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SOME GENERALIZED HERMITE-HADAMARD AND SIMPSON TYPE INEQUALITIES BY USING THE p-CONVEXITY OF DIFFERENTIABLE MAPPINGS

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Abstract. We derive a new integral identity for differentiable mappings. By using the obtained identity as an auxiliary result, we present new Hermite-Hadamard type and Simpson type inequalities for differentiable p-convex functions. Several special cases are also discussed.

1. Introduction

Following inequalities are well known in the literature termed as the Hermite-Hadamard's inequality and Simpson type inequality, respectively.

Theorem 1.1. *Let $Y : S \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on $[e_1, e_2]$ with $e_1, e_2 \in S$ and $e_1 < e_2$. Then the following inequalities hold:*

$$Y\left(\frac{e_1+e_2}{2}\right) \leq \frac{1}{e_2-e_1}\int_{e_1}^{e_2} Y(k)dk \leq \frac{Y(e_1)+Y(e_2)}{2}. \tag{1.1}$$

Theorem 1.2. *Let $Y : [e_1, e_2] \to \mathbb{R}$ be a four times continuously differentiable mapping on (e_1, e_2) and $\left\|Y^{(4)}\right\|_\infty := \sup_{k\in(e_1,e_2)} |Y^{(4)}(k)| < \infty$. Then the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{Y(e_1)+Y(e_2)}{2} + 2Y\left(\frac{e_1+e_2}{2}\right)\right] - \frac{1}{e_2-e_1}\int_{e_1}^{e_2} Y(k)dk\right| \leq \frac{1}{2880}\left\|Y^{(4)}\right\|_\infty (e_2-e_1)^4. \tag{1.2}$$

For some results which generalize, improve, and extend the Hermite-Hadamard and Simpson type inequalities, one refers the reader to the recent papers (see [6, 7, 12, 17–20]).

Let us evoke some basic definitions as follows.

The harmonically convex functions are defined as follows.

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Definition 1.1. ([10]) A function $Y : S \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be harmonically convex function, if

$$Y\left(\frac{kl}{(1-\eta)k+\eta l}\right) \leq \eta Y(k) + (1-\eta)Y(l),$$

for all $k, l \in S$, $\eta \in [0,1]$.

Some interesting and important inequalities related to harmonically convex functions can be found in [4, 5, 9, 11, 13–15, 21, 22, 24, 25] and references cited therein.

Definition 1.2. [26] An interval $S \subseteq \mathbb{R}$ is said to be a p-convex set, if

$$\theta_p(k,l;\eta) = [\eta k^p + (1-\eta)l^p]^{\frac{1}{p}} \in S,$$

for all $k, l \in S$, $\eta \in [0,1]$, where $p = 2k+1$ or $p = \frac{n}{m}, n = 2r+1, m = 2s+1$ and $k, r, s \in \mathbb{N}$.

Definition 1.3. [26] Let S be a p-convex set. A function $Y : S \to \mathbb{R}$ is said to be p-convex function, if

$$Y(\theta_p(k,l;\eta)) \leq \eta Y(k) + (1-\eta)Y(l),$$

for all $k, l \in S$, $\eta \in [0,1]$.

Clearly, for $p = 1$, Definition 1.3 reduces to the definition for classical convex functions. For $\eta = \frac{1}{2}$ in Definition 1.3, we have Jensen p-convex functions or mid p-convex functions.

Remark 1.1. [11] If $S \subset (0,\infty)$ and $p \in \mathbb{R}\setminus\{0\}$, then

$$\theta_p(k,l;\eta) = [\eta k^p + (1-\eta)\, l^p]^{\frac{1}{p}} \in S,$$

for all $k,\ l \in S$ and $\eta \in [0,1]$.

According to Remark 1.1, İşcan presented the following amendment in the definition of p-convex functions.

Definition 1.4. [11] Let $S \subset (0,\infty)$ and $p \in \mathbb{R}\setminus\{0\}$. A function $Y : S \to \mathbb{R}$ is said to be p-convex function or Y is said to belong to the class $PC(S)$, if

$$Y(\theta_p(k,l;\eta)) \leq \eta Y(k) + (1-\eta)Y(l), \tag{1.3}$$

for all $k,\ l \in S$ and $\eta \in [0,1]$. If the inequality (1.3) is reversed, then Y is said to be p-concave.

It should be observed from Definition 1.4 that the class of p-convex functions contains the class of convex functions and harmonic convex functions defined on $I \subset (0,\infty)$ when $p = 1$ and $p = -1$ respectively.

