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ON ULAM'S STABILITIES OF MACKEY-GLASS EQUATION WITH VARIABLE COEFFICIENTS

MAHER NAZMI QARAWANI[1]

ABSTRACT. This paper considers the Hyers-Ulam stability for Mackey-Glass differential equation with variable coefficents and a nonconstant delay. It also investigates the Hyers -Ulam-Rassias stability for Mackey-Glass Equation. The results are illustrated by given examples.

1. INTRODUCTION

In 1940, Ulam [28] posed an important problem before the Mathematics Club of the University of Wisconsin concerning the stability of group homomorphisms . A significant breakthrough came in 1941, when Hyers [6] gave an answer to Ulam's problem. During the last two decades very important contributions to the stability problems of functional equations were given by many mathematicians, e.g., [3, 5–9, 13, 14, 17–19, 24]. More than twenty years ago, a generalization of Ulam's problem was proposed by replacing functional equations with differential equations: The differential equation $F(t, y(t), y'(t), ..., y^{(n)}(t)) = 0$ has the Hyers-Ulam stability if for given $\varepsilon > 0$ and a function y such that

$$\left|F(t, y(t), y'(t), ..., y^{(n)}(t))\right| \leq \varepsilon$$

there exists a solution y_0 of the differential equation such that

$$|y(t) - y_0(t)| \leq K(\varepsilon)$$

and $\lim_{\varepsilon\to 0} K(\varepsilon) = 0$.

The first step in the direction of investigating the Hyers-Ulam stability of differential equations was taken by Obloza [15, 16]. Thereafter, Alsina and Ger [1] have studied the Hyers-Ulam stability of the linear differential equation $y'(t) = y(t)$. The Hyers-Ulam stability problems of linear differential equations of first order and second order with constant coefficients were studied in the papers [29] by using the method of integral factors. The

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results given in [10, 25, 26] have been generalized by Popa and Rus [21, 22] for the linear differential equations of nth order with constant coefficients.

In addition to above-mentioned studies, several authors have studied the Hyers-Ulam stability for differential equations of first and second order, e.g., [4, 11, 12, 27]. During the last few decades, it has been recognized that equations with delay more adequately describe various models of mathematical biology than equations without delay. For example, the delay equation (the Mackey-Glass, or the hematopoiesis, equation) was applied to model white blood cells production.

This paper investigates the Hyers-Ulam-Rassias and Hyers-Ulam stability of Mackey-Glass equation with variable coefficients :

$$x' + \alpha(t)x = \frac{\beta(t)x(g(t))}{1 + x^{\gamma}(g(t))} \tag{1.1}$$

with the initial function $x(t) = \varphi(t), \forall t \leq 0$, and the initial condition

$$x(0) = 0, \tag{1.2}$$

where $\varphi(t)$ is positive and continuous $\forall t < 0, x \in C^1(I)$, $I = [a, b]$, $0 < a < b \leq \infty$, $\gamma > 0$, $\alpha(t) : I \to [0, \infty)$ is a bounded function, $\beta(t)$ is a positive bounded function on I , and $g(t)$ is a continuous function such that $g(t) \leq t$, $\lim_{t\to\infty} g(t) = \infty$.

It should be noted, that the Hyers-Ulam stability for Mackey-Glass differential equation with constant coefficients and a constant delay $g(t) = t - \tau$,

$$x' + \gamma x = \frac{\beta x(t-\tau)}{1 + x^n(t-\tau)}$$

was established in [23].

2. Preliminaries

Definition 2.1. We say that equation (1.1) has the Hyers -Ulam stability with initial conditions (1.2) if there exists a positive constant $K > 0$ with the following property:

For every $\varepsilon > 0$, $x \in C^1(I)$, if

$$|x' + \alpha(t)x - \frac{\beta(t)x(g(t))}{1 + x^{\gamma}(g(t))}| \leq \varepsilon \tag{2.1}$$

and $x(0) = 0$, then there exists some $w \in C^1(I)$ satisfying the equation (1.1) and $w(0) = 0$, such that $|w(t) - x(t)| \leq K\varepsilon$.

Definition 2.2. We say that equation (1.1) with initial condition (1.2) has the Hyers-Ulam-Rassias (HUR) stability with respect to θ if there exists a positive constant $K > 0$ with the following property:

For each $x(t)$ satisfying

$$\left|x' + \alpha(t)x - \frac{\beta(t)x(g(t))}{1 + x^{\gamma}(g(t))}\right| \leq \theta(t) \tag{2.2}$$

then there exists some solution $w \in C^1(I)$ of the equation (1.1) with (1.2) such that $|x(t) - w(t)| \leq K\theta(t)$.

