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COEFFICIENT ESTIMATES FOR NEW SUBCLASS OF BI-UNIVALENT FUNCTIONS INVOLVING SĂLĂGEAN DIFFERENTIAL OPERATOR

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ABSTRACT. In the paper, the authors introduce a new subclass $\mathcal{M}_{\tau}^{n,\alpha}(\varphi,\phi)$ of the class $\mathcal{M}$ consisting of normalized analytic and bi-univalent functions associated with Sălăgean differential operator in $\mathbb{U}$. Initial Taylor series coefficients estimates are obtained. The results presented here extended some of the earlier results.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of all function of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ } and \text{ } |z| < 1\}.$$

For two functions $f(z)$ and $g(z)$, analytic functions in $\mathbb{U}$. the function $f(z)$ is said to be subordinate to $g(z)$ and written by

$$f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function $\omega(z)$, analytic in $\mathbb{U}$, with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1,$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \mathbb{U}).$$

It is well known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

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If $g(z)$ is univalent, then

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Further, let $\mathcal{P}$ be the class of functions

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{1.2}$$

which are analytic and convex in $\mathbb{U}$. If $p(z) \in \mathcal{P}$ satisfies the condition

$$\Re(p(z)) > 0 \quad (z \in \mathbb{U}),$$

then, $|c_n| \leq 2$, we call the functions as the Carath$\acute{e}$odory Lemma (e.g., see[8]).

The Koebe one-quarter theorem [8] states that the image of $\mathbb{U}$ under every function f from $\mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Thus, every univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f^{-1}(f(\omega)) = \omega \quad \left(|\omega| < r_0(f);\; r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(\omega) = f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \cdots . \tag{1.3}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Denote by $\mathcal{M}$ the class of bi-univalent functions in $\mathbb{U}$.

Let the functions $\varphi, \phi : \mathbb{U} \to \mathbb{C}$ be analytic and univalent with positive real part in $\mathbb{U}, \varphi(0) = \phi(0) = 1, \varphi'(0) > 0, \phi'(0) > 0$ and φ, ϕ maps the unit disk $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The TaylorâĂŹs series expansion of such functions are

$$\varphi(z) = 1 + \sum_{k=1}^{\infty} \Psi_k z^k \quad (\Psi_1 > 0) \text{ and } \phi(z) = 1 + \sum_{k=1}^{\infty} \Phi_k z^k \quad (\Phi_1 > 0). \tag{1.4}$$

Ma and Minda [11] unified various subclasses of starlike and convex functions. The class of Ma-Minda starlike functions consists of functions $f \in \mathcal{A}$ satisfying

$$\frac{zf'(z)}{f(z)} \prec \varphi(z).$$

Similarly, the class of Ma-Minda convex functions consists of functions $f \in \mathcal{A}$ satisfying

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z).$$

A function f is bi-starlike or bi-convex of Ma-Minda type if both f and f^{-1} are respectively Ma-Minda starlike or convex. These classes are denoted by $\mathcal{ST}_{\mathcal{M}}(\varphi)$ and $\mathcal{CV}_{\mathcal{M}}(\varphi)$.

For a function $f \in \mathcal{A}$, Sălăgean differential operator [14] defined the following differential operator

$$D^0 f(z) = f(z), \tag{1.5}$$
$$D^1 f(z) = zf'(z), \tag{1.6}$$
$$D^n f(z) = D(D^{n-1} f(z)) \quad (n \in \mathbb{N}). \tag{1.7}$$

If f is given by (1.1), then from (1.5) and (1.7), we get

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \quad (n \in \mathbb{N}), \tag{1.8}$$

In 1967, Lewin [10] investigated the class $\mathcal{M}$ and showed that $a_2 < 1.51$. Subsequently, Brannan and Clunie [2] conjectured that $a_2 \le \sqrt{2}$ for $f \in \mathcal{M}$. Later, Netanyahu [12] derived that $\max_{f\in\mathcal{M}} |a_2| = 3/4$. In 1981, Styer and Wright [21] showed that there exist functions for which $|a_2| > 4/3$. The best result for functions in $\mathcal{M}$ was obtained by Tan [22] in 1984, that was, $|a_2| \le 1.485$. However, the coefficient estimate problem of $|a_n|$ $(n \in \mathbb{N}\backslash\{1,2\})$ is still an open problem.