For some new recent investigations involving p-convex functions, see ([8, 11, 16, 23]).

The main aim of this article is to establish new Hermite-Hadamard type and Simpson type inequalities for mappings the absolute values of whose derivatives are p-convex. To begin with, the authors will derive an integral identity for differentiable mappings. By using this integral equality, the authors establish some new inequalities of the Hermite-Hadamard type and the Simpson type for these mappings.

2. Main Results

In this section, we define the following conditions

A Let the mapping $Y : S \subseteq \mathbb{R}\backslash\{0\} \rightarrow \mathbb{R}$ be differentiable on S° (the interior of S).

B Let $\alpha_k = \frac{e_2^p - k^p}{e_2^p - e_1^p}$, where $e_1,\ e_2 \in S^{\circ}$ with $e_1 < e_2$, $k \in [e_1, e_2]$ and $p \in \mathbb{R}\backslash\{0\}$.

C Let $Y' \in L[e_1, e_2]$, where $L[e_1, e_2]$ is the space of all integrable mappings over $[e_1, e_2]$.

D The mapping $\left|Y'\right|^j$ is p-convex on $[e_1, e_2]$ for $j \geq 1$ and $p \in \mathbb{R}\backslash\left\{-1, -\frac{1}{2}, 0\right\}$.

E The mapping $\left|Y'\right|^j$ is p-convex on $[e_1, e_2]$ for $j > 1$, $j \neq 2$ and $p \in \mathbb{R}\backslash\left\{\frac{j}{j-1}, \frac{j}{j-2}, 0\right\}$.

F The mapping $\left|Y'\right|^j$ is p-convex on $[e_1, e_2]$ for $j > 1$, $0 < n < j$ and $p \in \mathbb{R}\backslash\{0\}$.

We will consider the definition of p-convex functions given by İşcan [11] to prove our results.

Next we derive a new integral identity.

Lemma 2.1. *If the conditions A, B and C are satisfied and λ, $\mu \in \mathbb{R}$, then the following equality holds*

$$\begin{aligned}\Phi_k(\lambda,\mu) &\equiv \lambda Y(e_2) + (1-\mu)Y(e_1) + (\mu-\lambda)Y(k) \\ &\qquad - \frac{p}{e_2^p - e_1^p}\int_{e_1}^{e_2}\frac{Y(k)}{k^{1-p}}dk \\ &= \frac{e_2^p - e_1^p}{p}\left\{\int_0^{\alpha_k}(\lambda-\eta)\,\theta_{p-1}(e_1,e_2;\eta)\,Y'(\theta_p(e_1,e_2;\eta))\,d\eta\right. \\ &\qquad \left. + \int_{\alpha_k}^1(\mu-\eta)\,\theta_{p-1}(e_1,e_2;\eta)\,Y'(\theta_p(e_1,e_2;\eta))\,d\eta\right\}, \end{aligned} \tag{2.1}$$

where $\theta_{p-1}(e_1,e_2;\eta) = [\eta e_1^p + (1-\eta)e_2^p]^{\frac{1}{p}-1}$.

Proof. By integration by parts, we have

$$\begin{aligned}&\int_0^{\alpha_k}(\lambda-\eta)\,\theta_{p-1}(e_1,e_2;\eta)\,Y'(\theta_p(e_1,e_2;\eta))\,d\eta \\ &= -\frac{p}{e_2^p - e_1^p}\int_0^{\alpha_k}(\lambda-\eta)\,d\left[Y(\theta_p(e_1,e_2;\eta))\right] \\ &= -\frac{p}{e_2^p - e_1^p}(\lambda-\eta)\,Y(\theta_p(e_1,e_2;\eta))\Big|_0^{\alpha_k} \\ &\quad - \frac{p}{e_2^p - e_1^p}\int_0^{\alpha_k}Y(\theta_p(e_1,e_2;\eta))\,d\eta \\ &= -\frac{p(\lambda-\alpha_k)}{e_2^p - e_1^p}Y(\theta_p(e_1,e_2;\alpha_k)) + \frac{\lambda p Y(e_2)}{e_2^p - e_1^p} \\ &\quad - \frac{p}{e_2^p - e_1^p}\int_0^{\alpha_k}Y(\theta_p(e_1,e_2;\eta))\,d\eta \\ &= -\frac{p(\lambda-\alpha_k)\,Y(k)}{(e_2^p - e_1^p)} + \frac{\lambda p Y(e_2)}{e_2^p - e_1^p} - \frac{p}{e_2^p - e_1^p}\int_0^{\alpha_k}Y(\theta_p(e_1,e_2;\eta))\,d\eta \end{aligned} \tag{2.2}$$

