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A NEW GENERALIZATION FOR n-TIME DIFFERENTIABLE MAPPINGS WHICH ARE CONVEX

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Abstract. In this paper, we establish several new inequalities for n-time differentiable mappings that are connected with the celebrated Hermite-Hadamard integral inequality.

1. Introduction

On November 22, 1881, Hermite (1822-1901) sent a letter to the Journal Mathesis. This letter was published in Mathesis 3 (1883, p: 82) and in this letter an inequality presented which is well-known in the literature as Hermite-Hadamard integral inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I of a real numbers and $a, b \in I$ with $a < b$. If the function f is concave, the inequality in (1.1) is reversed.

The inequalities (1.1) have become an important cornerstone in mathematical analysis and optimization. Many uses of these inequalities have been discovered in a variety of settings. Moreover , many inequalities of special means can be obtained for a particular choice of the function f. Due to the rich geometrical significance of Hermite-Hadamard's inequality, there is growing literature providing its new proofs, extensions, refinements and generalizations, see for example ([4, 7–11, 15–19]) and the references therein.

Definition 1.1. A function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if whenever $x, y \in [a, b]$ and $t \in [0, 1]$, the following inequality holds:

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

We say that f is concave if $(-f)$ is convex. This definition has its origins in Jensen's results from [6] and has opened up the most extended, useful and multi-disciplinary domain of mathematics, namely, convex analysis. Convex curves and convex bodies have appeared

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in mathematical literature since antiquity and there are many important results related to them.

For other recent results concerning the n-time differentiable functions see [1–3,5,7,10,12, 18] where further references are given.

In [8], Kırmacı proved the following result:

Theorem 1.1. *Let $f : I^* \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^*, $a, b \in I^*$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then we have*

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{8} \left[|f'(a)| + |f'(b)| \right] \tag{1.2}$$

In [16], Sarıkaya and Aktan proved the following results for convex functions:

Theorem 1.2. *Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$ with $a < b$ and $f : I \to \mathbb{R}$ be a twice differentiable mapping such that f'' is integrable. If $|f''|$ is a convex on $[a, b]$, then the following inequalities hold:*

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{24} \left\{ \frac{|f''(a)| + |f''(b)|}{2} \right\} \tag{1.3}$$

Theorem 1.3. *Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$ with $a < b$ and $f : I \to \mathbb{R}$ be a twice differentiable mapping such that f'' is integrable. If $|f''|^q$ is a convex on $[a, b]$, $q \geq 1$, then the following inequalities hold:*

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \qquad (1.4) \\ \leq\ & \frac{(b-a)^2}{48} \left\{ \left(\frac{3|f'(a)|^q + 5|f'(b)|^q}{8} \right)^{\frac{1}{q}} + \left(\frac{5|f'(a)|^q + 3|f'(b)|^q}{8} \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

The main purpose of the present paper is to establish several new inequalities for n-time differentiable mappings that are connected with the celebrated Hermite-Hadamard integral inequality.

2. Main Results

Lemma 2.1. *([10])For $n \in \mathbb{N}$, let $f : I \subset \mathbb{R} \to \mathbb{R}$ be n-time differentiable function. If $a, b \in I$ with $a < b$ and $f^{(n)} \in L[a, b]$, then*

$$\begin{aligned} \int_a^b f(t)dt \;=\; & \sum_{k=0}^{n-1} \left(\frac{1+(-1)^k}{2^{k+1}(k+1)!} \right) (b-a)^{k+1} f^{(k)}\left(\frac{a+b}{2}\right) \qquad (2.1) \\ & + (b-a)^{n+1} \int_0^1 M_n(t) f^{(n)}(ta + (1-t)b)dt \end{aligned}$$

where

$$M_n(t) = \begin{cases} \frac{t^n}{n!}, & t \in \left[0, \frac{1}{2}\right] \\ \frac{(t-1)^n}{n!}, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

and n natural number, $n \geq 1$.

Proof. The proof is by mathematical induction.

The case $n = 1$ is [[8], Lemma 2.1].