In recent years, a huge amount of papers related to the class $\mathcal{M}$ due mainly to the pioneering work of Srivastava [19] in 2010, then follows [1,4,6,19,20,24]. In 2014, Sivasubramanian [17] answered some questions of covering theorem, distortion theorem, growth theorem and the radius of convexity for the functions of the class $\mathcal{M}$ which raised by Goodman [9].

Motivated by Çağlar and Deniz[6], Selvaraj and Thirupathi [15], Srivastava *et al.* [18] and, Xiong *et al.* [25] with the operator $D^n f$, we investigate the estimates for initial $|a_2|$ and $|a_3|$ of bi-univalent functions of Ma-Minda type belonging to the new subclass of analytic function $\mathcal{M}_\tau^{n,\alpha}(\varphi, \phi)$ following. The presented results here extended some of the earlier results.

Definition 1.1. Let f given by (1.1), then $f \in \mathcal{M}_\tau^{n,\alpha}(\varphi, \phi)(\tau \in \mathbb{C}\backslash\{0\}, 0 \le \alpha \le 1)$ if the following conditions are satisfied

$$1 + \frac{1}{\tau}\left[(1-\alpha)\frac{D^{n+1}f(z)}{D^n f(z)} + \alpha\frac{(D^{n+1}f(z))'}{(D^n f(z))'} - 1\right] \prec \varphi(z) \quad (z \in \mathbb{U}) \tag{1.9}$$

and

$$1 + \frac{1}{\tau}\left[(1-\alpha)\frac{D^{n+1}g(\omega)}{D^n g(\omega)} + \alpha\frac{(D^{n+1}g(\omega))'}{(D^n g(\omega))'} - 1\right] \prec \phi(\omega) \quad (\omega \in \mathbb{U}), \tag{1.10}$$

where $g(\omega) = f^{-1}(\omega)$ given by (1.3).

Remark 1.1. Noting that

$$\frac{D_1^1 f(z)}{D^0 f(z)} = \frac{zf'(z)}{f(z)}, \quad \frac{D^2 f(z)}{D^1 f(z)} = \frac{z(zf'(z))'}{zf'(z)} = 1 + \frac{zf''(z)}{f'(z)},$$

we see that

$$\mathcal{M}_1^{0,0}(\varphi, \varphi) = \mathcal{ST}_{\mathcal{M}}(\varphi), \quad \mathcal{M}_1^{0,1}(\varphi, \varphi) = \mathcal{CV}_{\mathcal{M}}(\varphi).$$

Remark 1.2. If we set

$$\varphi(z) = \phi(z) = \frac{1+Az}{1+Bz} \quad (-1 \le B < A \le 1),$$

then the class

$$\mathcal{M}_\tau^{n,\,\alpha}(\varphi,\,\phi) \equiv \mathcal{M}_\tau^{n,\,\alpha}(A,\,B),$$

which is defined by demanding that $f \in \mathcal{M}$,

$$1 + \frac{1}{\tau}\left[(1-\alpha)\frac{D^{n+1}f(z)}{D^n f(z)} + \alpha\frac{(D^{n+1}f(z))'}{(D^n f(z))'} - 1\right] \prec \frac{1+Az}{1+Bz} \tag{1.11}$$

and

$$1 + \frac{1}{\tau}\left[(1-\alpha)\frac{D^{n+1}g(\omega)}{D^n g(\omega)} + \alpha\frac{(D^{n+1}g(\omega))'}{(D^n g(\omega))'} - 1\right] \prec \frac{1+A\omega}{1+B\omega}. \tag{1.12}$$

where the function g is given by (1.3).

