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OPERATOR AND MATRIX INEQUALITIES FOR HEINZ MEAN

WENSHI LIAO[1] AND PUJUN LONG[1]

Abstract. Motivated by the refinements and reverses of the well-known Young inequality, in this article, we derive some new improvements of Heinz mean inequalities for positive invertible operators and the Hilbert-Schmidt norm.

1. Introduction

It is well-known that the Young inequality for scalars says that if $a, b > 0$ and $v \in [0, 1]$, then

$$a^{1-v}b^v \leq (1-v)a + vb \tag{1.1}$$

with equality if and only if $a = b$. The inequality (1.1) is also called weighted arithmetic-geometric mean inequality.

For $0 \leq v \leq 1$ and two nonnegative real numbers a and b, the Heinz mean interpolates between the arithmetic mean and geometric mean defined as

$$H_v(a, b) = \frac{a^v b^{1-v} + a^{1-v}b^v}{2}.$$

It is easy to see that the Heinz mean is convex as a function of v on the interval $[0, 1]$, attains minimum at $v = 1/2$, and attains maximum at $v = 0$ and $v = 1$, it is obvious that

$$\sqrt{ab} \leq H_v(a, b) \leq \frac{a+b}{2}. \tag{1.2}$$

Moreover, the function $H_v(a, b)$ is symmetric about the point $v = 1/2$, that is, $H_v(a, b) = H_{1-v}(a, b)$.

Kittaneh and Manasrah [5,6] improved Young inequality (1.1) and Heinz inequality (1.2), and obtained the following inequalities:

$$r(\sqrt{a} - \sqrt{b})^2 \leq (1-v)a + vb - a^{1-v}b^v \leq R(\sqrt{a} - \sqrt{b})^2, \tag{1.3}$$

$$2r(\sqrt{a} - \sqrt{b})^2 \leq a + b - \left(a^v b^{1-v} + a^{1-v}b^v\right) \leq 2R(\sqrt{a} - \sqrt{b})^2, \tag{1.4}$$

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$$2r(a-b)^2 \le (a+b)^2 - \left(a^v b^{1-v} + a^{1-v} b^v\right)^2 \le 2R(a-b)^2, \tag{1.5}$$

where $a, b > 0$, $v \in [0,1]$, $r = \min\{v, 1-v\}$ and $R = \max\{v, 1-v\}$.

Recently, the inequality (1.1) and the first one in (1.3) was refined by Zhao and Wu in the following forms, for the purpose of the study on operators inequalities.

Proposition 1.1. [10] *Let a, b be two nonnegative real numbers and $v \in (0,1)$.*

(I) *If $0 < v \le \frac{1}{2}$, then*

$$r_0(\sqrt[4]{ab} - \sqrt{a})^2 + v(\sqrt{a} - \sqrt{b})^2 + a^{1-v}b^v \le (1-v)a + vb, \tag{1.6}$$

(II) *if $\frac{1}{2} < v < 1$, then*

$$r_0(\sqrt[4]{ab} - \sqrt{b})^2 + (1-v)(\sqrt{a} - \sqrt{b})^2 + a^{1-v}b^v \le (1-v)a + vb, \tag{1.7}$$

where $r = \min\{v, 1-v\}$ and $r_0 = \min\{2r, 1-2r\}$.

And they also presented the reverse forms of the inequality (1.1), which are more precise than the second inequality in (1.3).

Proposition 1.2. [10] *Let a, b be two nonnegative real numbers and $v \in (0,1)$.*

(I) *If $0 < v \le \frac{1}{2}$, then*

$$(1-v)a + vb \le a^{1-v}b^v + (1-v)(\sqrt{a} - \sqrt{b})^2 - r_0(\sqrt[4]{ab} - \sqrt{b})^2, \tag{1.8}$$

(II) *if $\frac{1}{2} < v < 1$, then*

$$(1-v)a + vb \le a^{1-v}b^v + v(\sqrt{a} - \sqrt{b})^2 - r_0(\sqrt[4]{ab} - \sqrt{a})^2, \tag{1.9}$$

where $r = \min\{v, 1-v\}$ and $r_0 = \min\{2r, 1-2r\}$.

Now we explain the notation and historical background of the operator inequalities related to the previous classical inequalities.