and

$$\begin{aligned}
&\int_{\alpha_k}^{1}(\mu-\eta)\,\theta_{p-1}(e_1,e_2;\eta)\,Y'(\theta_p(e_1,e_2;\eta))\,d\eta \\
&\qquad\qquad = -\frac{p}{e_2^p-e_1^p}\int_{\alpha_k}^{1}(\mu-\eta)\,d\left[Y(\theta_p(e_1,e_2;\eta))\right] \\
&= -\frac{p(\mu-\eta)}{e_2^p-e_1^p}Y(\theta_p(e_1,e_2;\eta))\Big|_{\alpha_k}^{1}-\frac{p}{e_2^p-e_1^p}\int_{\alpha_k}^{1}Y(\theta_p(e_1,e_2;\eta))\,d\eta \\
&= \frac{p(1-\mu)Y(e_1)}{e_2^p-e_1^p}+\frac{p(\mu-\alpha_k)Y(k)}{e_2^p-e_1^p}-\frac{p}{e_2^p-e_1^p}\int_{\alpha_k}^{1}Y(\theta_p(e_1,e_2;\eta))\,d\eta. \qquad (2.3)
\end{aligned}$$

Adding (2.2) and (2.3), making use of the substitution $k=\left[\eta e_1^p+(1-\eta)e_2^p\right]^{\frac{1}{p}}$ and multiplying the resulting equality by $\frac{p}{e_2^p-e_1^p}$, we get the desired identity. □

Lemma 2.2. *Let $\xi,\ c\geq 0,\ 0<e_1<e_2,\ \alpha,\beta\in\mathbb{R}$ and $p\in\mathbb{R}\backslash\left\{-1,-\frac{1}{2},0\right\}$. Then*

$$\chi_1(p,\alpha,\beta;e_1,e_2,\xi,c)=\int_0^c(\alpha+\beta u)\,|\xi-u|\,\theta_{p-1}(e_1,e_2;u)\,du$$

$$=\begin{cases}\frac{e_1^{2p}\nu_1+e_1^pe_2^p\nu_2+e_2^{2p}\nu_3}{(2p+1)(p+1)\left(e_1^p-e_2^p\right)^3} & \xi\geq c,\\[2ex] -\frac{e_1^{2p}\nu_1+e_1^pe_2^p\nu_2+e_2^{2p}\nu_3}{(2p+1)(p+1)\left(e_1^p-e_2^p\right)^3} & 0\leq\xi\leq c,\end{cases}$$

where

$$\begin{aligned}
\nu_1 &= -\left(2p^2+3p+1\right)pe_2\alpha\xi-p\left(ce_1^p+(1-c)e_2^p\right)^{\frac{1}{p}} \\
&\quad\times\left(c^2\beta(p+1)-\alpha\xi\left(2p^2+3p+1\right)+c(\alpha-\beta\xi)(2p+1)\right),
\end{aligned}$$

$$\begin{aligned}
\nu_2 &= pe_2(2p+1)(2\alpha\xi+p(\beta\xi+\alpha(2\xi-1)))-p\left(ce_1^p+(1-c)e_2^p\right)^{\frac{1}{p}} \\
&\quad\times\left(-2c^2\beta(p+1)-2c(\alpha+2\alpha p+\beta(p-\xi-2p\xi))\right. \\
&\quad\left.+(2p+1)(2\alpha\xi+p(-\alpha+2\alpha\xi+\beta\xi))\right)
\end{aligned}$$

and

$$\begin{aligned}
\nu_3 &= -pe_2\left(2p^2(\alpha+\beta)(\xi-1)+\alpha\xi+p(\beta\xi+\alpha(3\xi-1))\right) \\
&\quad-p\left(ce_1^p+(1-c)e_2^p\right)^{\frac{1}{p}}\left(c^2\beta(p+1)-2p^2(\alpha+\beta)(\xi-1)-\alpha\xi\right. \\
&\quad\left.+p(\alpha-3\alpha\xi-\beta\xi)+c(\alpha+2p\alpha-\beta(2p(\xi-1)+\xi))\right).
\end{aligned}$$

