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FRACTIONAL HERMITE-HADAMARD TYPE INEQUALITIES FOR CO-ORDINATED MT-CONVEX FUNCTIONS

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Abstract. In this paper we establish a new fractional identity involving a function of two independent variables, and then we derive some fractional Hermite-Hadamard type integral inequalities for functions whose modulus of the mixed derivatives are co-ordinated MT-convex.

1. Introduction

Let f be a convex function, then we have

$$f\left(\tfrac{n+m}{2}\right) \leq \tfrac{1}{m-n}\int\limits_{n}^{m} f(x)dx \leq \tfrac{f(n)+f(m)}{2}, \tag{1.1}$$

holds in every finite interval $[n, m]$. In the case where f is concave, then (1.1) holds in the reverse direction. The above inequality is well-known as Hermite-Hadamard integral inequality (see [4]).

In [15] Dragomir established the bi-dimensional analogue of (1.1) given by

$$\begin{aligned} f\left(\tfrac{n+m}{2}, \tfrac{r+k}{2}\right) &\leq \tfrac{1}{2}\left(\tfrac{1}{m-n}\int\limits_{n}^{m} f(x, \tfrac{r+k}{2})dx + \tfrac{1}{k-r}\int\limits_{r}^{k} f(\tfrac{n+m}{2}, y)dy\right) \\ &\leq \tfrac{1}{(m-n)(k-r)}\int\limits_{n}^{m}\int\limits_{r}^{k} f(x, y)dydx \\ &\leq \tfrac{1}{4}\left(\tfrac{1}{m-n}\int\limits_{n}^{m} f(x, r)dx + \tfrac{1}{m-n}\int\limits_{n}^{m} f(x, k)dx\right. \end{aligned}$$

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$$+\frac{1}{k-r}\int_r^k f(n,y)dy+\frac{1}{k-r}\int_r^k f(m,y)dy\Bigg)$$
$$\leq \frac{f(n,r)+f(n,k)+f(m,r)+f(m,k)}{4}. \tag{1.2}$$

The inequalities (1.2) have attracted many researchers. Various variants and refinements of (1.2) have appeared in the literature, see [2, 3, 5–14] and references therein.

Sarikaya [12] gave the following fractional Hermite-Hadamard for co-ordinated convex functions.

Theorem 1.1. *Let $f:[n,m]\times[r,k]\subset\mathbb{R}^2\to\mathbb{R}$ be a function with $n<m, r<k$ and $\frac{\partial^2 f}{\partial s\partial t}\in L([n,m]\times[r,k])$. If $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ is co-ordinated convex function, then we have*

$$\Bigg|\frac{f(n,r)+f(n,k)+f(m,r)+f(m,k)}{4}+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(m-n)^{\alpha}(k-r)^{\beta}}\Big(J^{\alpha,\beta}_{n^+,r^+}f(m,k)+J^{\alpha,\beta}_{n^+,k^-}f(m,r)$$
$$+J^{\alpha,\beta}_{m^-,r^+}f(n,k)+J^{\alpha,\beta}_{m^-,k^-}f(n,r)-A\Bigg|$$
$$\leq \frac{(m-n)(k-r)}{4(\alpha+1)(\beta+1)}\left(\left|\frac{\partial^2 f}{\partial s\partial t}(n,r)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(n,k)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(m,r)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(m,k)\right|\right),$$

where

$$A = \frac{\Gamma(\beta+1)}{4(k-r)^{\beta}}\left(J^{\beta}_{r^+}f(n,k)+J^{\beta}_{r^+}f(m,k)+J^{\beta}_{k^-}f(n,r)+J^{\beta}_{k^-}f(m,r)\right)$$
$$+\frac{\Gamma(\alpha+1)}{4(m-n)^{\alpha}}\left(J^{\alpha}_{n^+}f(m,r)+J^{\alpha}_{n^+}f(m,k)+J^{\alpha}_{m^-}f(n,r)+J^{\alpha}_{m^-}f(n,k)\right). \tag{1.3}$$