Let $(\mathcal{B}(\mathcal{H}), \|\cdot\|)$ be the C^*-algebra of all bounded linear operators on a complex separable Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ and $\mathcal{B}_h(\mathcal{H})$ be the semi-space of all self-adjoint operators in $\mathcal{B}(\mathcal{H})$. Moreover, a self-adjoint operator $A \in \mathcal{B}_h(\mathcal{H})$ is called strictly positive if $\langle Ax, x\rangle > 0$ for every $x \in \mathcal{H} \setminus \{0\}$ and the cone of all positive invertible operators is denoted by $\mathcal{B}^{++}(\mathcal{H})$, I stands for the identity operator. In the case when $\dim \mathcal{H} = n$, we identify $\mathcal{B}(\mathcal{H})$ with the full matrix algebra $\mathcal{M}_n$ of all $n \times n$ matrices with entries in the complex field.

As a matter of convenience, we use the following notation to define the weighted arithmetic mean and geometric mean for operators:

$$A\nabla_v B = (1-v)A + vB, \ \ A\#_v B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v A^{\frac{1}{2}},$$

where $A, B \in \mathcal{B}^{++}(H)$ and $0 \le v \le 1$. When $v = \frac{1}{2}$, we write $A\nabla B$ and $A\#B$ for brevity, respectively.

The operator version of the Heinz mean, denoted by $H_v(A, B)$, is defined as

$$H_v(A,B) = \frac{A\#_v B + A\#_{1-v} B}{2},$$

for $A, B \in \mathcal{B}^{++}(H)$ and $0 \le v \le 1$.

It is easy to see that

$$A\#B \leq H_v(A,B) \leq A\nabla B. \tag{1.10}$$

The series of inequalities in (1.10) are the Heinz operator inequalities.

Kittaneh and Manasrah [6] obtained the following inequalities related to the inequalities (1.4)

$$2r(A\nabla B - A\#B) \leq A\nabla B - H_v(A,B) \leq 2R(A\nabla B - A\#B) \tag{1.11}$$

for positive definite matrices A and B and $0 \leq v \leq 1$, where $r = \min\{v, 1-v\}$ and $R = \max\{v, 1-v\}$, which of course remain valid for Hilbert space operators by a standard approximation argument.

The authors of [7] also obtain the same result of the first inequality in (1.11) for two positive operators.

In [8], we derived a reverse ratio form of the first inequality of (1.11), for $A, B \in \mathcal{B}^{++}(\mathcal{H})$ and $v \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, then

$$A\nabla B \leq \mathrm{K}(\sqrt{h}, 2)^{R_0} H_v(A,B) + 2r(A\nabla B - A\#B), \tag{1.12}$$

and from the inequalities of Theorem 3.4 in Wu and Zhao [9], by putting $C = A^{1/2}$, we deduce that

$$\begin{aligned}
\mathrm{K}(\sqrt{h}, 2)^{r_0} & H_v(A,B) + 2r(A\nabla B - A\#B) \\
&\leq A\nabla B \\
&\leq \mathrm{K}(\sqrt{h}, 2)^{-r_0} H_v(A,B) + 2R(A\nabla B - A\#B),
\end{aligned} \tag{1.13}$$

where $\mathrm{K}(\cdot, 2)$ is the Kantorovich constant, defined by $\mathrm{K}(t,2) = (t+1)^2/4t$ $(t > 0)$, $R = \max\{v, 1-v\}$, $r = \min\{v, 1-v\}$, $R_0 = \max\{1-2r, 2r\}$, $r_0 = \min\{1-2r, 2r\}$ and $h = M/m$. Note that the inequalities (1.13) are an improvement of (1.11) with $\mathrm{K}(t,2) \geq 1$ and the inequality (1.12) can not be compared with the second inequality in (1.13).

By Proposition 1.1 and 1.2, Zhao and Wu [10] presented the improvements of the inequalities (1.10) and (1.11) for two positive invertible operators.

Our main task in this article is to derive several new refinements and reverses of the Heinz operator inequalities. This article is organized in the following way: in Section 2, we establish the whole series of refinements and reverses of the Heinz inequalities via the improved Young inequality which will help us in deriving Heinz operator inequalities. In Section 3, we obtain related Heinz operator inequalities. In Section 4, the Hilbert-Schmidt norm inequalities are presented.

Throughout the paper, $R = \max\{v, 1-v\}$, $r = \min\{v, 1-v\}$, $R_0 = \max\{1-2r, 2r\}$, $r_0 = \min\{1-2r, 2r\}$.

2. Refinements and reverses of the Heinz mean inequalities

In this section, we give refinements and reverses of the Heinz inequality (for more details, the reader is referred to [8–10]).