Proof. The proof follows by straightforward computations. □

Lemma 2.3. *Let ξ, $c \geq 0$, $\alpha, \beta \in \mathbb{R}$, $r \in \{0\} \cup \mathbb{N}$. Then the following equality holds*

$$\chi_2(\alpha, \beta; \xi, c, r) = \int_0^c (\alpha + \beta\eta) |\xi - u|^r du$$
$$= \begin{cases} \frac{\xi^{r+1}((r+2)\alpha+\beta\xi)-(\xi-c)^{r+1}((r+2)\alpha+\beta(c+cr+\xi))}{(r+1)(r+2)}, & \xi \geq c, \\ \frac{(-\xi)^{r+1}((r+2)\alpha+\beta\xi)+(c-\xi)^{r+1}((r+2)\alpha+\beta(c+cr+\xi))}{(r+1)(r+2)}, & 0 \leq \xi \leq c. \end{cases}$$

Lemma 2.4. *Let $c \geq 0$, $\alpha, \beta \in \mathbb{R}$, $\alpha > 1$, $\alpha \neq 2$, $0 < e_1 < e_2$ and $p \in \mathbb{R} \backslash \left\{\frac{j}{j-1}, \frac{j}{j-2}, 0\right\}$. Then*

$$\begin{aligned} &\chi_3(\alpha, \beta, e_1, e_2; c, p, j) \\ &= \int_0^c (\alpha + \beta u)\, \theta_{p-1}^j(e_1, e_2; u)\, du \\ &= \frac{1}{(e_1^p - e_2^p)^2 (2p - pj + j)(p - pj + j)} \\ &\times \left\{-pe_2^{p-pj+j}\left[e_1^p(2p - pj + j)\alpha\right.\right. \\ &+ e_2^p(-\alpha j - 2\alpha p + pj\alpha - p\beta)] - p\left(ce_1^p + (1-c)e_2^p\right)^{1-j+\frac{j}{p}} \\ &\times [e_1^p(-\alpha j - jc\beta - 2p\alpha + \alpha pj - cp\beta + pjc\beta) \\ &+ e_2^p(\alpha j + jc\beta + 2\alpha p - \alpha pj + p\beta + cp\beta - c\beta pj)]\}. \end{aligned}$$

Let us start to establish the results of this paper.

Theorem 2.1. *If the conditions A, B, C and D are fulfilled, then the following inequality holds true for all λ, $\mu \in [0, 1]$*

$$\begin{aligned} &\left|\lambda Y(e_2) + (1-\mu) Y(e_1) + (\mu - \lambda) Y(k) - \frac{p}{e_2^p - e_1^p}\int_{e_1}^{e_2} \frac{Y(k)}{k^{1-p}} dk\right| \\ &\leq \frac{e_2^p - e_1^p}{p}\left\{[\chi_1(p, 1, 0; e_1, e_2, \lambda, \alpha_k)]^{1-\frac{1}{j}}\right. \\ &\times \left(\chi_1(p, 1, -1; e_1, e_2, \lambda, \alpha_k)\left|Y'(e_1)\right|^j + \chi_1(p, 0, 1; e_1, e_2, \lambda, \alpha_k)\left|Y'(e_2)\right|^j\right)^{\frac{1}{j}} \\ &+ [\chi_1(p, 1, 0; e_2, e_1, 1-\mu, 1-\alpha_k)]^{1-\frac{1}{j}}\left(\chi_1(p, 0, 1; e_2, e_1, 1-\mu, 1-\alpha_k)\left|Y'(e_1)\right|^j\right. \\ &\left.\left.+ \chi_1(p, 1, -1; e_2, e_1, 1-\mu, 1-\alpha_k)\left|Y'(e_2)\right|^j\right)^{\frac{1}{j}}\right\}, \qquad (2.4) \end{aligned}$$

where $\chi_1(p, \alpha, \beta; e_1, e_2, \xi, c)$ is defined in Lemma 2.2.