Lemma 2.1. *(Gronwall's lemma). Let $u(t)$ and $v(t)$ be nonnegative continuous functions on some interval $0 < t_0 \leq t \leq t_0 + a$. Also, let the function $f(t)$ be positive, continuous, and monotonically nondecreasing on $[t_0, t_0 + a]$ and satisfy the inequality*

$$u(t) \leq f(t) + \int_{t_0}^{t} u(s)v(s)ds$$

then, there holds the inequality

$$u(t) \leq f(t) \exp\left(\int_{t_0}^{t} v(s)ds\right), \text{ for } t_0 \leq t \leq t_0 + a$$

For the proof of Lemma 1, see [20].

3. On Hyers-Ulam Stability of Solutions

Theorem 3.1. *Suppose that $x : I \to \mathbb{R}$ is a continuously differentiable function and satisfies the inequality (2.1), and the initial condition (1.2).*

If $\sup\limits_{t \geq 0} \left[\exp\left(\int_0^t \alpha(\tau)d\tau\right)\right] < \infty$, then the equation (1.1) has the Hyers-Ulam stability with initial condition (1.2).

Proof. Suppose that $\varepsilon > 0$ and $x \in C^1(I)$ satisfies the inequality (2.1) and the initial condition $x(0) = 0$. We will show that there exists a function $w(t) \in C^1(I)$ satisfying the equation (1.1) such that $|x(t) - w(t)| \leq K\varepsilon$ and $w(0) = 0$, where K is a constant that never depends on ε nor on $w(t)$. Substituting

$$x(t) = u(t) \exp\left(-\int_0^t \alpha(s)ds\right) \tag{3.1}$$

into the equation (1.1) see [2], we get

$$u'(t) \exp\left(-\int_0^t \alpha(s)ds\right) = \frac{\beta(t)u(g(t)) \exp\left(-\int_0^{g(t)} \alpha(s)ds\right)}{1 + u^{\gamma}(g(t)) \exp\left(-\gamma \int_0^{g(t)} \alpha(s)ds\right)}. \tag{3.2}$$

The inequality (2.1) and (3.2) imply that

$$-\varepsilon \leq u'(t) \exp\left(-\int_0^t \alpha(s)ds\right) - \frac{\beta(t)u(g(t)) \exp\left(-\int_0^{g(t)} \alpha(s)ds\right)}{1 + u^{\gamma}(g(t)) \exp\left(-\gamma \int_0^{g(t)} \alpha(s)ds\right)} \leq \varepsilon. \tag{3.3}$$

Multiplying (3.3) by $\exp\left(\int\limits_0^t \alpha(s)ds\right)$ and then integrating with respect to t, we get

$$-\varepsilon\int\limits_0^t \exp\left(\int\limits_0^s \alpha(\tau)d\tau\right)ds \leq u(t) - \int\limits_0^t \frac{\beta(t)u(g(s))\exp\left(\int\limits_{g(s)}^s \alpha(\tau)d\tau\right)}{1+u^\gamma(g(s))\exp\left(-\gamma\int\limits_0^{g(s)} \alpha(\tau)d\tau\right)}ds \tag{3.4}$$

$$\leq \varepsilon\int\limits_0^t \exp\left(\int\limits_0^s \alpha(\tau)d\tau\right)ds.$$

Since $\varphi(t) > 0$, then from (3.1) $u'(t) > 0, \forall t > 0$, and hence the function $u(t)$ is an increasing and positive $\forall t > 0$.

Let us denote by $k = \sup\limits_{t\ \geq 0}\left[\exp\left(\int\limits_0^t \alpha(\tau)d\tau\right)\right] < \infty$. Then, by (3.4), and the boundedness of $\beta(t)$ on I , we obtain

$$u(t) - \beta_0 k\int\limits_0^t u(s)ds < u(t) - \int\limits_0^t \frac{\beta(t)u(g(s))\exp\left(\int\limits_{g(s)}^s \alpha(\tau)d\tau\right)}{1+u^\gamma(g(s)\exp\left(-\gamma\int\limits_0^{g(s)} \alpha(\tau)d\tau\right)}ds \leq k\varepsilon t \leq k\varepsilon b$$

where $k = \sup\limits_{t\ \geq 0}\left[\exp\left(\int\limits_0^t \alpha(\tau)d\tau\right)\right] \geq \sup\limits_{t\ \geq g(t)}\left[\exp\left(\int\limits_{g(t)}^t \alpha(\tau)d\tau\right)\right]$.