Remark 1.3. If we choose

$$\varphi(z) = \phi(z) = \frac{1+(1-2\beta)z}{1-z} \quad (0 \le \beta < 1),$$

then the class

$$\mathcal{M}_\tau^{n,\,\alpha}(\varphi,\,\phi) \equiv \mathcal{M}_\tau^{n,\,\alpha}(\beta),$$

which is defined by demanding that $f \in \mathcal{M}$,

$$\Re\left(1 + \frac{1}{\tau}\left[(1-\alpha)\frac{D^{n+1}f(z)}{D^n f(z)} + \alpha\frac{(D^{n+1}f(z))'}{(D^n f(z))'} - 1\right]\right) > \beta \tag{1.13}$$

and

$$\Re\left(1 + \frac{1}{\tau}\left[(1-\alpha)\frac{D^{n+1}g(\omega)}{D^n g(\omega)} + \alpha\frac{(D^{n+1}g(\omega))'}{(D^n g(\omega))'} - 1\right]\right) > \beta, \tag{1.14}$$

where the function g is given by (1.3).

Remark 1.4. By putting

$$\varphi(z) = \phi(z) = \left(\frac{1+z}{1-z}\right)^\gamma \; (0 < \gamma \le 1),$$

then the class

$$\mathcal{M}_\tau^{n,\,\alpha}(\varphi,\,\phi) \equiv \mathcal{M}_\tau^{n,\,\alpha}(\gamma),$$

which is defined by demanding that $f \in \mathcal{M}$,

$$\left|\arg\left(1 + \frac{1}{\tau}\left[(1-\alpha)\frac{D^{n+1}f(z)}{D^n f(z)} + \alpha\frac{(D^{n+1}f(z))'}{(D^n f(z))'} - 1\right]\right)\right| < \frac{\gamma\pi}{2} \tag{1.15}$$

and

$$\left|\arg\left(1 + \frac{1}{\tau}\left[(1-\alpha)\frac{D^{n+1}g(\omega)}{D^n g(\omega)} + \alpha\frac{(D^{n+1}g(\omega))'}{(D^n g(\omega))'} - 1\right]\right)\right| < \frac{\gamma\pi}{2}, \tag{1.16}$$

where the function g is given by (1.3).

A function in the class $\mathcal{M}_\tau^{n,\,\alpha}(\varphi,\,\phi)$ is called both bi-(α) convex function and bi-(α) starlike function of complex order τ of Ma-Minda type. The related research of the general class associated with subclasses could be found in [7, 23].

Remark 1.5. We note that, denote by $\sigma, \Sigma, \mathcal{T}$ the same class of bi-univalent functions as $\mathcal{M}$ in $\mathbb{U}$. For suitable choices n, α, τ, the class $\mathcal{M}_{\tau}^{n,\alpha}(\varphi, \phi)$ reduces to several known classes and bring about new classes.