Proof. Taking the absolute value on both sides of the result in Lemma 2.1, applying the power-mean inequality and using the p-convexity of $\left|Y'\right|^j$, $j \geq 1$, we have

$$\begin{aligned}
\left|\Phi_k(\lambda,\mu)\right| \leq \frac{e_2^p - e_1^p}{p} \bigg\{ \int_0^{\alpha_k} |\lambda-\eta|\,\theta_{p-1}(e_1,e_2;\eta)\left|Y'\left(\theta_p(e_1,e_2;\eta)\right)\right| d\eta \\
+ \int_{\alpha_k}^1 |\mu-\eta|\,\theta_{p-1}(e_1,e_2;\eta)\left|Y'\left(\theta_p(e_1,e_2,\eta)\right)\right| d\eta \bigg\} \\
\leq \left(\frac{e_2^p - e_1^p}{p}\right) \left\{ \left(\int_0^{\alpha_k} |\lambda-\eta|\,\theta_{p-1}(e_1,e_2;\eta)\,d\eta\right)^{1-\frac{1}{j}} \right. \\
\times \left(\int_0^{\alpha_k} |\lambda-\eta|\,\theta_{p-1}(e_1,e_2;\eta)\left[\eta\left|Y'(e_1)\right|^j + (1-\eta)\left|Y'(e_2)\right|^j\right] d\eta\right)^{\frac{1}{j}} \\
+ \left(\int_{\alpha_k}^1 |\mu-\eta|\,\theta_{p-1}(e_1,e_2;\eta)\,d\eta\right)^{1-\frac{1}{j}} \\
\left. \times \left(\int_{\alpha_k}^1 |\mu-\eta|\,\theta_{p-1}(e_1,e_2;\eta)\left[\eta\left|Y'(e_1)\right|^j + (1-\eta)\left|Y'(e_2)\right|^j\right] d\eta\right)^{\frac{1}{j}} \right\}. \quad (2.5)
\end{aligned}$$

By applying Lemma 2.2, we observe that

$$\int_0^{\alpha_k} |\lambda-\eta|\,\theta_{p-1}(e_1,e_2;\eta)\,d\eta = \chi_1(p,1,0;e_1,e_2,\lambda,\alpha_k),$$

$$\begin{aligned}
&\int_{\alpha_k}^1 |\mu-\eta|\,\theta_{p-1}(e_1,e_2,\eta)\,d\eta \\
&= \int_0^{1-\alpha_k} |1-\mu-\eta|\,\theta_{p-1}(e_2,e_1,\eta)\,d\eta = \chi_1(p,1,0;e_2,e_1,1-\mu,1-\alpha_k),
\end{aligned}$$

$$\int_0^{\alpha_k} \eta\,|\lambda-\eta|\,\theta_{p-1}(e_1,e_2,\eta)\,d\eta = \chi_1(p,0,1;e_1,e_2,\lambda,\alpha_k),$$

$$\int_0^{\alpha_k} (1-\eta)\,|\lambda-\eta|\,\theta_{p-1}(e_1,e_2,\eta)\,d\eta = \chi_1(p,1,-1;e_1,e_2,\lambda,\alpha_k),$$

$$\begin{aligned}
\int_{\alpha_k}^1 \eta\,|\mu-\eta|\,\theta_{p-1}(e_1,e_2,\eta)\,d\eta &= \int_0^{1-\alpha_k} (1-\eta)\,|1-\mu-\eta|\,\theta_{p-1}(e_2,e_1,\eta)\,d\eta \\
&= \chi_1(p,1,-1;e_2,e_1,1-\mu,1-\alpha_k)
\end{aligned}$$

and

$$\begin{aligned}
\int_{\alpha_k}^1 (1-\eta)\,|\mu-\eta|\,\theta_{p-1}(e_1,e_2,\eta)\,d\eta &= \int_0^{1-\alpha_k} \eta\,|1-\mu-\eta|\,\theta_{p-1}(e_2,e_1,\eta)\,d\eta \\
&= \chi_1(p,0,1;e_2,e_1,1-\mu,1-\alpha_k).
\end{aligned}$$

By applying the above integrals in (2.5), we get the required inequality. □

Corollary 2.1. *If* $0 \leq \lambda \leq \frac{1}{2} \leq \mu \leq 1$, $j = 1$ *and* $k = \left[\frac{e_1^p + e_2^p}{2}\right]^{\frac{1}{p}}$ *in Theorem 2.1, the following inequality holds valid for* $p \in \mathbb{R} \backslash \left\{-1, -\frac{1}{2}, 0\right\}$