Lemma 2.1. *Let a, b be two nonnegative real numbers and $v \in (0,1)$. Then*

$$\frac{1}{2}r_0\left[(\sqrt[4]{ab}-\sqrt{a})^2+(\sqrt[4]{ab}-\sqrt{b})^2\right]+r(\sqrt{a}-\sqrt{b})^2+H_v(a,b)\le\frac{a+b}{2}. \tag{2.1}$$

Proof. If $0<v\le\frac{1}{2}$, interchanging a and b in the inequality (1.6), then

$$r_0(\sqrt[4]{ab}-\sqrt{b})^2+v(\sqrt{a}-\sqrt{b})^2+a^vb^{1-v}\le va+(1-v)b, \tag{2.2}$$

adding the inequalities (1.6) and (2.2) together, we obtain

$$r_0\left[(\sqrt[4]{ab}-\sqrt{a})^2+(\sqrt[4]{ab}-\sqrt{b})^2\right]+2v(\sqrt{a}-\sqrt{b})^2+a^vb^{1-v}+a^{1-v}b^v\le a+b.$$

If $\frac{1}{2}<v<1$, by the inequality (1.7) with the similar method, we get

$$r_0\left[(\sqrt[4]{ab}-\sqrt{a})^2+(\sqrt[4]{ab}-\sqrt{b})^2\right]+2(1-v)(\sqrt{a}-\sqrt{b})^2+a^vb^{1-v}+a^{1-v}b^v\le a+b.$$

So we conclude that for $v\in(0,1)$

$$\frac{1}{2}r_0\left[(\sqrt[4]{ab}-\sqrt{a})^2+(\sqrt[4]{ab}-\sqrt{b})^2\right]+r(\sqrt{a}-\sqrt{b})^2+H_v(a,b)\le\frac{a+b}{2}.$$

The proof is completed. □

Lemma 2.2. *Let a,b be two nonnegative real numbers and $v\in(0,1)$. Then*

$$\frac{a+b}{2}\le H_v(a,b)+R(\sqrt{a}-\sqrt{b})^2-\frac{1}{2}r_0\left[(\sqrt[4]{ab}-\sqrt{a})^2+(\sqrt[4]{ab}-\sqrt{b})^2\right]. \tag{2.3}$$

Proof. By the inequalities (1.8) and (1.9), the proof is similar to that of Lemma 2.1, so we omit it. □

Remark 2.1. The inequalities in Lemma 2.1 and Lemma 2.2 improve the inequalities (1.4).

Lemma 2.3. *Let a,b be two nonnegative real numbers and $v\in(0,1)$. Then*

$$r_0\left[(\sqrt{ab}-a)^2+(\sqrt{ab}-b)^2\right]+2r(a-b)^2+4H_v(a,b)^2\le(a+b)^2. \tag{2.4}$$

Proof. If $0<v\le\frac{1}{2}$, by virtue of replacing a by a^2 and b by b^2 in (1.6), respectively, then we have

$$\begin{aligned}
&(a+b)^2-4H_v(a,b)^2\\
&\qquad=(a+b)^2-\left(a^{1-v}b^v+a^vb^{1-v}\right)^2\\
&\qquad=a^2+b^2-(a^{1-v}b^v)^2-(a^vb^{1-v})^2\\
&\qquad=(1-v)\,a^2+vb^2-(a^{1-v}b^v)^2+va^2+(1-v)\,b^2-(a^vb^{1-v})^2\\
&\qquad\ge r_0(\sqrt{ab}-a)^2+v(a-b)^2+r_0(\sqrt{ab}-b)^2+v(a-b)^2\\
&\qquad=r_0\left[(\sqrt{ab}-a)^2+(\sqrt{ab}-b)^2\right]+2v(a-b)^2,
\end{aligned}$$

hence

$$r_0\left[(\sqrt{ab}-a)^2+(\sqrt{ab}-b)^2\right]+2v(a-b)^2+4H_v(a,b)^2\le(a+b)^2.$$

If $\frac{1}{2}<v<1$, by the inequality (1.7) with the similar method, we get

$$r_0\left[(\sqrt{ab}-a)^2+(\sqrt{ab}-b)^2\right]+2(1-v)(a-b)^2+4H_v(a,b)^2\le(a+b)^2.$$

So we obtain the desired result based on the above discussion. □

Lemma 2.4. *Let a, b be two nonnegative real numbers and $v \in (0,1)$. Then*

$$(a+b)^2 \leq 4H_v(a,b)^2 + 2R(a-b)^2 - r_0\left[(\sqrt{ab}-a)^2 + (\sqrt{ab}-b)^2\right]. \tag{2.5}$$