Theorem 1.2. *Let $f:[n,m]\times[r,k]\subset\mathbb{R}^2\to\mathbb{R}$ be a function with $n<m, r<k$ and $\frac{\partial^2 f}{\partial s\partial t}\in L([n,m]\times[r,k])$. If $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$, $q>1$ is co-ordinated convex function, then we have*

$$\Bigg|\frac{f(n,r)+f(n,k)+f(m,r)+f(m,k)}{4}+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(m-n)^{\alpha}(k-r)^{\beta}}\Big(J^{\alpha,\beta}_{n^+,r^+}f(m,k)+J^{\alpha,\beta}_{n^+,k^-}f(m,r)$$
$$+J^{\alpha,\beta}_{m^-,r^+}f(n,k)+J^{\alpha,\beta}_{m^-,k^-}f(n,r)-A\Bigg|$$
$$\leq \frac{(m-n)(k-r)}{((\alpha p+1)(\beta p+1))^{\frac{1}{p}}}\left(\frac{1}{4}\right)^{\frac{1}{q}}$$
$$\times\left(\left|\frac{\partial^2 f}{\partial s\partial t}(n,r)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(n,k)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(m,r)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(m,k)\right|^q\right)^{\frac{1}{q}},$$

where A is defined by (1.3) and $\frac{1}{p}+\frac{1}{q}=1$.

In this paper we establish a new fractional identity involving a function of two independent variables, and then we derive some fractional Hermite-Hadamard type integral inequalities for functions whose modulus of the mixed derivatives are co-ordinated MT-convex.

2. Preliminaries

In this section we recall some definitions that are well known in the literature, and assume that $\Lambda := [n,m]\times[r,k]$ be an interval in $\mathbb{R}^2$ with $n<m$ and $r<k$.

Definition 2.1. [9] A function $g:\Lambda\rightarrow\mathbb{R}$ is said to be co-ordinated MT-convex on Λ, if

$$\begin{aligned} f\left(tu+(1-t)v,\lambda w+(1-\lambda)x\right) &\leq \frac{\sqrt{t\lambda}}{4\sqrt{(1-t)(1-\lambda)}}f(u,w)+\frac{\sqrt{t(1-\lambda)}}{4\sqrt{(1-t)\lambda}}f(u,x) \\ &\quad +\frac{\sqrt{(1-t)\lambda}}{4\sqrt{t(1-\lambda)}}f(v,w)+\frac{\sqrt{(1-t)(1-\lambda)}}{4\sqrt{t\lambda}}f(v,x) \end{aligned}$$

holds for all $t,\lambda\in[0,1]$ and $(u,w),(u,x),(v,w),(v,x)\in\Lambda$.

Definition 2.2. [1] The Riemann-Liouville integrals $J_{a+}^{\alpha}g$ and $J_{b-}^{\alpha}g$ of order $\alpha>0$ with $a\geq 0$ where $g\in L([a,b])$ are defined by

$$\begin{aligned} J_{a+}^{\alpha}g(u) &= \frac{1}{\Gamma(\alpha)}\int_{a}^{u}(u-t)^{\alpha-1}g(t)dt, \qquad u>a \\ J_{b-}^{\alpha}g(u) &= \frac{1}{\Gamma(\alpha)}\int_{u}^{b}(t-u)^{\alpha-1}g(t)dt, \qquad b>u \end{aligned}$$

respectively and $\Gamma(\alpha)=\int_{0}^{\infty}e^{-t}t^{\alpha-1}dt$, is the Gamma function and $J_{a+}^{0}g(u)=J_{b-}^{0}g(u)=g(u)$.

Definition 2.3. [1] Let $f\in L(\Lambda)$. The Riemann–Liouville integrals $J_{n^{+},r^{+}}^{\alpha,\beta}$, $J_{n^{+},k^{-}}^{\alpha,\beta}$, $J_{m^{-},r^{+}}^{\alpha,\beta}$, and $J_{m^{-},k^{-}}^{\alpha,\beta}$ of order $\alpha,\beta>0$ with $n,r\geq 0$, $n<m$ and $r<k$ are defined by

$$J_{n^{+},r^{+}}^{\alpha,\beta}f(m,k)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{n}^{m}\int_{r}^{k}(m-x)^{\alpha-1}(k-y)^{\beta-1}f(x,y)\,dydx, \tag{2.1}$$

$$J_{n^{+},k^{-}}^{\alpha,\beta}f(m,r)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{n}^{m}\int_{r}^{k}(m-x)^{\alpha-1}(y-r)^{\beta-1}f(x,y)\,dydx, \tag{2.2}$$

$$J_{m^{-},r^{+}}^{\alpha,\beta}f(n,k)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{n}^{m}\int_{r}^{k}(x-n)^{\alpha-1}(k-y)^{\beta-1}f(x,y)\,dydx, \tag{2.3}$$

$$J_{m^{-},k^{-}}^{\alpha,\beta}f(n,r)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{n}^{m}\int_{r}^{k}(x-n)^{\alpha-1}(y-r)^{\beta-1}f(x,y)\,dydx, \tag{2.4}$$

where Γ is the Gamma function, and

$$J_{n^{+},r^{+}}^{0,0}f(m,k)=J_{n^{+},k^{-}}^{0,0}f(m,r)=J_{m^{-},r^{+}}^{0,0}f(n,k)=J_{m^{-},k^{-}}^{0,0}f(n,r)=f(x,y).$$