$$\left|\lambda Y(e_2) + (1-\mu) Y(e_1) + (\mu - \lambda) Y\left(\left[\frac{e_1^p + e_2^p}{2}\right]^{\frac{1}{p}}\right) - \frac{p}{e_2^p - e_1^p} \int_{e_1}^{e_2} \frac{Y(k)}{k^{1-p}} dk\right|$$
$$\leq \frac{e_2^p - e_1^p}{p} \left\{\left[\chi_1\left(p, 1, -1; e_1, e_2, \lambda, \frac{1}{2}\right) + \chi_1\left(p, 0, 1; e_2, e_1, 1-\mu, \frac{1}{2}\right)\right] \left|Y'(e_1)\right| \right.$$
$$\left. + \left[\chi_1\left(p, 0, 1; e_1, e_2, \lambda, \frac{1}{2}\right) + \chi_1\left(p, 1, -1; e_2, e_1, 1-\mu, \frac{1}{2}\right)\right] \left|Y'(e_2)\right| \right\}, \quad (2.6)$$

where $\chi_1(p, \alpha, \beta; e_1, e_2, \xi, c)$ *is defined in Lemma 2.2.*

Proof. Since for $k = \left[\frac{e_1^p + e_2^p}{2}\right]^{\frac{1}{p}}$, $\alpha_k = \frac{1}{2}$ and $1 - \alpha_k = \frac{1}{2}$. Hence the proof follows from the result of Theorem 2.1. □

Corollary 2.2. *If* $\lambda = 0$ *and* $\mu = 1$ *in Corollary 2.1, the following Hermite-Hadamard type inequality for p-convex functions holds for* $p \in \mathbb{R} \backslash \left\{-1, -\frac{1}{2}, 0\right\}$

$$\left|Y\left(\left[\frac{e_1^p + e_2^p}{2}\right]^{\frac{1}{p}}\right) - \frac{p}{e_2^p - e_1^p} \int_{e_1}^{e_2} \frac{Y(k)}{k^{1-p}} dk\right|$$
$$\leq \frac{e_2^p - e_1^p}{p} \left\{\left[\chi_1\left(p, 1, -1; e_1, e_2, 0, \frac{1}{2}\right) + \chi_1\left(p, 0, 1; e_2, e_1, 0, \frac{1}{2}\right)\right] \left|Y'(e_1)\right| \right.$$
$$\left. + \left[\chi_1\left(p, 0, 1; e_1, e_2, 0, \frac{1}{2}\right) + \chi_1\left(p, 1, -1; e_2, e_1, 0, \frac{1}{2}\right)\right] \left|Y'(e_2)\right| \right\}, \quad (2.7)$$

where $\chi_1(p, \alpha, \beta; e_1, e_2, \xi, c)$ *is defined in Lemma 2.2.*

Proof. It is a direct consequence of Corollary 2.1. □

Corollary 2.3. *If we set* $\lambda = \mu = \frac{1}{2}$ *in Corollary 2.1, the following Hermite-Hadamard type inequality for p-convex functions holds for* $p \in \mathbb{R} \backslash \left\{-1, -\frac{1}{2}, 0\right\}$

$$\left|\frac{Y(e_2) + Y(e_1)}{2} - \frac{p}{e_2^p - e_1^p} \int_{e_1}^{e_2} \frac{Y(k)}{k^{1-p}} dk\right|$$
$$\leq \frac{e_2^p - e_1^p}{p} \left\{\left[\chi_1\left(p, 1, -1; e_1, e_2, \frac{1}{2}, \frac{1}{2}\right) + \chi_1\left(p, 0, 1; e_2, e_1, \frac{1}{2}, \frac{1}{2}\right)\right] \left|Y'(e_1)\right| \right.$$
$$\left. + \left[\chi_1\left(p, 0, 1; e_1, e_2, \frac{1}{2}, \frac{1}{2}\right) + \chi_1\left(p, 1, -1; e_2, e_1, \frac{1}{2}, \frac{1}{2}\right)\right] \left|Y'(e_2)\right| \right\}, \quad (2.8)$$

where $\chi_1(p, \alpha, \beta; e_1, e_2, \xi, c)$ *is defined in Lemma 2.2.*

Proof. It follows from Corollary 2.1. □

Corollary 2.4. *If we set $\lambda=\frac{5}{6}$ and $\mu=\frac{5}{6}$ in Corollary 2.1, the following Simpson type inequality for p-convex functions holds for $p\in\mathbb{R}\backslash\left\{-1,-\frac{1}{2},0\right\}$*