(1) $\mathcal{M}_{\tau}^{0,0}(\varphi, \varphi) = \mathcal{S}_{\Sigma}(\tau, 0, 1, 0; \varphi)$ and
$\mathcal{M}_{\tau}^{1,0}(\varphi, \varphi) = \mathcal{S}_{\Sigma}(\tau, 1, 1, 0; \varphi)$ (see Srivastava *et al.* [18, Definition 1]);
(2) $\mathcal{M}_{1}^{0,0}(\varphi, \varphi) = \mathcal{S}\mathcal{T}_{\sigma}(0, \varphi)$, $\mathcal{M}_{1}^{0,0}(\varphi, \varphi) = \mathcal{M}_{\sigma}(0, \varphi)$, $\mathcal{M}_{1}^{1,0}(\varphi, \varphi) = \mathcal{M}_{\sigma}(1, \varphi)$,
$\mathcal{M}_{1}^{0,0}(\varphi, \varphi) = \mathcal{L}_{\sigma}(1, \varphi)$ and $\mathcal{M}_{1}^{1,0}(\varphi, \varphi) = \mathcal{L}_{\sigma}(0, \varphi)$ (see Ali *et al.* [1]);
(3) $\mathcal{M}_{\tau}^{0,0}(\varphi, \varphi) = \mathcal{S}_{\Sigma}(0, \tau; \varphi)$ and $\mathcal{M}_{\tau}^{0,1}(\varphi, \varphi) = \mathcal{S}_{\Sigma}(1, \tau; \varphi)$ (see Deniz [7]);
(4) $\mathcal{M}_{\tau}^{n,\alpha}(\varphi, \varphi) = \mathcal{S}_{\Sigma}^{a,1,a}(\tau, 1, \varphi)(a \in \mathbb{C})$ (see Peng *et al.* [13, Example 4]);
(5) $\mathcal{M}_{1}^{0,0}(\varphi, \phi) = \mathcal{M}_{\Sigma}^{a,1,a}(1, 1, \varphi, \phi)$ (see Sharma [16, Definition 1]);
(6) $\mathcal{M}_{\tau}^{n,0}(\varphi, \phi) = \mathcal{H}_{\mathcal{T}}(\varphi, \phi, n, \tau)$ (see Xiong *et al.* [25]);
(7) $\mathcal{M}_{\tau}^{n,0}(\varphi, \varphi) = ST_{\Sigma}(\tau, \varphi)$ (see Selvaraj *et al.* [15, Definition 1]),
$\mathcal{M}_{\tau}^{n,1}(\varphi, \varphi) = \mathcal{C}\mathcal{V}_{\Sigma}(\tau, \varphi)$ (see Selvaraj *et al.* [15, Definition 2]);
(8) $\mathcal{M}_{1}^{0,0}(\varphi, \varphi) = \mathcal{H}_{\sigma}^{0}(1, \varphi)$ (see Tang *et al.* [23, Definition 2.1]),
$\mathcal{M}_{1}^{0,0}(\beta, \beta) = \mathcal{M}_{\sigma}^{\tau}(0, 0, \beta)$ $(0 \le \beta < 1)$ (see Tang *et al.* [23, Definition 2.2]);
(9) $\mathcal{M}_{1}^{0,0}(\beta, \beta) = \mathcal{S}_{\sigma}^{*}(\beta)$ $(0 \le \beta < 1)$ (see Brannan and Taha [3, Definition 3.1]);

2. Coefficient estimates for the function class $\mathcal{M}_{\tau}^{n,\alpha}(\varphi, \phi)$

In this section, we will find the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{M}_{\tau}^{n,\alpha}(\varphi, \phi)$.

Theorem 2.1. *Let f be given by (1.1) and $f \in \mathcal{M}_{\tau}^{n,\alpha}(\varphi, \phi)$. Then*

$$|a_2| \le \frac{|\tau|\,\Psi_1\Phi_1\sqrt{\Psi_1+\Phi_1}}{\sqrt{\left|\tau\Psi_1^2\Phi_1^2\cdot\Omega_\alpha - [(\Psi_2-\Psi_1)\Phi_1^2 + (\Phi_2-\Phi_1)\Psi_1^2](1+\alpha)^2 4^n\right|}} \tag{2.1}$$

and

$$|a_3| \le \frac{|\tau|\,\Psi_1^3\cdot|\Theta_\alpha| + |\tau|\,(1+3\alpha)4^n\Psi_1^2\Phi_1 + |\tau|\left|(\Psi_2-\Psi_1)\Phi_1^2\cdot\Theta_\alpha + (1+3\alpha)4^n(\Phi_2-\Phi_1)\Psi_1^2\right|}{2(1+2\alpha)3^n\,|\Theta_\alpha - (1+3\alpha)4^n|\,\Psi_1^2}, \tag{2.2}$$

where

$$\Omega_\alpha := 4(1+2\alpha)3^n - 2(1+3\alpha)4^n \tag{2.3}$$

and

$$\Theta_\alpha := 4(1+2\alpha)3^n - (1+3\alpha)4^n. \tag{2.4}$$

Proof. Since $f \in \mathcal{M}_{\tau}^{n,\alpha}(\varphi, \phi)$, there exist two Schwarz functions $u(z), \upsilon(\omega) : \mathbb{U} \to \mathbb{U}$, with $u(0) = \upsilon(0) = 0$, such that