Definition 2.4. [12] Let $f\in L(\Lambda)$. The Riemann–Liouville integrals $J_{m^{-}}^{\alpha}f(n,r)$, $J_{n^{+}}^{\alpha}f(m,r)$, $J_{k^{-}}^{\beta}f(n,r)$, and $J_{r^{+}}^{\alpha}f(n,k)$ of order $\alpha,\beta>0$ with $n,r\geq 0$, $n<m$, and $r<k$ are defined by

$$J_{m^{-}}^{\alpha}f(n,r)=\frac{1}{\Gamma(\alpha)}\int_{n}^{m}(x-n)^{\alpha-1}f(x,r)\,dx, \tag{2.5}$$

$$J_{n^+}^{\alpha} f(m,r) = \frac{1}{\Gamma(\alpha)} \int_n^m (m-x)^{\alpha-1} f(x,r)\,dx, \tag{2.6}$$

$$J_{k^-}^{\beta} f(n,r) = \frac{1}{\Gamma(\beta)} \int_r^k (y-r)^{\beta-1} f(n,y)\,dy, \tag{2.7}$$

$$J_{r^+}^{\alpha} f(n,k) = \frac{1}{\Gamma(\beta)} \int_r^k (k-y)^{\beta-1} f(n,y)\,dy, \tag{2.8}$$

where Γ is the Gamma function.

3. Main results

Lemma 3.1. *Let $f : \Lambda \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a differentiable mapping on $\Lambda = [n,m] \times [r,k]$ with $n < m$ and $r < k$. If $\frac{\partial^2 f}{\partial t \partial \lambda} \in L(\Lambda)$, then the following fractional equality holds*

$$\begin{aligned}
& f\left(\tfrac{n+m}{2}, \tfrac{r+k}{2}\right) - \frac{f\left(n,\frac{r+k}{2}\right)+f\left(\frac{n+m}{2},r\right)+f\left(m,\frac{r+k}{2}\right)+f\left(\frac{n+m}{2},k\right)}{2} + A \\
& -\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(m-n)^{\alpha}(k-r)^{\beta}} \\
& \times \left(J_{n^+,r^+}^{\alpha,\beta} f(m,k) + J_{m^-,r^+}^{\alpha,\beta} f(n,k) + J_{n^+,k^-}^{\alpha,\beta} f(m,r) + J_{m^-,k^-}^{\alpha,\beta} f(n,r)\right) \\
= \; & \tfrac{(m-n)(k-r)}{4} \left(\int_0^1 \int_0^1 lh \tfrac{\partial^2 f}{\partial t \partial \lambda}(tn + (1-t)m, \lambda r + (1-\lambda)k)\,dt d\lambda \right. \\
& - \int_0^1 \int_0^1 \left((1-t)^{\alpha} - t^{\alpha}\right)\left((1-\lambda)^{\beta} - \lambda^{\beta}\right) \\
& \left. \times \tfrac{\partial^2 f}{\partial t \partial \lambda}(tn + (1-t)m, \lambda r + (1-\lambda)k)\,dt d\lambda \right),
\end{aligned} \tag{3.1}$$

where

$$h = \begin{cases} 1 \text{ if } 0 \le t < \frac{1}{2}, \\ -1 \text{ if } \frac{1}{2} \le t < 1, \end{cases} \tag{3.2}$$

$$l = \begin{cases} 1 \text{ if } 0 \le \lambda < \frac{1}{2}, \\ -1 \text{ if } \frac{1}{2} \le \lambda < 1, \end{cases} \tag{3.3}$$

and

$$\begin{aligned}
A = \; & \tfrac{\Gamma(\beta+1)}{4(k-r)^{\beta}} \left(J_{k^-}^{\beta} f(n,r) + J_{k^-}^{\beta} f(m,r) + J_{r^+}^{\alpha} f(n,k) + J_{r^+}^{\alpha} f(m,k)\right) \\
& + \tfrac{\Gamma(\alpha+1)}{4(m-n)^{\alpha}} \left(J_{m^-}^{\alpha} f(n,r) + J_{m^-}^{\alpha} f(n,k) + J_{n^+}^{\alpha} f(m,r) + J_{n^+}^{\alpha} f(m,k)\right).
\end{aligned} \tag{3.4}$$