$$\left|\frac{1}{3}\left[\frac{Y(e_2)+Y(e_1)}{2}+2Y\left(\left[\frac{e_1^p+e_2^p}{2}\right]^{\frac{1}{p}}\right)\right]-\frac{p}{e_2^p-e_1^p}\int_{e_1}^{e_2}\frac{Y(k)}{k^{1-p}}dk\right|$$
$$\leq\frac{e_2^p-e_1^p}{p}\left\{\left[\chi_1\left(p,1,-1;e_1,e_2,\frac{1}{6},\frac{1}{2}\right)+\chi_1\left(p,0,1;e_2,e_1,\frac{1}{6},\frac{1}{2}\right)\right]\left|Y'(e_1)\right|\right.$$
$$\left.+\left[\chi_1\left(p,0,1;e_1,e_2,\frac{1}{6},\frac{1}{2}\right)+\chi_1\left(p,1,-1;e_2,e_1,\frac{1}{6},\frac{1}{2}\right)\right]\left|Y'(e_2)\right|\right\},\quad (2.9)$$

where $\chi_1(p,\alpha,\beta;e_1,e_2,\xi,c)$ is defined in Lemma 2.2.

Remark 2.1. By choosing $p=1$, we can get some new Hermite-Hadamard and Simpson type inequalities from the result of Theorem 2.1 and its related inequalities.

Theorem 2.2. *If the assumptions A, B, D and E are fulfilled, then the following inequality holds true for all λ, $\mu\in[0,1]$*

$$|\Phi_k(\lambda,\mu)|\leq\left(\frac{e_2^p-e_1^p}{p}\right)\left\{\left[\chi_2\left(1,0;\lambda,\alpha_k,\left\lfloor\frac{j}{j-1}\right\rfloor\right)\right]^{1-\frac{1}{j}}\right.$$
$$\times\left(\chi_3(0,1;e_1,e_2,\alpha_k,p,j)\left|Y'(e_1)\right|^j+\chi_3(1,-1;e_1,e_2,\alpha_k,p,j)\left|Y'(e_2)\right|^j\right)^{\frac{1}{j}}$$
$$+\left[\chi_2\left(1,0;1-\mu,1-\alpha_k,\left\lfloor\frac{j}{j-1}\right\rfloor\right)\right]^{1-\frac{1}{j}}\left(\chi_3(1,-1;e_2,e_1,1-\alpha_k,p,j)\left|Y'(e_1)\right|^j\right.$$
$$\left.\left.+\chi_3(0,1;e_2,e_1,1-\alpha_k,p,j)\left|Y'(e_2)\right|^j\right)^{\frac{1}{j}}\right\},\quad (2.10)$$

where $\chi_2(\alpha,\beta;\xi,c,r)$ is defined in Lemma 2.3, $\chi_3(\alpha,\beta,e_1,e_2;c,p,j)$ is defined in Lemma 2.4 and $\lfloor k\rfloor$ is the floor function.

Proof. From the result in Lemma 2.1, applying the Hölder's inequality and using the p-convexity of $\left|Y'\right|^j$, $j>1$, we have

$$|\Phi_k(\lambda,\mu)|\leq\left(\frac{e_2^p-e_1^p}{p}\right)\left\{\int_0^{\alpha_k}|\lambda-\eta|\,\theta_{p-1}(e_1,e_2,\eta)\left|Y'(\theta_p(e_1,e_2,\eta))\right|d\eta\right.$$
$$\left.+\int_{\alpha_k}^1|\mu-\eta|\,\theta_{p-1}(e_1,e_2,\eta)\left|Y'(\theta_p(e_1,e_2,\eta))\right|d\eta\right\}\leq\left(\frac{e_2^p-e_1^p}{p}\right)$$
$$\times\left\{\left(\int_0^{\alpha_k}|\lambda-\eta|^{\frac{j}{j-1}}d\eta\right)^{1-\frac{1}{j}}\left(\int_0^{\alpha_k}\theta_{p-1}^j(e_1,e_2,\eta)\left[\eta\left|Y'(e_1)\right|^j+(1-\eta)\left|Y'(e_2)\right|^j\right]d\eta\right)^{\frac{1}{j}}\right.$$
$$\left.+\left(\int_{\alpha_k}^1|\mu-\eta|^{\frac{j}{j-1}}d\eta\right)^{1-\frac{1}{j}}\left(\int_{\alpha_k}^1\theta_{p-1}^j(e_1,e_2,\eta)\left[\eta\left|Y'(e_1)\right|^j+(1-\eta)\left|Y'(e_2)\right|^j\right]d\eta\right)^{\frac{1}{j}}\right\}.\quad (2.11)$$