Proof. If $0 < v \leq \frac{1}{2}$, replacing a by a^2 and b by b^2 in (1.8), respectively, then we have

$$\begin{aligned}
&(a+b)^2 - 4H_v(a,b)^2 \\
&\quad = (1-v)\,a^2 + vb^2 - (a^{1-v}b^v)^2 + va^2 + (1-v)\,b^2 - (a^vb^{1-v})^2 \\
&\quad \leq (1-v)(a-b)^2 - r_0(\sqrt{ab}-b)^2 + (1-v)(a-b)^2 - r_0(\sqrt{ab}-a)^2 \\
&\quad = 2(1-v)(a-b)^2 - r_0\left[(\sqrt{ab}-a)^2 + (\sqrt{ab}-b)^2\right],
\end{aligned}$$

hence

$$(a+b)^2 \leq 4H_v(a,b)^2 + 2(1-v)(a-b)^2 - r_0\left[(\sqrt{ab}-a)^2 + (\sqrt{ab}-b)^2\right].$$

If $\frac{1}{2} < v < 1$, by the inequality (1.9) with the similar method, we get

$$(a+b)^2 \leq 4H_v(a,b)^2 + 2v(a-b)^2 - r_0\left[(\sqrt{ab}-a)^2 + (\sqrt{ab}-b)^2\right].$$

So we conclude the desired result. □

Remark 2.2. The inequalities in Lemma 2.3 and Lemma 2.4 improve the inequalities (1.5).

3. Refinements and reverses of the Heinz mean operator inequalities

In this section, we present the operator version for the refinements and reverses of the Heinz inequalities in section 2. The techniques are based on the monotonicity property of operator functions described in the following lemma (for more details, see [1,3]).

Lemma 3.1. *Let $X \in \mathcal{B}(\mathcal{H})$ be self-adjoint operator and if f and g are both continuous functions with $f(t) \geq g(t)$ for $t \in \operatorname{Sp}(X)$ (the spectrum of X), then $f(X) \geq g(X)$ with equality if and only if $f(t) = g(t)$ for all $t \in \operatorname{Sp}(X)$.*

Theorem 3.1. *Let $A, B \in \mathcal{B}^{++}(\mathcal{H})$ and $v \in (0,1)$. Then*

$$r_0\left(A\nabla B + A\#B - 2H_{\frac{1}{4}}(A,B)\right) + 2r(A\nabla B - A\#B) + H_v(A,B) \leq A\nabla B. \tag{3.1}$$

Equality holds if and only if $A = B$ or equivalently $A^{-\frac{1}{2}}BA^{-\frac{1}{2}} = I$.

Proof. By Lemma 2.1, if $0 < v \leq \frac{1}{2}$, then for any $b > 0$ we have

$$r_0\left[\frac{b+1}{2} + \sqrt{b} - 2H_{\frac{1}{4}}(1,b)\right] + 2v\left(\frac{b+1}{2} - \sqrt{b}\right) + H_v(1,b) \leq \frac{1+b}{2}.$$

Making use of Lemma 3.1, for a positive invertible operator T and $v \in (0,1)$, it follows that

$$r_0\left[\frac{T+I}{2} + T^{\frac{1}{2}} - 2H_{\frac{1}{4}}(I,T)\right] + 2v\left(\frac{T+I}{2} - T^{\frac{1}{2}}\right) + H_v(I,T) \leq \frac{I+T}{2}.$$

Substituting $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ for T in the above inequality, we obtain

$$\begin{aligned} &r_0\left[\frac{A^{-\frac{1}{2}}BA^{-\frac{1}{2}}+I}{2}+\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}-2H_{\frac{1}{4}}\left(I,A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right]\\ &+2v\left(\frac{A^{-\frac{1}{2}}BA^{-\frac{1}{2}}+I}{2}-\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}\right)+H_v\left(I,A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\\ &\leq\frac{I+A^{-\frac{1}{2}}BA^{-\frac{1}{2}}}{2}. \end{aligned}\tag{3.2}$$

Applying the $*$-conjugation $\bullet\mapsto A^{1/2}\bullet A^{1/2}$ to (3.2) (see [4, Theorem 7.7.2])

$$r_0\left(A\nabla B+A\#B-2H_{\frac{1}{4}}(A,B)\right)+2v(A\nabla B-A\#B)+H_v(A,B)\leq A\nabla B.$$