Proof. Let

$$I = \tfrac{(m-n)(k-r)}{4}(I_1 - I_2), \tag{3.5}$$

where

$$I_1=\int_0^1\int_0^1 lh\frac{\partial^2 f}{\partial t\partial\lambda}\left(tn+(1-t)\,m,\lambda r+(1-\lambda)\,k\right)dtd\lambda,$$

and

$$\begin{aligned}I_2 &= \int_0^1\int_0^1\left((1-t)^{\alpha}-t^{\alpha}\right)\left((1-\lambda)^{\beta}-\lambda^{\beta}\right)\\&\quad\times\frac{\partial^2 f}{\partial t\partial\lambda}\left(tn+(1-t)\,m,\lambda r+(1-\lambda)\,k\right)dtd\lambda.\end{aligned}$$

Clearly, we have

$$\begin{aligned}I_1 &= \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\frac{\partial^2 f}{\partial t\partial\lambda}\left(tn+(1-t)\,m,\lambda r+(1-\lambda)\,k\right)dtd\lambda\\&\quad-\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1}\frac{\partial^2 f}{\partial t\partial\lambda}\left(tn+(1-t)\,m,\lambda r+(1-\lambda)\,k\right)dtd\lambda\\&\quad-\int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}}\frac{\partial^2 f}{\partial t\partial\lambda}\left(tn+(1-t)\,m,\lambda r+(1-\lambda)\,k\right)dtd\lambda\\&\quad+\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}\frac{\partial^2 f}{\partial t\partial\lambda}\left(tn+(1-t)\,m,\lambda r+(1-\lambda)\,k\right)dtd\lambda\\&= \frac{1}{(m-n)(k-r)}\left(\left(f\left(\frac{n+m}{2},\frac{r+k}{2}\right)-f\left(m,\frac{r+k}{2}\right)-f\left(\frac{n+m}{2},k\right)+f\left(m,k\right)\right)\right.\\&\quad-f\left(n,\frac{r+k}{2}\right)+f\left(\frac{n+m}{2},\frac{r+k}{2}\right)+f\left(n,k\right)-f\left(\frac{n+m}{2},k\right)\\&\quad-f\left(\frac{n+m}{2},r\right)+f\left(m,r\right)+f\left(\frac{n+m}{2},\frac{r+k}{2}\right)-f\left(m,\frac{r+k}{2}\right)\\&\quad\left.+f\left(n,r\right)-f\left(\frac{n+m}{2},r\right)-f\left(n,\frac{r+k}{2}\right)+f\left(\frac{n+m}{2},\frac{r+k}{2}\right)\right)\\&= \frac{4}{(m-n)(k-r)}\left(\left(f\left(\frac{n+m}{2},\frac{r+k}{2}\right)\right)+\frac{f(m,k)+f(n,k)+f(m,r)+f(n,r)}{4}\right.\\&\quad\left.-\frac{f\left(n,\frac{r+k}{2}\right)+f\left(\frac{n+m}{2},r\right)+f\left(m,\frac{r+k}{2}\right)+f\left(\frac{n+m}{2},k\right)}{2}\right).\end{aligned}\tag{3.6}$$

Now, by integration by parts, I_2 gives

$$\begin{aligned}I_2 &= \int_0^1\left((1-\lambda)^{\beta}-\lambda^{\beta}\right)\\&\quad\times\left(\int_0^1\left((1-t)^{\alpha}-t^{\alpha}\right)\frac{\partial^2 f}{\partial t\partial\lambda}\left(tn+(1-t)\,m,\lambda r+(1-\lambda)\,k\right)dt\right)d\lambda\end{aligned}$$