From which we obtain the estimation

$$u(t) \leq kb\varepsilon + \beta_0 k\int\limits_0^t u(s)ds. \tag{3.5}$$

By applying Gronwall inequality to (3.5), we have

$$u(t) \leq kb\varepsilon\exp(\beta_0 t) \leq K\varepsilon$$

where $K = ke^{\beta_0 b}$.

Now, since $\alpha(t) \geq 0$ and $\sup\limits_{t\ \geq 0}\left[\exp\left(\int\limits_0^t \alpha(\tau)d\tau\right)\right] < \infty$,then by virtue of the substitution (3.1) we infer that

$$\max_{a\leq t\leq b}(x(t)) \leq K\varepsilon.$$

Obviously, $w(t) \equiv 0$ is a solution of the equation (1.1) satisfying the initial condition (1.2) and such that $|x(t) - w(t)| \leq K\varepsilon$, which completes the proof. □

Now we give an example illustrating the Theorem 3.1.

Example 3.1. Consider the equation

$$x' + 2t\ x = \frac{|\sin t|\ \ x(t-1)}{1 + x^{20}(t-1)} \qquad,\ 0 \le t \le 100 \tag{3.6}$$

with the initial condition

$$x(0) = 0 \tag{3.7}$$

and the history function $\varphi(t) = -t,\ t \in [-1, 0]$.

Substituting $x(t) = u(t) \exp\left(-\int\limits_0^t 2sds\right)$ into equation (3.6), we get

$$u'(t)e^{-t^2} = \frac{e^{-(t-1)^2}\ |\sin t|\, u(t-1)}{1 + e^{-20(t-1)^2}\ [u(t-1)]^{20}}.$$

Multiplying the following inequality by e^{t^2}

$$-\varepsilon \le u'(t)e^{-t^2} - \frac{e^{-(t-1)^2}\ |\sin t|\ \ u(t-1)}{1 + e^{-20(t-1)^2}\ [u(t-1)]^{20}} \le \varepsilon$$

after then we integrate from zero to t, we obtain

$$u(t) - e^{2b} \int\limits_0^t u(s-1)ds < u(t) - \int\limits_0^t \frac{e^{2t-1}\, |\sin t|\, u(s-1)}{1 + e^{-20(s-1)^2} u^{20}(s-1)} ds \le \varepsilon \int\limits_0^t e^{s^2} ds.$$

Using Gronwall inequality we obtain

$$u(t) \le \varepsilon e^{b^2} t \exp(e^{2b}), \qquad 0 < a \le t\ \ \le b$$

which it follows that $0 < u(t) \le K_1\varepsilon$, where $K_1 = b \exp(b^2 + e^{2b})$.

Now, since $\sup\limits_{t\ \ge 0} \left[\exp\left(\int\limits_0^t \alpha(\tau)d\tau\right)\right] = \sup\limits_{t\ \ge 0} \left[\exp\left(\int\limits_0^t 2sds\right)\right] \le e^{b^2}$, then by virtue of the substitution

$$x(t) = u(t) \exp\left(-\int\limits_0^t 2sds\right)$$

we get that

$$\max_{0 \le t \le 100} (x(t)) \le K\varepsilon, \qquad K = be^{2b}.$$

Obviously, $w(t) \equiv 0$ is a solution of the equation (3.6) satisfying the initial condition (3.7) and such that $|x(t) - w(t)| \le k\varepsilon$.

Hence, the problem (3.5)-(3.6) is stable in the sense of Hyers and Ulam.

In the following theorem we establish the Hyers-Ulam-Rassias (HUR) stability for (1.1) in the interval $0 \le a < b \le \infty$.

Theorem 3.2. *Let $x\ (t)$ be a continuously differentiable function for $0 \le t \le \infty$, satisfying the inequality (2.1), and the initial condition (1.2) and suppose that*

$$\sup_{0 \le t\ \le \infty} \left[\exp\left(\int\limits_0^t \alpha(\tau)d\tau\right)\right] < \infty.$$

and

$$\int_0^\infty \beta(s)ds < \infty. \tag{3.8}$$

If there exists a continuous function $\theta(t) : [0, \infty) \to (0, \infty)$ such that

$$\int_0^t \theta(s)ds \le C\ \theta(t)$$

then the problem (1.1)-(1.2) is stable in the sense of HUR as $t \to \infty$.