If $\frac{1}{2}<v<1$, then for any $b>0$ we have

$$r_0\left[\frac{b+1}{2}+\sqrt{b}-2H_{\frac{1}{4}}(1,b)\right]+2(1-v)\left(\frac{b+1}{2}-\sqrt{b}\right)+H_v(1,b)\leq\frac{1+b}{2}.$$

By the similar process of the case $0<v\leq\frac{1}{2}$, we can deduce

$$r_0\left(A\nabla B+A\#B-2H_{\frac{1}{4}}(A,B)\right)+2(1-v)(A\nabla B-A\#B)+H_v(A,B)\leq A\nabla B.$$

So we conclude that the inequality (3.1) holds for $v\in(0,1)$. □

Theorem 3.2. *Let $A,B\in\mathcal{B}^{++}(\mathcal{H})$ and $v\in(0,1)$. Then*

$$A\nabla B\leq H_v(A,B)+2R(A\nabla B-A\#B)-r_0\left(A\nabla B+A\#B-2H_{\frac{1}{4}}(A,B)\right).\tag{3.3}$$

Equality holds if and only if $A=B$ or equivalently $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}=I$.

Proof. If $0<v\leq\frac{1}{2}$, by Lemma 2.2, then for any $a>0$ we have

$$\frac{a+1}{2}\leq H_v(a,1)+2(1-v)\left(\frac{a+1}{2}-\sqrt{a}\right)-r_0\left[\frac{a+1}{2}+\sqrt{a}-2H_{\frac{1}{4}}(a,1)\right].$$

Making use of Lemma 3.1, for a positive invertible operator R and $v\in(0,1)$, it follows that

$$\frac{R+I}{2}\leq H_v(R,I)+2(1-v)\left(\frac{R+I}{2}-R^{\frac{1}{2}}\right)-r_0\left[\frac{R+I}{2}+R^{\frac{1}{2}}-2H_{\frac{1}{4}}(R,I)\right].$$

Substituting $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ for R in the above inequality, we have

$$\begin{aligned} &\frac{I+B^{-\frac{1}{2}}AB^{-\frac{1}{2}}}{2}\\ &\leq H_v\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}},I\right)+2(1-v)\left[\frac{B^{-\frac{1}{2}}AB^{-\frac{1}{2}}+I}{2}-\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)^{\frac{1}{2}}\right]\\ &-r_0\left[\frac{B^{-\frac{1}{2}}AB^{-\frac{1}{2}}+I}{2}+\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)^{\frac{1}{2}}-2H_{\frac{1}{4}}\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}},I\right)\right]. \end{aligned}\tag{3.4}$$

Applying the $*$-conjugation $\bullet\mapsto B^{1/2}\bullet B^{1/2}$ to (3.4) and since $A\#_vB=B\#_{1-v}A$ for $0<v<1$, we get the following order relation

$$A\nabla B\leq H_v(A,B)+2(1-v)(A\nabla B-A\#B)-r_0\left(A\nabla B+A\#B-2H_{\frac{1}{4}}(A,B)\right).$$

If $\frac{1}{2} < v < 1$, by Lemma 2.2, then for any $a > 0$ we have

$$\frac{a+1}{2} \le H_v(a,1) + 2v\left(\frac{a+1}{2} - \sqrt{a}\right) - r_0\left[\frac{a+1}{2} + \sqrt{a} - 2H_{\frac{1}{4}}(a,1)\right].$$

By the similar process of the case $0 < v \le \frac{1}{2}$, we can deduce

$$A\nabla B \le H_v(A,B) + 2v(A\nabla B - A\#B) - r_0\left(A\nabla B + A\#B - 2H_{\frac{1}{4}}(A,B)\right).$$

So we conclude that the inequality (3.3) holds for $v \in (0,1)$.

□

Remark 3.1. Combining (3.1) and (3.3) with (1.11), we obtain

$$\begin{aligned} A\#B &\le H_v(A,B) \\ &\le 2r(A\nabla B - A\#B) + H_v(A,B) \\ &\le r_0\left(A\nabla B + A\#B - 2H_{\frac{1}{4}}(A,B)\right) + 2r(A\nabla B - A\#B) + H_v(A,B) \\ &\le A\nabla B \\ &\le H_v(A,B) + 2R(A\nabla B - A\#B) - r_0\left(A\nabla B + A\#B - 2H_{\frac{1}{4}}(A,B)\right) \\ &\le H_v(A,B) + 2R(A\nabla B - A\#B) \\ &\le A\nabla B + 2R(A\nabla B - A\#B). \end{aligned}$$

The inequalities (3.1) and (3.3) improve the inequalities (1.11). But neither the bounds of $A\nabla B$ in (3.1) and (3.3) nor the corresponding bounds in (1.13) are uniformly better than the other.