$$\begin{aligned}
&= \tfrac{1}{(m-n)(k-r)}\left(f(n,r)+f(n,k)+f(m,r)+f(m,k)\right) \\
&-\tfrac{\beta}{(m-n)(k-r)}\left(\int_0^1(1-\lambda)^{\beta-1}f(n,\lambda r+(1-\lambda)k)\,d\lambda\right. \\
&+\int_0^1\lambda^{\beta-1}f(n,\lambda r+(1-\lambda)k)\,d\lambda+\int_0^1\lambda^{\beta-1}f(m,\lambda r+(1-\lambda)k)\,d\lambda \\
&\left.+\int_0^1(1-\lambda)^{\beta-1}f(m,\lambda r+(1-\lambda)k)\,d\lambda\right) \\
&-\tfrac{\alpha}{(m-n)(k-r)}\left(\int_0^1(1-t)^{\alpha-1}f(tn+(1-t)m,r)\,dt\right. \\
&+\int_0^1t^{\alpha-1}f(tn+(1-t)m,r)\,dt+\int_0^1t^{\alpha-1}f(tn+(1-t)m,k)\,dt \\
&\left.+\int_0^1(1-t)^{\alpha-1}f(tn+(1-t)m,k)\,dt\right) \\
&+\tfrac{\alpha\beta}{(m-n)(k-r)}\left(\int_0^1\int_0^1t^{\alpha-1}\lambda^{\beta-1}f(tn+(1-t)m,\lambda r+(1-\lambda)k)\,d\lambda dt\right. \\
&+\int_0^1\int_0^1(1-t)^{\alpha-1}\lambda^{\beta-1}f(tn+(1-t)m,\lambda r+(1-\lambda)k)\,d\lambda dt \\
&+\int_0^1\int_0^1t^{\alpha-1}(1-\lambda)^{\beta-1}f(tn+(1-t)m,\lambda r+(1-\lambda)k)\,d\lambda dt \\
&\left.+\int_0^1\int_0^1(1-t)^{\alpha-1}(1-\lambda)^{\beta-1}f(tn+(1-t)m,\lambda r+(1-\lambda)k)\,d\lambda dt\right).
\end{aligned} \tag{3.7}$$

Substituting (3.6) and (3.7) in (3.5), and putting $x=tn+(1-t)m$ and $y=\lambda r+(1-\lambda)k$, we get

$$\begin{aligned}
I &= f\left(\tfrac{n+m}{2},\tfrac{r+k}{2}\right)-\frac{f\left(n,\frac{r+k}{2}\right)+f\left(\frac{n+m}{2},r\right)+f\left(m,\frac{r+k}{2}\right)+f\left(\frac{n+m}{2},k\right)}{2} \\
&+\tfrac{\beta}{4(k-r)^{\beta}}\left(\int_r^k(y-r)^{\beta-1}f(n,y)\,dy+\int_r^k(y-r)^{\beta-1}f(m,y)\,dy\right.
\end{aligned}$$

$$+\int_{r}^{k}(k-y)^{\beta-1}f(n,y)\,dy+\int_{r}^{k}(k-y)^{\beta-1}f(m,y)\,dy\Bigg)$$
$$+\tfrac{\alpha}{4(m-n)^{\alpha}}\Bigg(\int_{n}^{m}(x-n)^{\alpha-1}f(x,r)\,dx+\int_{n}^{m}(x-n)^{\alpha-1}f(x,k)\,dx$$
$$+\int_{n}^{m}(m-x)^{\alpha-1}f(x,r)\,dx+\int_{n}^{m}(m-x)^{\alpha-1}f(x,k)\,dx\Bigg)$$
$$-\tfrac{\alpha\beta}{4(m-n)^{\alpha}(k-r)^{\beta}}\Bigg(\int_{n}^{m}\int_{r}^{k}(m-x)^{\alpha-1}(k-y)^{\beta-1}f(x,y)\,dydx$$
$$+\int_{n}^{m}\int_{r}^{k}(x-n)^{\alpha-1}(k-y)^{\beta-1}f(x,y)\,dydx$$
$$+\int_{n}^{m}\int_{r}^{k}(m-x)^{\alpha-1}(y-r)^{\beta-1}f(x,y)\,dydx$$
$$+\int_{n}^{m}\int_{r}^{k}(x-n)^{\alpha-1}(y-r)^{\beta-1}f(x,y)\,dydx\Bigg). \tag{3.8}$$