Assume that (2.1) holds for "n" and let us prove it for "$n+1$". That is, we have to prove the equality

$$\int_a^b f(t)dt = \sum_{k=0}^{n}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right) + (b-a)^{n+2}\int_0^1 M_{n+1}(t)f^{(n+1)}(ta+(1-t)b)dt \tag{2.2}$$

where, obviously,

$$M_{n+1}(t) = \begin{cases} \frac{t^{n+1}}{(n+1)!}, & t \in \left[0, \frac{1}{2}\right] \\ \frac{(t-1)^{n+1}}{(n+1)!}, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Then, we have

$$\begin{aligned} &(b-a)^{n+2}\int_0^1 M_{n+1}(t)f^{(n+1)}(ta+(1-t)b)dt \\ = &(b-a)^{n+2}\left\{\int_0^{\frac{1}{2}} \frac{t^{n+1}}{(n+1)!}f^{(n+1)}(ta+(1-t)b)dt + \int_{\frac{1}{2}}^1 \frac{(t-1)^{n+1}}{(n+1)!}f^{(n+1)}(ta+(1-t)b)dt\right\} \end{aligned}$$

and integrating by parts gives

$$\begin{aligned} &(b-a)^{n+2}\int_0^1 M_{n+1}(t)f^{(n+1)}(ta+(1-t)b)dt \\ = &(b-a)^{n+2}\left\{\frac{t^{n+1}}{(n+1)!}\left.\frac{f^{(n)}(ta+(1-t)b)}{a-b}\right|_0^{\frac{1}{2}} - \frac{1}{a-b}\int_0^{\frac{1}{2}}\frac{t^n}{n!}f^{(n)}(ta+(1-t)b)dt \right. \\ &\left. + \frac{(t-1)^{n+1}}{(n+1)!}\left.\frac{f^{(n)}(ta+(1-t)b)}{a-b}\right|_{\frac{1}{2}}^1 - \frac{1}{a-b}\int_{\frac{1}{2}}^1\frac{(t-1)^n}{n!}f^{(n)}(ta+(1-t)b)dt\right\} \\ = &-\frac{1+(-1)^n}{2^{n+1}(n+1)!}f^{(n)}\left(\frac{a+b}{2}\right)(b-a)^{n+1} + (b-a)^{n+1}\int_0^1 M_n(t)f^{(n)}(ta+(1-t)b)dt. \end{aligned}$$

That is

$$\begin{aligned} &(b-a)^{n+1}\int_0^1 M_n(t)f^{(n)}(ta+(1-t)b)dt \\ = &\frac{1+(-1)^n}{2^{n+1}(n+1)!}f^{(n)}\left(\frac{a+b}{2}\right)(b-a)^{n+1} \\ &+(b-a)^{n+2}\int_0^1 M_{n+1}(t)f^{(n+1)}(ta+(1-t)b)dt. \end{aligned}$$

Now, using the mathematical induction hypothesis, we get

$$\begin{aligned}
\int_a^b f(t)dt &= \sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\\
&+\frac{1+(-1)^n}{2^{n+1}(n+1)!}(b-a)^{n+1}f^{(n)}\left(\frac{a+b}{2}\right)\\
&+(b-a)^{n+2}\int_0^1 M_{n+1}(t)f^{(n+1)}(ta+(1-t)b)dt\\
&= \sum_{k=0}^{n}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\\
&+(b-a)^{n+2}\int_0^1 M_{n+1}(t)f^{(n+1)}(ta+(1-t)b)dt.
\end{aligned}$$

Thus, the identity (2.2) and the lemma is proved. □

Theorem 2.1. *For $n \geq 1$, let $f : I \subset \mathbb{R} \to \mathbb{R}$ be n-time differentiable function, $a, b \in I$ and $a < b$. If $f^{(n)} \in L[a,b]$ and $\left|f^{(n)}\right|$ is convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\int_a^b f(t)dt - \sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right| \qquad (2.3)\\
\leq\ & \frac{(b-a)^{n+1}}{2^n(n+1)!}\left(\frac{\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|}{2}\right).
\end{aligned}$$