Theorem 3.3. *Let $A, B \in \mathcal{B}^{++}(\mathcal{H})$ and $v \in (0,1)$. Then*

$$\begin{aligned} 2r^2(A\nabla B - A\#B) + H_v(A,B) + r_0\left(A\nabla B + A\#B - 2H_{\frac{1}{4}}(A,B)\right) \\ \le [1 - 2v(1-v)]\, A\nabla B + 2v(1-v)A\#B \\ \le A\nabla B. \end{aligned} \tag{3.5}$$

Equality holds if and only if $A = B$ or equivalently $A^{-\frac{1}{2}}BA^{-\frac{1}{2}} = I$.

Proof. By the inequalities (18) and (19) in [10], replacing a^2 and b^2 by a and b, respectively, in each inequality, we obtain

$$\begin{aligned} r_0\left(\sqrt{ab} + a - 2a^{\frac{3}{4}}b^{\frac{1}{4}}\right) + 2v^2\left(\frac{a+b}{2} - \sqrt{ab}\right) + a^{1-v}b^v \\ \le (1-v)^2a + v^2b + 2v(1-v)\sqrt{ab}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} r_0\left(\sqrt{ab} + b - 2b^{\frac{3}{4}}a^{\frac{1}{4}}\right) + 2(1-v)^2\left(\frac{a+b}{2} - \sqrt{ab}\right) + a^{1-v}b^v \\ \le (1-v)^2a + v^2b + 2v(1-v)\sqrt{ab}. \end{aligned} \tag{3.7}$$

Replacing a by b and b by a in (3.6) and (3.7), then combining (3.6) and (3.7) we have

$$\begin{aligned} r_0\left(\sqrt{ab}+\frac{a+b}{2}-2H_{\frac{1}{4}(a,b)}\right)+2v^2\left(\frac{a+b}{2}-\sqrt{ab}\right)+H_v(a,b) \\ \leq \left[(1-v)^2+v^2\right]\frac{a+b}{2}+2v(1-v)\sqrt{ab}, \\ r_0\left(\sqrt{ab}+\frac{a+b}{2}-2H_{\frac{1}{4}(a,b)}\right)+2(1-v)^2\left(\frac{a+b}{2}-\sqrt{ab}\right)+H_v(a,b) \\ \leq \left[(1-v)^2+v^2\right]\frac{a+b}{2}+2v(1-v)\sqrt{ab}. \end{aligned}$$

The rest of the proof is similar to that of Theorem 3.1, so we omit it. □

Theorem 3.4. *Let $A, B \in \mathcal{B}^{++}(\mathcal{H})$ and $v \in (0,1)$. Then*

$$\begin{aligned} 2R^2(A\nabla B - A\#B) + H_v(A,B) - r_0\left(A\nabla B + A\#B - 2H_{\frac{1}{4}}(A,B)\right) \\ \geq \left[1-2v(1-v)\right]A\nabla B + 2v(1-v)A\#B \\ \geq A\#B. \end{aligned} \tag{3.8}$$

Equality holds if and only if $A = B$ or equivalently $A^{-\frac{1}{2}}BA^{-\frac{1}{2}} = I$.

Proof. By the inequality (20) and (21) in [10], the proof of the inequality (3.8) can be completed by an argument similar to that used in the proof of Theorem 3.3. □

Remark 3.2. Combining (3.5) with (3.8), we obtain

$$\begin{aligned} A\#B &\leq H_v(A,B) \\ &\leq 2r^2(A\nabla B - A\#B) + H_v(A,B) \\ &\leq 2r^2(A\nabla B - A\#B) + H_v(A,B) + r_0\left(A\nabla B + A\#B - 2H_{\frac{1}{4}}(A,B)\right) \\ &\leq \left[1-2v(1-v)\right]A\nabla B + 2v(1-v)A\#B \\ &\leq 2R^2(A\nabla B - A\#B) + H_v(A,B) - r_0\left(A\nabla B + A\#B - 2H_{\frac{1}{4}}(A,B)\right) \\ &\leq 2R^2(A\nabla B - A\#B) + H_v(A,B) \\ &\leq 2R^2(A\nabla B - A\#B) + A\nabla B, \end{aligned}$$

which is an improvement of $A\#B \leq H_v(A,B) \leq A\nabla B$.