Using (2.1)-(2.8) in (3.8), we obtain the desired result. □

Theorem 3.1. *Let $f:\Lambda\to\mathbb{R}$ be a partially differentiable function on Δ. If $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|$ is co-ordinated MT-convex function on Λ, then the following fractional inequality holds*

$$\Bigg|f\left(\tfrac{n+m}{2},\tfrac{r+k}{2}\right)-\frac{f\left(n,\frac{r+k}{2}\right)+f\left(\frac{n+m}{2},r\right)+f\left(m,\frac{r+k}{2}\right)+f\left(\frac{n+m}{2},k\right)}{2}+A$$
$$-\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(m-n)^{\alpha}(k-r)^{\beta}}$$
$$\times\left(J^{\alpha,\beta}_{n^+,r^+}f(m,k)+J^{\alpha,\beta}_{m^-,r^+}f(n,k)+J^{\alpha,\beta}_{n^+,k^-}f(m,r)+J^{\alpha,\beta}_{m^-,k^-}f(n,r)\right)\Bigg|$$
$$\leq\tfrac{(m-n)(k-r)}{16}\left(\frac{\pi^2}{4}+B\left(\frac{1}{2},\alpha+\frac{1}{2}\right)B\left(\frac{1}{2},\beta+\frac{1}{2}\right)\right)$$
$$\times\left(\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(n,r)\right|+\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(n,k)\right|+\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(m,r)\right|+\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(m,k)\right|\right),$$

where A is defined as in (3.4) and $B(.,.)$ is the beta function.

Proof. From Lemma 3.1, and properties of modulus, we have

$$\begin{aligned}
&\Bigg| f\left(\tfrac{n+m}{2},\tfrac{r+k}{2}\right)-\frac{f\left(n,\frac{r+k}{2}\right)+f\left(\frac{n+m}{2},r\right)+f\left(m,\frac{r+k}{2}\right)+f\left(\frac{n+m}{2},k\right)}{2}+A\\
&-\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(m-n)^{\alpha}(k-r)^{\beta}}\\
&\times\left(J^{\alpha,\beta}_{n^+,r^+}f(m,k)+J^{\alpha,\beta}_{m^-,r^+}f(n,k)+J^{\alpha,\beta}_{n^+,k^-}f(m,r)+J^{\alpha,\beta}_{m^-,k^-}f(n,r)\right)\Bigg|\\
\leq\ &\tfrac{(m-n)(k-r)}{4}\Bigg(\int_0^1\int_0^1\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(tn+(1-t)m,\lambda r+(1-\lambda)k\right)\right|dtd\lambda\\
&+\int_0^1\int_0^1(1-t)^{\alpha}(1-\lambda)^{\beta}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(tn+(1-t)m,\lambda r+(1-\lambda)k\right)\right|dtd\lambda\\
&+\int_0^1\int_0^1 t^{\alpha}(1-\lambda)^{\beta}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(tn+(1-t)m,\lambda r+(1-\lambda)k\right)\right|dtd\lambda\\
&+\int_0^1\int_0^1(1-t)^{\alpha}\lambda^{\beta}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(tn+(1-t)m,\lambda r+(1-\lambda)k\right)\right|dtd\lambda\\
&+\int_0^1\int_0^1 t^{\alpha}\lambda^{\beta}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(tn+(1-t)m,\lambda r+(1-\lambda)k\right)\right|dtd\lambda. \qquad (3.9)
\end{aligned}$$

Since $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|$ is co-ordinated MT-convex, we have

$$\begin{aligned}
&\Bigg| f\left(\tfrac{n+m}{2},\tfrac{r+k}{2}\right)-\frac{f\left(n,\frac{r+k}{2}\right)+f\left(\frac{n+m}{2},r\right)+f\left(m,\frac{r+k}{2}\right)+f\left(\frac{n+m}{2},k\right)}{2}+A\\
&-\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(m-n)^{\alpha}(k-r)^{\beta}}\\
&\times\left(J^{\alpha,\beta}_{n^+,r^+}f(m,k)+J^{\alpha,\beta}_{m^-,r^+}f(n,k)+J^{\alpha,\beta}_{n^+,k^-}f(m,r)+J^{\alpha,\beta}_{m^-,k^-}f(n,r)\right)\Bigg|\\
\leq\ &\tfrac{(m-n)(k-r)}{4}\Bigg(\int_0^1\int_0^1\Bigg(\tfrac{\sqrt{t\lambda}}{4\sqrt{(1-t)(1-\lambda)}}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(n,r)\right|+\tfrac{\sqrt{t(1-\lambda)}}{4\sqrt{(1-t)\lambda}}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(n,k)\right|\\
&+\tfrac{\sqrt{(1-t)\lambda}}{4\sqrt{t(1-\lambda)}}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(m,r)\right|+\tfrac{\sqrt{(1-t)(1-\lambda)}}{4\sqrt{t\lambda}}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(m,k)\right|\Bigg)dtd\lambda\\
&+\int_0^1\int_0^1(1-t)^{\alpha}(1-\lambda)^{\beta}\Bigg(\tfrac{\sqrt{t\lambda}}{4\sqrt{(1-t)(1-\lambda)}}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(n,r)\right|+\tfrac{\sqrt{t(1-\lambda)}}{4\sqrt{(1-t)\lambda}}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(n,k)\right|
\end{aligned}$$