$$1 + \frac{1}{\tau}\left[(1-\alpha)\frac{D^{n+1}f(z)}{D^n f(z)} + \alpha\frac{(D^{n+1}f(z))'}{(D^n f(z))'} - 1\right] = \varphi(u(z)) \quad (z \in \mathbb{U}) \tag{2.5}$$

and

$$1 + \frac{1}{\tau}\left[(1-\alpha)\frac{D^{n+1}g(\omega)}{D^n g(\omega)} + \alpha\frac{(D^{n+1}g(\omega))'}{(D^n g(\omega))'} - 1\right] = \phi(\upsilon(\omega)) \quad (\omega \in \mathbb{U}). \tag{2.6}$$

Define the functions ξ and η as following

$$\xi(z) = \frac{1+u(z)}{1-u(z)} = 1 + r_1 z + r_2 z^2 + r_3 z^3 + \cdots$$

and

$$\eta(\omega) = \frac{1+\upsilon(\omega)}{1-\upsilon(\omega)} = 1 + s_1 \omega + s_2 \omega^2 + b_3 \omega^3 + \cdots$$

or equivalently

$$u(z) = \frac{\xi(z)-1}{\xi(z)+1} = \frac{r_1}{2} z + \frac{1}{2}\left(r_2 - \frac{r_1^2}{2}\right) z^2 + \frac{1}{2}\left(r_3 + \frac{r_1}{2}\left(\frac{r_1^2}{2} - r_2\right) - \frac{r_1 r_2}{2}\right) z^3 + \cdots \quad (2.7)$$

and

$$\upsilon(\omega) = \frac{\eta(\omega)-1}{\eta(\omega)+1} = \frac{s_1}{2}\omega + \frac{1}{2}\left(s_2 - \frac{s_1^2}{2}\right)\omega^2 + \frac{1}{2}\left(s_3 + \frac{s_1}{2}\left(\frac{s_1^2}{2} - s_2\right) - \frac{s_1 s_2}{2}\right)\omega^3 + \cdots \quad (2.8)$$

Obviously, u and $\upsilon \in \mathcal{P}$ and $u(0) = \upsilon(0) = 1$. Also u and υ have positive real part in $\mathbb{U}$, hence $|r_i| \leq 2$ and $|s_i| \leq 2$ $(i = 1, 2, 3, \cdots)$. Combining (1.4), (2.7) and (2.8), we obtain

$$\varphi(u(z)) = \varphi\left(\frac{\xi(z)-1}{\xi(z)+1}\right) = 1 + \frac{1}{2}\Psi_1 r_1 z + \left(\frac{1}{2}\Psi_1\left(r_2 - \frac{r_1^2}{2}\right) + \frac{1}{4}\Psi_2 r_1^2\right) z^2 + \cdots \quad (2.9)$$

and

$$\phi(\upsilon(\omega)) = \phi\left(\frac{\eta(\omega)-1}{\eta(\omega)+1}\right) = 1 + \frac{1}{2}\Phi_1 s_1 \omega + \left(\frac{1}{2}\Phi_1\left(s_2 - \frac{s_1^2}{2}\right) + \frac{1}{4}\Phi_2 s_1^2\right)\omega^2 + \cdots \quad (2.10)$$

Follows from (2.5), (2.6), (2.9) and (2.10) comparing coefficients of z, z^2, respectively, we get

$$\frac{1}{\tau}(1+\alpha)2^n a_2 = \frac{1}{2}\Psi_1 r_1, \quad (2.11)$$

$$\frac{1}{\tau}[2(1+2\alpha)3^n a_3 - (1+3\alpha)4^n a_2^2] = \frac{1}{2}\Psi_1\left(r_2 - \frac{r_1^2}{2}\right) + \frac{1}{4}\Psi_2 r_1^2, \quad (2.12)$$

$$-\frac{1}{\tau}(1+\alpha)2^n a_2 = \frac{1}{2}\Phi_1 s_1, \quad (2.13)$$

and

$$\frac{1}{\tau}\Big\{[4(1+2\alpha)3^n - (1+3\alpha)4^n]a_2^2 - 2(1+2\alpha)3^n a_3\Big\} = \frac{1}{2}\Phi_1\left(s_2 - \frac{s_1^2}{2}\right) + \frac{1}{4}\Phi_2 s_1^2. \quad (2.14)$$