4. Matrix version of the Heinz norm inequality

In this section, we will discuss the improved Heinz inequality for the Hilbert-Schmidt norm.

Let $M_n(\mathbb{C})$ denote the algebra of all $n\times n$ complex matrices. The Hilbert-Schmidt norm of $A=[a_{ij}]\in M_n(\mathbb{C})$ is defined by $\|A\|_F^2 = \left(\sum\limits_{i,j=1}^{n}|a_{ij}|^2\right)^{1/2} = \left(\sum\limits_{i=1}^{n}s_i^2(A)\right)^{1/2}$ (see [4, p.341-342]), where $s_i(A), i=1,2,\cdots,n$ denote the singular values of A. It is well-known that each unitarily invariant norm is a symmetric guage function of singular values [1, p.91], so the Hilbert-Schmidt norm is unitarily invariant in the sense that $\|UAV\|_F^2 = \|A\|_F^2$ for all unitary matrices $U, V \in M_n(\mathbb{C})$. The Schur product of two matrices $A, B \in M_n(\mathbb{C})$ is the entrywise product and denoted by $A \circ B$.

Based on the refined Heinz inequality (2.3) in [5] and the reverse Heinz inequality (2.4) in [6], Kittaneh and Manasrah have showed that if $A, B, X \in M_n(\mathrm{C})$ with A and B positive semidefinite matrices and $v \in [0,1]$, then

$$\|AX + XB\|_F^2 \geq \left\|A^{1-v}XB^v + A^vXB^{1-v}\right\|_F^2 + 2r\,\|AX - XB\|_F^2\,, \tag{4.1}$$

$$\|AX + XB\|_F^2 \leq \left\|A^{1-v}XB^v + A^vXB^{1-v}\right\|_F^2 + 2R\,\|AX - XB\|_F^2\,. \tag{4.2}$$

Now, we derive the following two theorems which improve the inequalities (4.1) and (4.2).

Theorem 4.1. *Let $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive semidefinite and $0 < v < 1$. Then*

$$\begin{aligned} 2r\,\|AX - XB\|_F^2 + r_0\left(\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}} - AX\right\|_F^2 + \left\|A^{\frac{1}{2}}XB^{\frac{1}{2}} - XB\right\|_F^2\right) \\ \leq \|AX + XB\|_F^2 - \left\|A^vXB^{1-v} + A^{1-v}XB^v\right\|_F^2. \end{aligned} \tag{4.3}$$

Proof. Since A and B are positive semidefinite, it follows by the spectral theorem that there exist unitary matrices $U, V \in M_n(\mathrm{C})$ such that

$$A = U\Lambda_1U^*, B = V\Lambda_2V^*,$$

where $\Lambda_1 = \operatorname{diag}(\mu_1, \mu_2, \cdots, \mu_n)$, $\Lambda_2 = \operatorname{diag}(\nu_1, \nu_2, \cdots, \nu_n)$, $\mu_i, \nu_i \geq 0$, $i = 1, 2, \cdots, n$.

Let $Y = U^*XV = [y_{ij}]$, then

$$\begin{aligned} A^vXB^{1-v} + A^{1-v}XB^v &= U\left(\Lambda_1^vY\Lambda_2^{1-v} + \Lambda_1^{1-v}Y\Lambda_2^v\right)V^* \\ &= U\left[\left(\mu_i^{1-v}\nu_j^v + \mu_i^v\nu_j^{1-v}\right)\circ Y\right]V^*, \\ A^{\frac{1}{2}}XB^{\frac{1}{2}} - AX &= U\left[\left((\mu_i\nu_j)^{\frac{1}{2}} - \mu_i\right)\circ Y\right]V^*, \\ A^{\frac{1}{2}}XB^{\frac{1}{2}} - XB &= U\left[\left((\mu_i\nu_j)^{\frac{1}{2}} - \nu_i\right)\circ Y\right]V^*, \\ AX \pm XB &= U\left[(\mu_i \pm \nu_j)\circ Y\right]V^*. \end{aligned}$$