$$+\frac{\sqrt{(1-t)\lambda}}{4\sqrt{t(1-\lambda)}}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(m,r)\right|+\frac{\sqrt{(1-t)(1-\lambda)}}{4\sqrt{t\lambda}}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(m,k)\right|\Bigg)dtd\lambda$$

$$+\int_0^1\int_0^1 t^\alpha(1-\lambda)^\beta\Bigg(\frac{\sqrt{t\lambda}}{4\sqrt{(1-t)(1-\lambda)}}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(n,r)\right|+\frac{\sqrt{t(1-\lambda)}}{4\sqrt{(1-t)\lambda}}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(n,k)\right|$$

$$+\frac{\sqrt{(1-t)\lambda}}{4\sqrt{t(1-\lambda)}}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(m,r)\right|+\frac{\sqrt{(1-t)(1-\lambda)}}{4\sqrt{t\lambda}}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(m,k)\right|\Bigg)dtd\lambda$$

$$+\int_0^1\int_0^1 (1-t)^\alpha\lambda^\beta\Bigg(\frac{\sqrt{t\lambda}}{4\sqrt{(1-t)(1-\lambda)}}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(n,r)\right|+\frac{\sqrt{t(1-\lambda)}}{4\sqrt{(1-t)\lambda}}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(n,k)\right|$$

$$+\frac{\sqrt{(1-t)\lambda}}{4\sqrt{t(1-\lambda)}}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(m,r)\right|+\frac{\sqrt{(1-t)(1-\lambda)}}{4\sqrt{t\lambda}}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(m,k)\right|\Bigg)dtd\lambda$$

$$+\int_0^1\int_0^1 t^\alpha\lambda^\beta\Bigg(\frac{\sqrt{t\lambda}}{4\sqrt{(1-t)(1-\lambda)}}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(n,r)\right|+\frac{\sqrt{t(1-\lambda)}}{4\sqrt{(1-t)\lambda}}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(n,k)\right|$$

$$+\frac{\sqrt{(1-t)\lambda}}{4\sqrt{t(1-\lambda)}}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(m,r)\right|+\frac{\sqrt{(1-t)(1-\lambda)}}{4\sqrt{t\lambda}}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(m,k)\right|\Bigg)dtd\lambda\Bigg)$$

$$=\frac{(m-n)(k-r)}{16}\left(\frac{\pi^2}{4}+B\left(\frac{1}{2},\alpha+\frac{1}{2}\right)B\left(\frac{1}{2},\beta+\frac{1}{2}\right)\right)$$

$$\times\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(n,r)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(n,k)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(m,r)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(m,k)\right|\right),$$

which is the desired result. □

Theorem 3.2. *Let $f:\Lambda\to\mathbb{R}$ be a partially differentiable function on Λ. If $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$ with $q>1$ is co-ordinated MT-convex function on Λ, then the following fractional inequality holds*

$$\Bigg|f\left(\frac{n+m}{2},\frac{r+k}{2}\right)-\frac{f\left(n,\frac{r+k}{2}\right)+f\left(\frac{n+m}{2},r\right)+f\left(m,\frac{r+k}{2}\right)+f\left(\frac{n+m}{2},k\right)}{2}+A$$

$$-\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(m-n)^\alpha(k-r)^\beta}$$

$$\times\left(J^{\alpha,\beta}_{n^+,r^+}f(m,k)+J^{\alpha,\beta}_{m^-,r^+}f(n,k)+J^{\alpha,\beta}_{n^+,k^-}f(m,r)+J^{\alpha,\beta}_{m^-,k^-}f(n,r)\right)\Bigg|$$

$$\leq\frac{(m-n)(k-r)}{4}\left(1+\frac{4}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\right)\left(\frac{\pi}{4}\right)^{\frac{2}{q}}$$

$$\times\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(n,r)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(n,k)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(m,r)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(m,k)\right|^q\right)^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$ and A is defined as in (3.4).