By adding (2.11) and (2.13), we easily know that

$$r_1 = -\frac{\Phi_1}{\Psi_1} s_1. \quad (2.15)$$

By virtue of (2.12) to (2.14), we have

$$a_2^2 = \frac{\tau^2 \Psi_1^2 \Phi_1^2 (\Psi_1 r_2 + \Phi_1 s_2)}{2\tau \Psi_1^2 \Phi_1^2 \cdot \Omega_\alpha - 2[(\Psi_2 - \Psi_1)\Phi_1^2 + (\Phi_2 - \Phi_1)\Psi_1^2](1+\alpha)^2 4^n}, \quad (2.16)$$

where Ω_α given by (2.3). As $|r_1| \leq 2$, $|s_1| \leq 2$ mentioned above, we derive the desired estimation on $|a_2|$ as asserted in (2.2).

From (2.12) and (2.14), we get

$$\frac{2}{\tau}(1+2\alpha)3^n[4(1+2\alpha)3^n-(1+3\alpha)4^n]a_3=\frac{1}{2}[4(1+2\alpha)3^n-(1+3\alpha)4^n]\Psi_1\left(r_2-\frac{r_1^2}{2}\right)$$
$$+\frac{1}{4}[4(1+2\alpha)3^n-(1+3\alpha)4^n]\Psi_2 r_1^2+\frac{1}{2}(1+3\alpha)4^n\Phi_1\left(s_2-\frac{s_1^2}{2}\right)$$
$$+\frac{1}{4}(1+3\alpha)4^n B_2 s_1^2+\frac{2}{\tau}(1+2\alpha)(1+3\alpha)4^n3^n a_3.$$

Let $\Theta_\alpha := 4(1+2\alpha)3^n-(1+3\alpha)4^n$, such that

$$a_3=\frac{\begin{array}{l}2\tau\Psi_1^3 r_2\cdot\Theta_\alpha+2\tau(1+3\alpha)4^n\Psi_1^2\Phi_1 s_2\\ \qquad+\tau[(\Psi_2-\Psi_1)\Phi_1^2\cdot\Theta_\alpha+(1+3\alpha)4^n(\Phi_2-\Phi_1)\Psi_1^2]s_1^2\end{array}}{8(1+2\alpha)3^n\Psi_1^2\cdot\Theta_\alpha-8(1+2\alpha)(1+3\alpha)3^n4^n\Psi_1^2},\tag{2.17}$$

thus, the result of (2.5) follows. We complete the the proof of Theorem 2.1.

3. Applications of the main result

This section is aimed to give some application of the main result. Different choices of the functions φ and ϕ which would provide interesting subclasses of analytic functions.

At first, by putting

$$\varphi(z)=\phi(z)=1+\Psi_1 z+\Psi_2 z^2+\Psi_3 z^3+\cdots \quad (\Psi_1>0).\tag{3.1}$$

in Theorem 2.1, we get following corollary.

Corollary 3.1. *Let f be given by (1.1) and $f\in\mathcal{M}_\tau^{n,\alpha}(\varphi)$. Then*

$$|a_2|\le\frac{|\tau|\,\Psi_1\sqrt{2\Psi_1}}{\sqrt{|\tau\Psi_1^2\cdot\Omega_\alpha-2(\Psi_2-\Psi_1)(1+\alpha)^2 4^n|}}\tag{3.2}$$

and

$$|a_3|\le\frac{|\tau|\,\Psi_1\,[|\Theta_\alpha|+(1+3\alpha)4^n]+4\,|\tau|\,(1+2\alpha)3^n\,|\Psi_2-\Psi_1|}{2(1+2\alpha)3^n\,|\Theta_\alpha-(1+3\alpha)4^n|},\tag{3.3}$$

where Ω_α and Θ_α given by (2.3) and (2.4), respectively.