Utilizing the inequality (2.4) and the unitary invariance of the Hilbert-Schmidt norm, if $0 < v \leq \frac{1}{2}$, we have

$$\begin{aligned} &\|AX + XB\|_F^2 - \left\|A^vXB^{1-v} + A^{1-v}XB^v\right\|_F^2 \\ &= \sum_{i,j=1}^n (\mu_i + \nu_j)^2\,|y_{ij}|^2 - \sum_{i,j=1}^n \left(\mu_i^{1-v}\nu_j^v + \mu_i^v\nu_j^{1-v}\right)^2|y_{ij}|^2 \\ &= \sum_{i,j=1}^n \left[(\mu_i + \nu_j)^2 - \left(\mu_i^{1-v}\nu_j^v + \mu_i^v\nu_j^{1-v}\right)^2\right]|y_{ij}|^2 \end{aligned}$$

$$\geq \sum_{i,j=1}^{n} \left[2v\left(\mu_i - \nu_j\right)^2 + r_0\left(((\mu_i\nu_j)^{\frac{1}{2}} - \mu_i)^2 + ((\mu_i\nu_j)^{\frac{1}{2}} - \nu_i)^2\right)\right]|y_{ij}|^2$$
$$= 2v\sum_{i,j=1}^{n}\left(\mu_i - \nu_j\right)^2|y_{ij}|^2 + r_0\sum_{i,j=1}^{n}\left(((\mu_i\nu_j)^{\frac{1}{2}} - \mu_i)^2 + ((\mu_i\nu_j)^{\frac{1}{2}} - \nu_i)^2\right)|y_{ij}|^2$$
$$= 2v\left\|AX - XB\right\|_F^2 + r_0\left(\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}} - AX\right\|_F^2 + \left\|A^{\frac{1}{2}}XB^{\frac{1}{2}} - XB\right\|_F^2\right).$$

Similarly, if $\frac{1}{2} < v < 1$, we can deduce

$$\left\|AX + XB\right\|_F^2 - \left\|A^vXB^{1-v} + A^{1-v}XB^v\right\|_F^2$$
$$\geq 2(1-v)\left\|AX - XB\right\|_F^2 + r_0\left(\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}} - AX\right\|_F^2 + \left\|A^{\frac{1}{2}}XB^{\frac{1}{2}} - XB\right\|_F^2\right).$$

So we conclude that the desired result hold for $0 < v < 1$. □

Theorem 4.2. *Let $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive semidefinite and $0 < v < 1$. Then*

$$\begin{aligned} 2R\left\|AX - XB\right\|_F^2 - r_0\left(\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}} - AX\right\|_F^2 + \left\|A^{\frac{1}{2}}XB^{\frac{1}{2}} - XB\right\|_F^2\right) \\ \geq \left\|AX + XB\right\|_F^2 - \left\|A^vXB^{1-v} + A^{1-v}XB^v\right\|_F^2. \end{aligned} \tag{4.4}$$

Proof. By Lemma 2.4, the proof of the inequality (4.4) can be completed by an argument similar to that used in the proof of Theorem 4.1. □

Remark 4.1. Combining (4.3) with (4.4), we obtain

$$\begin{aligned} 4\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_F^2 \leq & \left\|A^vXB^{1-v} + A^{1-v}XB^v\right\|_F^2 \\ \leq & 2r\left\|AX - XB\right\|_F^2 + \left\|A^vXB^{1-v} + A^{1-v}XB^v\right\|_F^2 \\ \leq & 2r\left\|AX - XB\right\|_F^2 + \left\|A^vXB^{1-v} + A^{1-v}XB^v\right\|_F^2 \\ & + r_0\left(\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}} - AX\right\|_F^2 + \left\|A^{\frac{1}{2}}XB^{\frac{1}{2}} - XB\right\|_F^2\right) \\ \leq & \left\|AX + XB\right\|_F^2 \\ \leq & 2R\left\|AX - XB\right\|_F^2 + \left\|A^vXB^{1-v} + A^{1-v}XB^v\right\|_F^2 \\ & - r_0\left(\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}} - AX\right\|_F^2 + \left\|A^{\frac{1}{2}}XB^{\frac{1}{2}} - XB\right\|_F^2\right) \\ \leq & 2R\left\|AX - XB\right\|_F^2 + \left\|A^vXB^{1-v} + A^{1-v}XB^v\right\|_F^2 \\ \leq & 2R\left\|AX - XB\right\|_F^2 + \left\|AX + XB\right\|_F^2, \end{aligned}$$

which is an improvement of $4\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_F^2 \leq \left\|A^vXB^{1-v} + A^{1-v}XB^v\right\|_F^2 \leq \left\|AX + XB\right\|_F^2$ (see [2]).

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[1]College of Mathematics and Physics,
Chongqing University of Science and Technology,
Chongqing, 401331, P. R.China
E-mail address: liaowenshi@gmail.com,
E-mail address: longpujun@gmail.com