Proof. From Lemma 3.1, properties of modulus, and Hölder inequality, we have

$$\begin{aligned}
&\Bigg| f\left(\tfrac{n+m}{2},\tfrac{r+k}{2}\right) - \frac{f\left(n,\frac{r+k}{2}\right)+f\left(\frac{n+m}{2},r\right)+f\left(m,\frac{r+k}{2}\right)+f\left(\frac{n+m}{2},k\right)}{2} + A \\
&-\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(m-n)^{\alpha}(k-r)^{\beta}} \\
&\times\left(J^{\alpha,\beta}_{n^+,r^+}f(m,k)+J^{\alpha,\beta}_{m^-,r^+}f(n,k)+J^{\alpha,\beta}_{n^+,k^-}f(m,r)+J^{\alpha,\beta}_{m^-,k^-}f(n,r)\right)\Bigg| \\
\leq\ & \tfrac{(m-n)(k-r)}{4}\left(\left(\int_0^1\int_0^1 dtd\lambda\right)^{\frac{1}{p}} + \left(\int_0^1\int_0^1 (1-t)^{\alpha p}(1-\lambda)^{\beta p}\,dtd\lambda\right)^{\frac{1}{p}}\right. \\
&+\left(\int_0^1\int_0^1 t^{\alpha p}(1-\lambda)^{\beta p}\,dtd\lambda\right)^{\frac{1}{p}} + \left(\int_0^1\int_0^1 (1-t)^{\alpha p}\lambda^{\beta p}dtd\lambda\right)^{\frac{1}{p}} \\
&\left.+\left(\int_0^1\int_0^1 t^{\alpha p}\lambda^{\beta p}dtd\lambda\right)^{\frac{1}{p}}\right) \\
&\times\left(\int_0^1\int_0^1 \left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(tn+(1-t)m,\lambda r+(1-\lambda)k)\right|^q dtd\lambda\right)^{\frac{1}{q}} \\
=\ & \tfrac{(m-n)(k-r)}{4}\left(1+\tfrac{4}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\right) \\
&\times\left(\int_0^1\int_0^1 \left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(tn+(1-t)m,\lambda r+(1-\lambda)k)\right|^q dtd\lambda\right)^{\frac{1}{q}}. \qquad (3.10)
\end{aligned}$$

Since $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$ is co-ordinated MT-convex, we deduce

$$\begin{aligned}
&\Bigg| f\left(\tfrac{n+m}{2},\tfrac{r+k}{2}\right) - \frac{f\left(n,\frac{r+k}{2}\right)+f\left(\frac{n+m}{2},r\right)+f\left(m,\frac{r+k}{2}\right)+f\left(\frac{n+m}{2},k\right)}{2} + A \\
&-\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(m-n)^{\alpha}(k-r)^{\beta}} \\
&\times\left(J^{\alpha,\beta}_{n^+,r^+}f(m,k)+J^{\alpha,\beta}_{m^-,r^+}f(n,k)+J^{\alpha,\beta}_{n^+,k^-}f(m,r)+J^{\alpha,\beta}_{m^-,k^-}f(n,r)\right)\Bigg| \\
\leq\ & \tfrac{(m-n)(k-r)}{4}\left(1+\tfrac{4}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\right)\left(\int_0^1\int_0^1\left(\tfrac{\sqrt{(1-t)(1-\lambda)}}{4\sqrt{t\lambda}}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(m,k)\right|^q\right.\right. \\
&+\tfrac{\sqrt{t(1-\lambda)}}{4\sqrt{(1-t)\lambda}}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(n,k)\right|^q + \tfrac{\sqrt{(1-t)\lambda}}{4\sqrt{t(1-\lambda)}}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(m,r)\right|^q
\end{aligned}$$

$$+\frac{\sqrt{t\lambda}}{4\sqrt{(1-t)(1-\lambda)}}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(n,r)\right|^q\Bigg)dtd\lambda\Bigg)^{\frac{1}{q}}$$

$$=\frac{(m-n)(k-r)}{4}\left(1+\frac{4}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\right)\left(\frac{\pi}{4}\right)^{\frac{2}{q}}$$

$$\times\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(n,r)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(n,k)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(m,r)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(m,k)\right|^q\right)^{\frac{1}{q}},$$

which is the desired result. □

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