing sequence $(x_k)_{k=1}^{n} \in I$, we have*

$$f\left(x_1 + x_n - \frac{1}{W_n}\sum_{k=1}^{n} w_k x_k\right) \leq f(x_1) + f(x_n) - \sum_{k=1}^{n} h\left(\frac{w_k}{W_n}\right) f(x_k). \tag{2.2}$$

If h is submultiplicative function, $\sum\limits_{k=1}^{n} h\left(\frac{w_k}{W_n}\right) \geq 1$ and f is an h-concave then inequality (2.2) is reversed.

Proof. Since $\frac{1}{W_n}\sum\limits_{k=1}^{n} w_k = 1$, we have

$$\begin{aligned}
f\left(x_1+x_n-\frac{1}{W_n}\sum_{k=1}^{n} w_k x_k\right) &= f\left(\sum_{k=1}^{n}\frac{w_k}{W_n}\left(x_1+x_n-x_k\right)\right) \\
&\le \sum_{k=1}^{n} h\left(\frac{w_k}{W_n}\right) f\left(x_1+x_n-x_k\right) \qquad \text{by (1.1)} \\
&\le \sum_{k=1}^{n} h\left(\frac{w_k}{W_n}\right)\left[f\left(x_1\right)+f\left(x_n\right)-f\left(x_k\right)\right] \qquad \text{by (2.1)} \\
&= \left[f\left(x_1\right)+f\left(x_n\right)\right]\sum_{k=1}^{n} h\left(\frac{w_k}{W_n}\right) - \sum_{k=1}^{n} h\left(\frac{w_k}{W_n}\right) f\left(x_k\right) \\
&\le f\left(x_1\right)+f\left(x_n\right) - \sum_{k=1}^{n} h\left(\frac{w_k}{W_n}\right) f\left(x_k\right) \qquad \text{by assumption}
\end{aligned}$$

and this proves the result in (2.2). □

One of the direct application and interesting benefit of (2.2) is to offer an upper bound for the converse of h-Jensen inequality (1.1), by rearranging the terms in (2.2) we get

$$\sum_{k=1}^{n} h\left(\frac{w_k}{W_n}\right) f\left(x_k\right) \le f\left(x_1\right)+f\left(x_n\right) - f\left(x_1+x_n-\frac{1}{W_n}\sum_{k=1}^{n} w_k x_k\right). \tag{2.3}$$

For instance, if $f(x)=|x|$, $h(t)=t$ and $W_n=1$, then we have the following refinement of the celebrated triangle inequality which is of great interests itself

$$\sum_{k=1}^{n} w_k\left|x_k\right| \le \left|x_1\right|+\left|x_n\right| - \left|x_1+x_n-\sum_{k=1}^{n} w_k x_k\right|. \tag{2.4}$$

This inequality can be generalized for norms by considering the mapping $f(\mathbf{x}) = \|\mathbf{x}\|$ ($\mathbf{x}\in L$), where L is a linear space.

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