Proof. From Lemma 2.1 and using the properties of modulus, we write

$$\begin{aligned}
&\left|\int_a^b f(t)dt - \sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right|\\
\leq\ & (b-a)^{n+1}\int_0^1 |M_n(t)|\left|f^{(n)}(ta+(1-t)b)\right|dt\\
=\ & (b-a)^{n+1}\left\{\int_0^{\frac{1}{2}}\frac{t^n}{n!}\left|f^{(n)}(ta+(1-t)b)\right|dt\right.\\
&\left.+\int_{\frac{1}{2}}^1\frac{(1-t)^n}{n!}\left|f^{(n)}(ta+(1-t)b)\right|dt\right\}.
\end{aligned}$$

Since $\left|f^{(n)}\right|$ is convex on $[a,b]$, it follows that

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$

$$
\begin{aligned}
&\leq (b-a)^{n+1}\left\{\int_0^{\frac{1}{2}}\frac{t^n}{n!}\left[t\left|f^{(n)}(a)\right|+(1-t)\left|f^{(n)}(b)\right|\right]dt\right.\\
&\quad\left.+\int_{\frac{1}{2}}^1\frac{(1-t)^n}{n!}\left[t\left|f^{(n)}(a)\right|+(1-t)\left|f^{(n)}(b)\right|\right]dt\right\}\\
&= \frac{(b-a)^{n+1}}{2^n(n+1)!}\left(\frac{\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|}{2}\right).
\end{aligned}
$$

This completes the proof. □

Remark 2.1. In the inequalities (2.3), if we choose $n=1$, then we have the inequality (1.2).

Remark 2.2. In the inequalities (2.3), if we choose $n=2$, then we have the inequality (1.3).

Theorem 2.2. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be n-time differentiable function and $a<b$. If $f^{(n)}\in L[a,b]$ and $\left|f^{(n)}\right|^q$ is convex on $[a,b]$, then we have*

$$
\begin{aligned}
&\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right| \qquad (2.4)\\
&\leq \frac{(b-a)^{n+1}}{2^{n+1}n!}\left(\frac{1}{np+1}\right)^{\frac{1}{p}}\\
&\quad\times\left\{\left(\frac{\left|f^{(n)}(a)\right|^q+3\left|f^{(n)}(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(b)\right|^q}{4}\right)^{\frac{1}{q}}\right\}
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1 and Hölder integral inequality, we obtain

$$
\begin{aligned}
&\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right|\\
&\leq (b-a)^{n+1}\int_0^1|M_n(t)|\left|f^{(n)}(ta+(1-t)b)\right|dt\\
&\leq \frac{(b-a)^{n+1}}{n!}\left\{\left(\int_0^{\frac{1}{2}}t^{np}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_{\frac{1}{2}}^1(1-t)^{np}dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Since $\left|f^{(n)}\right|^q$ is convex on $[a,b]$, then

$$\begin{aligned}
&\left|\int_a^b f(t)dt - \sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{(b-a)^{n+1}}{n!}\left\{\left(\int_0^{\frac{1}{2}} t^{np}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left[t\left|f^{(n)}(a)\right|^q+(1-t)\left|f^{(n)}(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1 (1-t)^{np}dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left[t\left|f^{(n)}(a)\right|^q+(1-t)\left|f^{(n)}(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right\} \\
\leq\ & \frac{(b-a)^{n+1}}{2^{n+1}n!}\left(\frac{1}{np+1}\right)^{\frac{1}{p}} \\
&\times\left\{\left(\frac{\left|f^{(n)}(a)\right|^q+3\left|f^{(n)}(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(b)\right|^q}{4}\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

which completes the proof. □

Theorem 2.3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be n-time differentiable function and $a < b$. If $f^{(n)} \in L[a,b]$ and $\left|f^{(n)}\right|^q$ is convex on $[a,b]$, then we get*

$$\begin{aligned}
&\left|\int_a^b f(t)dt - \sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right| \qquad (2.5)\\
\leq\ & \frac{(b-a)^{n+1}}{n!}\left(\frac{q-1}{nq+q-p-1}\right)^{1-\frac{1}{q}}\frac{1}{2^{n+1+1/q}} \\
&\times\left\{\left(\frac{1}{p+2}\left|f^{(n)}(a)\right|^q+\frac{3p+5}{(p+1)(p+2)}\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{3p+5}{(p+1)(p+2)}\left|f^{(n)}(a)\right|^q+\frac{1}{p+2}\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1 and Hölder inequality, we have