Proof. Suppose that $x \in C^1([0, \infty))$ satisfies the inequality (2.1) and the initial condition $x(0) = 0$. We will show that there exists a function $w(t) \in C^1([0, \infty))$ satisfying the equation (1.1) such that $|x(t) - w(t)| \le K\theta(t)$ and $w(0) = 0$, where K is a constant that never depends on ε nor on $w(t)$. Substituting

$$x(t) = u(t) \exp\left(-\int_0^t \alpha(s)ds\right)$$

into the equation (1.1), we get

$$u'(t) \exp\left(-\int_0^t \alpha(s)ds\right) = \frac{\beta(t)u(g(t)) \exp\left(-\int_0^{g(t)} \alpha(s)ds\right)}{1 + u^\gamma(g(t)) \exp\left(-\gamma \int_0^{g(t)} \alpha(s)ds\right)}. \tag{3.9}$$

The inequality (2.2) and the equation (3.9) implies that

$$-\theta(t) \ \le u'(t) \exp\left(-\int_0^t \alpha(s)ds\right) - \frac{\beta(t)u(g(t)) \exp\left(-\int_0^{g(t)} \alpha(s)ds\right)}{1 + u^\gamma(g(t)) \exp\left(-\gamma \int_0^{g(t)} \alpha(s)ds\right)} \le \theta(t). \tag{3.10}$$

Multiplying (3.10) by $\exp\left(\int_0^t \alpha(s)ds\right)$ and then integrating with respect to t, we get

$$-\int_0^t \theta(s) \exp\left(\int_0^s \alpha(\tau)d\tau\right) ds \le u(t) - \int_0^t \frac{\beta(t)u(g(s)) \exp\left(\int_{g(s)}^s \alpha(\tau)d\tau\right)}{1 + u^\gamma(g(s)) \exp\left(-\gamma \int_0^{g(s)} \alpha(\tau)d\tau\right)} ds$$

$$\le \int_0^t \theta(s) \exp\left(\int_0^s \alpha(\tau)d\tau\right) ds. \tag{3.11}$$

Since $\varphi(t) > 0$, then from (3.9) $u'(t) > 0, \forall t > 0$, and hence the function $u(t)$ is an increasing and positive $\forall t > 0$. Let us denote by $k = \sup\limits_{t \geq 0}\left[\exp\left(\int\limits_0^t \alpha(\tau)d\tau\right)\right] < \infty$. Then, by (3.11) we obtain

$$u(t) - k\int\limits_0^t u(s)\beta(s)ds \leq u(t) - \int\limits_0^t \frac{\beta(t)u(g(s))\exp\left(\int\limits_{g(s)}^s \alpha(\tau)d\tau\right)}{1 + u^\gamma(g(s))\exp\left(-\gamma\int\limits_0^{g(s)} \alpha(\tau)d\tau\right)}ds \leq Ck\theta(t).$$

From which we obtain the estimation

$$u(t) \leq Ck\theta(t) + k\int\limits_0^t u(s)\beta(s)ds. \tag{3.12}$$

By applying Gronwall inequality to (3.12), we have

$$u(t) \leq Ck\theta(t)\exp\left(\int\limits_0^t \beta(s)ds\right) \leq Ck\theta(t)\exp\left(\int\limits_0^\infty \beta(s)ds\right).$$

From (3.8) it follows that the integral $\int\limits_0^\infty \beta(s)ds$ converges, and then we can set

$$L = \exp\left(\int\limits_0^\infty \beta(s)ds\right).$$

Therefore

$$u(t) \leq Ck\theta(t)\exp\left(\int\limits_0^\infty \beta(s)ds\right) \leq K_0\theta(t)$$

where $K_0 = LCk$.

Now, since $\alpha(t) \geq 0$ and $\sup\limits_{t\ \geq\ 0}\left[\exp\left(\int\limits_0^t \alpha(\tau)d\tau\right)\right] < \infty$, then by virtue of the substitution (3.1) we infer that

$$\max_{a\leq t\leq b}(x(t)) \leq K\theta(t).$$

Obviously, $w(t) \equiv 0$ is a solution of the equation (1.1) satisfying the initial condition (1.2) and such that $|x(t) - w(t)| \leq K\theta(t)$, which completes the proof. ☐

Example 3.2. Consider the equation

$$x' + 2t\ e^{-t}x = \frac{x(t-2)/(t^2+1)}{1 + x^{20}(t-2)} \qquad , \ 0 \leq t \leq b \leq \infty \tag{3.13}$$

with the initial condition

$$x(0) = 0 \tag{3.14}$$

and the history function $\varphi(t) = -t,\ t \in [-2, 0]$.