Putting $\alpha=1$ in the Theorem 2.1 yields following corollary.

Corollary 3.2. *Let f be given by (1.1) and $f\in\mathcal{M}_\tau^{n,1}(\varphi,\phi)$. Then*

$$|a_2|\le\frac{|\tau|\,\Psi_1\Phi_1\sqrt{\Psi_1+\Phi_1}}{\sqrt{|\tau(4\cdot3^{n+1}-8\cdot4^n)\Psi_1^2\Phi_1^2-4^{n+1}[(\Psi_2-\Psi_1)\Phi_1^2+(\Phi_2-\Phi_1)\Psi_1^2]|}}\tag{3.4}$$

and

$$|a_3|\le\frac{|\tau|\,\Psi_1^3\cdot|\Theta_1|+|\tau|\,4^{n+1}\,\Psi_1^2\Phi_1+|\tau|\,|(\Psi_2-\Psi_1)\Phi_1^2\cdot\Theta_1+4^{n+1}(\Phi_2-\Phi_1)\Psi_1^2|}{6\cdot3^n\,|\Theta_1-6\cdot4^{n+1}|\,\Psi_1^2},\tag{3.5}$$

where $\Theta_1 := 4\cdot3^{n+1}-4^{n+1}$.

Put $\alpha=0$ in the Theorem 2.1, the following corollary coincides with the result of Xiong [25, Theorem 2.1].

Corollary 3.3. *(see [25]) Let f be given by (1.1) and $f \in \mathcal{M}_{\tau}^{n,0}(\varphi, \phi)$. Then*

$$|a_2| \leq \frac{|\tau|\, \Psi_1\Phi_1\sqrt{\Psi_1+\Phi_1}}{\sqrt{|\tau(4\cdot 3^n - 2\cdot 4^n)\Psi_1^2\Phi_1^2 - 4^n[(\Psi_2-\Psi_1)\Phi_1^2 + (\Phi_2-\Phi_1)\Psi_1^2]|}} \tag{3.6}$$

and

$$|a_3| \leq \frac{|\tau|\,\Psi_1^3\cdot|\Theta_0| + |\tau|\,4^n\,\Psi_1^2\Phi_1 + |\tau|\,|(\Psi_2-\Psi_1)\Phi_1^2\cdot\Theta_0 + 4^n(\Phi_2-\Phi_1)\Psi_1^2|}{2\cdot 3^n\,|\Theta_0 - 4^n|\;\Psi_1^2}, \tag{3.7}$$

where $\Theta_0 := 4\cdot 3^n - 4^n$.

Remark 3.1. If we put $\alpha = 0$ with the condition (3.1), then we get the result coinciding with Xiong [25, Corollary 2.1 of Theorem 2.1] and Selvaraj [15, Theorem 2]. Combining the conditions $\alpha = 0$, $\tau = 1$, choosing $n = 0$, $n = 1$ respectively, we we get the results which coincides with the assertion Ali [1]. We display here.

Corollary 3.4. *(see [1]) Let f be given by (1.1) and $f \in \mathcal{M}_{1}^{0,0}(\varphi, \phi)$. Then*

$$|a_2| \leq \frac{\Psi_1\sqrt{\Psi_1}}{\sqrt{|\Psi_1^2 + \Psi_1 - \Psi_2|}} \tag{3.8}$$

and

$$|a_3| \leq \Psi_1 + |\Psi_2 - \Psi_1|\,. \tag{3.9}$$

Corollary 3.5. *(see [1]) Let f be given by (1.1) and $f \in \mathcal{M}_{1}^{0,0}$. Then*

$$|a_2| \leq \frac{\Psi_1\sqrt{\Psi_1}}{\sqrt{2\,|\Psi_1^2 + 2\Psi_1 - 2\Psi_2|}} \tag{3.10}$$

and

$$|a_3| \leq \frac{1}{2}(\Psi_1 + |\Psi_2 - \Psi_1|). \tag{3.11}$$

By letting

$$\varphi(z) = \phi(z) = \left(\frac{1+z}{1-z}\right)^{\gamma} \quad (0 < \gamma \leq 1) \tag{3.12}$$

in Theorem 2.1 guides us the following theorem.