$$\begin{aligned}
&\left|\int_a^b f(t)dt - \sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right| \\
\leq\ & (b-a)^{n+1}\int_0^1 |M_n(t)|\left|f^{(n)}(ta+(1-t)b)\right|dt \\
=\ & \frac{(b-a)^{n+1}}{n!}\left\{\int_0^{\frac{1}{2}}\frac{t^n t^{\frac{p}{q}}}{t^{\frac{p}{q}}}\left|f^{(n)}(ta+(1-t)b)\right|dt\right. \\
&\left.+\int_{\frac{1}{2}}^1\frac{(1-t)^n(1-t)^{\frac{p}{q}}}{(1-t)^{\frac{p}{q}}}\left|f^{(n)}(ta+(1-t)b)\right|dt\right\}
\end{aligned}$$

$$\leq \frac{(b-a)^{n+1}}{n!}\left\{\left(\int_0^{\frac{1}{2}}\left[\frac{t^n}{t^{\frac{p}{q}}}\right]^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}t^p\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1\left[\frac{(1-t)^n}{(1-t)^{\frac{p}{q}}}\right]^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1(1-t)^p\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right\}.$$

Since $\left|f^{(n)}\right|^q$ is convex on $[a,b]$, then

$$\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{(b-a)^{n+1}}{n!}\left\{\left(\int_0^{\frac{1}{2}}t^{\frac{nq-p}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}t^p\left[t\left|f^{(n)}(a)\right|^q+(1-t)\left|f^{(n)}(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1(1-t)^{\frac{nq-p}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1(1-t)^p\left[t\left|f^{(n)}(a)\right|^q+(1-t)\left|f^{(n)}(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right\}$$
$$=\frac{(b-a)^{n+1}}{n!}\left(\frac{q-1}{nq+q-p-1}\right)^{1-\frac{1}{q}}\frac{1}{2^{n+1+1/q}}$$
$$\times\left\{\left(\frac{1}{p+2}\left|f^{(n)}(a)\right|^q+\frac{3p+5}{(p+1)(p+2)}\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{3p+5}{(p+1)(p+2)}\left|f^{(n)}(a)\right|^q+\frac{1}{p+2}\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}\right\}$$

which completes the proof of the theorem. □

Corollary 2.1. *In Theorem 2.3, if we choose $n=1$, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)}{4}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\frac{q-1}{2q-p-1}\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left(\frac{1}{p+2}\left|f'(a)\right|^q+\frac{3p+5}{(p+1)(p+2)}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{3p+5}{(p+1)(p+2)}\left|f'(a)\right|^q+\frac{1}{p+2}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right\}.$$

Corollary 2.2. *In Theorem 2.3, if we choose $n=2$, we obtain*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{16}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\frac{q-1}{3q-p-1}\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left(\frac{1}{p+2}\left|f''(a)\right|^q+\frac{3p+5}{(p+1)(p+2)}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{3p+5}{(p+1)(p+2)}\left|f''(a)\right|^q+\frac{1}{p+2}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right\}.$$

Theorem 2.4. *For $n \geq 1$, let $f : I \subset \mathbb{R} \to \mathbb{R}$ be n-time differentiable function and $a < b$. If $f^{(n)} \in L[a, b]$ and $\left|f^{(n)}\right|^q$ is convex on $[a, b]$, for $q \geq 1$, then the following inequality holds:*

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right| \tag{2.6}$$
$$\leq \frac{(b-a)^{n+1}}{2^{n+1}(n+1)!}\left\{\left[\frac{n+1}{2n+4}\left|f^{(n)}(a)\right|^q + \frac{n+3}{2n+4}\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}} + \left[\frac{n+3}{2n+4}\left|f^{(n)}(a)\right|^q + \frac{n+1}{2n+4}\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}\right\}.$$