Substituting $x(t)=u(t)\exp\left(-\int\limits_0^t 2se^{-s}ds\right)$ into equation (3.13), we get

$$u'(t)\exp\left(-\int\limits_0^t 2se^{-s}ds\right)=\frac{\exp\left(-\int\limits_0^{t-2} 2se^{-s}ds\right)\ u(t-2)/(t^2+1)}{1+\exp\left(-40\int\limits_0^t se^{-s}ds\right)[u(t-2)]^{20}}.$$

Instead of $\theta(t)$ we take e^t , then multiplying the following inequality by $\exp\left(\int\limits_0^t 2se^{-s}ds\right)$

$$-e^t\leq u'(t)\exp\left(-\int\limits_0^t 2se^{-s}ds\right)-\frac{\exp\left(-\int\limits_0^{t-2} 2se^{-s}ds\right)u(t-2)/(t^2+1)}{1+\exp\left(-40\int\limits_0^{t-2} se^{-s}ds\right)[u(t-2)]^{20}}\leq e^t$$

after then we integrate from zero to t, we obtain

$$\begin{aligned}
&0\leq u(t)-\frac{\int\limits_0^t \exp\left(\int\limits_0^t 2se^{-s}ds\right)\exp\left(-\int\limits_0^{t-2} 2se^{-s}ds\right)u(s-2)/(s^2+1)}{1+e^{(s-1)^{40}}u^{20}(s-2)}ds\\
&\leq \int\limits_0^t e^s\exp\left(\int\limits_0^t 2se^{-s}ds\right)ds\\
&\leq \int\limits_0^t e^s\exp\left(\int\limits_0^\infty 2ve^{-v}dv\right)ds\leq e^2\int\limits_0^t e^sds=e^2(e^t-1)\leq e^2e^t
\end{aligned}$$

and because of

$$\begin{aligned}
&\max_{t\ \geq\ 0}\exp\left(-\int\limits_0^{t-2} 2se^{-s}ds\right)\exp\left(\int\limits_0^t 2se^{-s}ds\right)\\
&=\max_{t\ \geq\ 0}\left(\exp\left(2te^2e^{-t}-2te^{-t}-2e^2e^{-t}-2e^{-t}\right)\right)=3.5414
\end{aligned}$$

we obtain the estimation

$$u(t)\leq e^2e^t+3.5414\int\limits_0^t\frac{u(s)}{s^2+1}ds.$$

Using Gronwall inequality we obtain

$$\begin{aligned}
u(t)&\leq e^2e^t\exp\left(3.5414\int\limits_0^t\frac{ds}{s^2+1}\right)\leq e^2e^t\exp\left(3.5414\int\limits_0^\infty\frac{ds}{s^2+1}\right)\\
&=e^2e^t\exp\left(3.5414\left(\frac{\pi}{2}\right)\right)=e^2e^te^{1.7707\pi}\qquad\qquad 0\leq t\leq\infty,
\end{aligned}$$

which it follows that $0 < u(t) \leq K\theta(t)$, for any t ,where $K = e^2 e^{1.7707\pi}$.

Now, since $\sup\limits_{t\ \geq 0} \left[\exp\left(-\int\limits_0^t \alpha(\tau)d\tau \right) \right] = \sup\limits_{t\ \geq 0} \left[\exp\left(-\int\limits_0^t 2sds \right) \right] = 1$, then by virtue of the substitution

$$x(t) = u(t) \exp\left(-\int\limits_0^t 2se^{-s}ds \right),$$

we get that

$$\max_{0 \leq t \leq b} (x(t)) \leq K\theta(t).$$

Obviously, $w(t) \equiv 0$ is a solution of the equation (3.13) satisfying the initial condition (3.14) and such that $|x(t) - w(t)| \leq K\theta(t)$.

Hence, the problem (3.13)-(3.14) is stable in the Hyers-Ulam-Rassias sense.

Conclusion.

Here we have established the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of Mackey-Glass differential equation with initial conditions. The results were achieved by integrating the differential equations and then estimating the maximum of solutions. To illustrate our theoretical results we have given an example.

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[1]Department of Mathematics,
Al-Quds Open University,
Salfit, West-Bank, Palestine.
E-mail address: mkerawani@qou.edu