Theorem 3.1. *Let f be given by (1.1) and $f \in \mathcal{M}_{\tau}^{n,\alpha}(\gamma)$. Then*

$$|a_2| \leq \frac{\sqrt{2}\,|\tau|\,\gamma}{\sqrt{|\tau\gamma\cdot\Omega_\alpha - 2(1+\alpha)^2\gamma(\gamma-1)4^n|}} \tag{3.13}$$

and

$$|a_3| \leq |\tau|\,\gamma\cdot\frac{[|\Theta_\alpha| + (1+3\alpha)4^n] + (1+2\alpha)3^n(1-\gamma)}{(1+2\alpha)3^n\,|\Theta_\alpha - (1+3\alpha)4^n|}, \tag{3.14}$$

where Ω_α and Θ_α given by (2.3) and (2.4), respectively.

Proof. From (3.12), we know that

$$\varphi(z) = \phi(z) = 1 + 2\gamma z + 2\gamma^2 z^2 + \cdots$$

Obviously $\varphi(z), \phi(z)$ satisfy the conditions

$$\varphi(0) = \phi(0) = 1, \quad \varphi'(0) = \phi'(0) = 2\gamma > 0.$$

Combining (3.1) and set

$$\Psi_1 = \Phi_1 = 2\gamma, \quad \Psi_2 = \Phi_2 = 2\gamma^2$$

in Theorem 2.1, we easily get the Theorem 3.1.

If we put

$$\varphi(z) = \phi(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1) \tag{3.15}$$

in Theorem 2.1, we get following corollary.

Corollary 3.6. *Let f be given by (1.1) and $f \in \mathcal{M}_\tau^{n,\alpha}(A, B)$. Then*

$$|a_2| \leq \frac{\sqrt{2}\,|\tau|\,(A - B)}{\sqrt{|\tau(A - B)\cdot\Omega_\alpha + 2(1 + \alpha)^2(1 + B)4^n|}} \tag{3.16}$$

and

$$|a_3| \leq (A - B)\,|\tau| \cdot \frac{[|\Theta_\alpha| + (1 + 3\alpha)4^n] + (1 + 3\alpha)4^n(1 + B)}{2(1 + 2\alpha)3^n\,|\Theta_\alpha - (1 + 3\alpha)4^n|}, \tag{3.17}$$

where Ω_α and Θ_α given by (2.3) and (2.4), respectively.

Putting

$$\varphi(z) = \phi(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \quad (0 \leq \beta < 1) \tag{3.18}$$

in Theorem 2.1 comes following corollary.

Corollary 3.7. *Let f be given by (1.1) and $f \in \mathcal{M}_\tau^{n,\alpha}(\beta)$. Then*

$$|a_2| \leq \frac{\sqrt{|2\tau(1 - \beta)|}}{\sqrt{|\Omega_\alpha|}} \tag{3.19}$$

and

$$|a_3| \leq |\tau|\,(1 - \beta)\frac{[|\Theta_\alpha| + 2(1 + 3\alpha)4^n]}{(1 + 2\alpha)3^n\,|\Theta_\alpha - (1 + 3\alpha)4^n|}, \tag{3.20}$$

where Ω_α and Θ_α given by (2.3) and (2.4), respectively.

4. Conclusions

In the first section, we introduce a new subclass $\mathcal{M}_\tau^{n,\alpha}(\varphi, \phi)$ with Sălăgean operator $D^n f$ in $\mathbb{U}$. With the operator we make some connections with related field's works. In the following section, we find estimates on the initial Taylor-Maclaurin coefficients a_2 and a_3. Then, we make some applications with the desired results. Some results extend and coincide with earlier works. It needs to point out that it is interesting and deserve us to explore more different results by setting the varied parameter α, τ, n.

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