Proof. From Lemma 2.1 and using the well known Power-mean integral inequality, we have

$$\begin{aligned}
&\left|\int_a^b f(t)dt - \sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right| \\
\leq\ & (b-a)^{n+1}\int_0^1 |M_n(t)|\left|f^{(n)}(ta+(1-t)b)\right|dt \\
\leq\ & \frac{(b-a)^{n+1}}{n!}\left\{\left(\int_0^{\frac{1}{2}} t^n dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}} t^n\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right. \\
& \left. + \left(\int_{\frac{1}{2}}^1 (1-t)^n dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1 (1-t)^n\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Since $\left|f^{(n)}\right|^q$ is convex on $[a, b]$, for $q \geq 1$, then we obtain

$$\begin{aligned}
&\left|\int_a^b f(t)dt - \sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{(b-a)^{n+1}}{n!}\left\{\left(\int_0^{\frac{1}{2}} t^n dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}} t^n\left[t\left|f^{(n)}(a)\right|^q + (1-t)\left|f^{(n)}(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right. \\
& \left. + \left(\int_{\frac{1}{2}}^1 (1-t)^n dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1 (1-t)^n\left[t\left|f^{(n)}(a)\right|^q + (1-t)\left|f^{(n)}(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right\} \\
=\ & \frac{(b-a)^{n+1}}{2^{n+1}(n+1)!}\left\{\left[\frac{n+1}{2n+4}\left|f^{(n)}(a)\right|^q + \frac{n+3}{2n+4}\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}\right. \\
& \left. + \left[\frac{n+3}{2n+4}\left|f^{(n)}(a)\right|^q + \frac{n+1}{2n+4}\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Hence, the proof of the theorem is completed. □

Corollary 2.3. *In Theorem 2.4, if we choose $n=1$, then*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right|\\ \leq\ &\frac{(b-a)}{8}\left\{\left(\frac{|f'(a)|^q+2|f'(b)|^q}{3}\right)^{\frac{1}{q}}+\left(\frac{2|f'(a)|^q+|f'(b)|^q}{3}\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Remark 2.3. In Theorem 2.4, if we choose $n=2$, we obtain the inequality (1.4).

Theorem 2.5. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be n-time differentiable function and $a<b$. If $f^{(n)}\in L[a,b]$ and $\left|f^{(n)}\right|^q$ is concave on $[a,b]$, then we obtain*

$$\begin{aligned}&\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right|\\ \leq\ &\frac{(b-a)^{n+1}}{2^{np+1+\frac{1}{q}}(np+1)n!}\left\{\left|f^{(n)}\left(\frac{a+3b}{4}\right)\right|+\left|f^{(n)}\left(\frac{3a+b}{4}\right)\right|\right\}\end{aligned}\tag{2.7}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1 and Hölder integral inequality, we obtain

$$\begin{aligned}&\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right|\\ \leq\ &(b-a)^{n+1}\int_0^1|M_n(t)|\left|f^{(n)}(ta+(1-t)b)\right|dt\\ \leq\ &\frac{(b-a)^{n+1}}{n!}\left\{\left(\int_0^{\frac{1}{2}}t^{np}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_{\frac{1}{2}}^1(1-t)^{np}dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right\}.\end{aligned}\tag{2.8}$$

Since $\left|f^{(n)}\right|^q$ is concave on $[a,b]$, we obtain the following inequalities via Jensen inequality:

$$\begin{aligned}\int_0^{\frac{1}{2}}\left|f^{(n)}(ta+(1-t)b)\right|^q dt &= \int_0^{\frac{1}{2}}t^0\left|f^{(n)}(ta+(1-t)b)\right|^q dt\\ &\leq \left(\int_0^{\frac{1}{2}}t^0dt\right)\left|f^{(n)}\left(\frac{\int_0^{\frac{1}{2}}(ta+(1-t)b)dt}{\int_0^{\frac{1}{2}}t^0dt}\right)\right|^q\\ &= \frac{1}{2}\left|f^{(n)}\left(\frac{a+3b}{4}\right)\right|^q\end{aligned}\tag{2.9}$$

and similarly

$$\int_{\frac{1}{2}}^1\left|f^{(n)}(ta+(1-t)b)\right|^q dt\leq\frac{1}{2}\left|f^{(n)}\left(\frac{3a+b}{4}\right)\right|^q.\tag{2.10}$$

Thus, if we use (2.9)–(2.10) in (2.8), we obtain the inequality of (2.7). This completes the proof. □

Corollary 2.4. *Under conditions of Theorem 2.5, if we choose $n = 1$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)}{2^{p+1+\frac{1}{q}}(p+1)}\left\{\left|f'\left(\frac{a+3b}{4}\right)\right| + \left|f'\left(\frac{3a+b}{4}\right)\right|\right\}.$$

Corollary 2.5. *Under conditions of Theorem 2.5, if we choose $n = 2$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{4^{p+1}(2p+1)}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left\{\left|f''\left(\frac{a+3b}{4}\right)\right| + \left|f''\left(\frac{3a+b}{4}\right)\right|\right\}.$$

Theorem 2.6. *For $n \geq 1$, let $f : I \subset \mathbb{R} \to \mathbb{R}$ be n-time differentiable function and $a < b$. If $f^{(n)} \in L[a,b]$ and $\left|f^{(n)}\right|^q$ is concave on $[a,b]$, for $q \geq 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\int_a^b f(t)dt - \sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right| \\
\leq\ &\frac{(b-a)^{n+1}}{2^{n+1}(n+1)!}\left\{\left|f^{(n)}\left(\frac{(n+1)a+(n+3)b}{2(n+2)}\right)\right| + \left|f^{(n)}\left(\frac{(n+3)a+(n+1)b}{2(n+2)}\right)\right|\right\}.
\end{aligned} \tag{2.11}$$

Proof. From Lemma 2.1 and using the well known Power-mean inequality, we have

$$\begin{aligned}
&\left|\int_a^b f(t)dt - \sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right| \\
\leq\ &(b-a)^{n+1}\int_0^1 |M_n(t)|\left|f^{(n)}(ta+(1-t)b)\right|dt \\
\leq\ &\frac{(b-a)^{n+1}}{n!}\left\{\left(\int_0^{\frac{1}{2}} t^n dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}} t^n\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\int_{\frac{1}{2}}^1 (1-t)^n dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1 (1-t)^n\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Using the Jensen inequality, we have

$$\begin{aligned}
&\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right|\\
\leq\ & \frac{(b-a)^{n+1}}{n!}\\
&\times\left\{\left(\int_0^{\frac{1}{2}}t^n dt\right)^{1-\frac{1}{q}}\left[\left(\int_0^{\frac{1}{2}}t^n dt\right)\left|f^{(n)}\left(\frac{\int_0^{\frac{1}{2}}t^n(ta+(1-t)b)dt}{\int_0^{\frac{1}{2}}t^n dt}\right)\right|^q\right]^{\frac{1}{q}}\right.\\
&\left.+\left(\int_{\frac{1}{2}}^1(1-t)^n dt\right)^{1-\frac{1}{q}}\left[\left(\int_{\frac{1}{2}}^1(1-t)^n dt\right)\left|f^{(n)}\left(\frac{\int_{\frac{1}{2}}^1(1-t)^n(ta+(1-t)b)dt}{\int_{\frac{1}{2}}^1(1-t)^n dt}\right)\right|^q\right]^{\frac{1}{q}}\right\}\\
=\ & \leq\frac{(b-a)^{n+1}}{2^{n+1}(n+1)!}\left\{\left|f^{(n)}\left(\frac{(n+1)a+(n+3)b}{2(n+2)}\right)\right|+\left|f^{(n)}\left(\frac{(n+3)a+(n+1)b}{2(n+2)}\right)\right|\right\}.
\end{aligned}$$

Hence, the proof of the theorem is completed. □

Corollary 2.6. *In the inequality (2.11), if we choose $n=1$, then we obtain*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|\leq\frac{(b-a)}{8}\left\{\left|f'\left(\frac{a+2b}{3}\right)\right|+\left|f'\left(\frac{2a+b}{3}\right)\right|\right\}.$$

Corollary 2.7. *In the inequality (2.11), if we choose $n=2$, then we obtain*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|\leq\frac{(b-a)^2}{48}\left\{\left|f''\left(\frac{3a+5b}{8}\right)\right|+\left|f''\left(\frac{5a+3b}{8}\right)\right|\right\}.$$

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