By applying Lemma 2.3 and Lemma 2.4 we have that

$$\int_0^{\alpha_k} |\lambda-\eta|^{\frac{j}{j-1}}\,d\eta \le \int_0^{\alpha_k} |\lambda-\eta|^{\left\lfloor\frac{j}{j-1}\right\rfloor}\,d\eta = \chi_2\left(1,0;\lambda,\alpha_k,\left\lfloor\frac{j}{j-1}\right\rfloor\right),$$

$$\begin{aligned}\int_{\alpha_k}^1 |\mu-\eta|^{\frac{j}{j-1}}\,d\eta &= \int_0^{1-\eta_k} |1-\mu-\eta|^{\frac{j}{j-1}}\,d\eta \\ &\le \chi_2\left(1,0;1-\mu,1-\alpha_k,\left\lfloor\frac{j}{j-1}\right\rfloor\right),\end{aligned}$$

$$\int_0^{\alpha_k} \eta\theta_{p-1}^j(e_1,e_2,\eta)\,d\eta = \chi_3(0,1;e_1,e_2,\alpha_k,p,j),$$

$$\int_0^{\alpha_k} (1-\eta)\,\theta_{p-1}^j(e_1,e_2,\eta)\,d\eta = \chi_3(1,-1;e_1,e_2,\alpha_k,p,j),$$

$$\begin{aligned}\int_{\alpha_k}^1 \eta\theta_{p-1}^j(e_1,e_2,\eta)\,d\eta &= \int_0^{1-\alpha_k} (1-\eta)\,\theta_{p-1}^j(e_2,e_1,\eta)\,d\eta \\ &= \chi_3(1,-1;e_2,e_1,1-\alpha_k,p,j)\end{aligned}$$

and

$$\begin{aligned}\int_{\alpha_k}^1 (1-\eta)\,\theta_{p-1}^j(e_1,e_2,\eta)\,d\eta &= \int_0^{1-\alpha_k} \eta\theta_{p-1}^j(e_2,e_1,\eta)\,d\eta \\ &= \chi_3(0,1;e_2,e_1,1-\alpha_k,p,j).\end{aligned}$$

By using the values of the above integrals in (2.11), we get the required inequality. □

Theorem 2.3. *If the assumptions A, B, C and E are fulfilled, then the following inequality holds true for all $\lambda,\ \mu\in[0,1]$*

$$\begin{aligned}|\Phi_k(\lambda,\mu)| \le \left(\frac{e_2^p-e_1^p}{p}\right)\Bigg\{&\left[\chi_3\left(1,0;e_1,e_2,\alpha_k,p,\frac{j}{j-1}\right)\right]^{1-\frac{1}{j}} \\ &\times\left(\chi_2(0,1;\alpha_k,\lfloor j\rfloor)\left|Y'(e_1)\right|^j+\chi_2(1,-1;\alpha_k,\lfloor j\rfloor)\left|Y'(e_2)\right|^j\right)^{\frac{1}{j}} \\ &+\left[\chi_3\left(1,0;e_2,e_1,1-\alpha_k,p,\frac{j}{j-1}\right)\right]^{1-\frac{1}{j}}\left(\chi_3(1,-1;1-\alpha_k,\lfloor j\rfloor)\left|Y'(e_1)\right|^j\right. \\ &\left.+\chi_3(0,1;1-\alpha_k,\lfloor j\rfloor)\left|Y'(e_2)\right|^j\right)^{\frac{1}{j}}\Bigg\},\end{aligned} \tag{2.12}$$

where $\chi_2(\alpha,\beta;\xi,c,r)$ is defined in Lemma 2.3, $\chi_3(\alpha,\beta,e_1,e_2;c,p,j)$ is defined in Lemma 2.4 and $\lfloor k\rfloor$ is the floor function.

Proof. Taking the absolute value on both sides of the result in Lemma 2.1, applying the Hölder's inequality and using the p-convexity of $\left|Y'\right|^j$, $j>1$, we have

$$\begin{aligned}\left|\Phi_k(\lambda,\mu)\right| \le \left(\frac{e_2^p-e_1^p}{p}\right)&\Bigg\{\left(\int_0^{\alpha_k}\theta_{p-1}^{\frac{j}{j-1}}(e_1,e_2,\eta)\,d\eta\right)^{1-\frac{1}{j}}\\ &\times\left(\int_0^{\alpha_k}|\lambda-\eta|^j\left[\eta\left|Y'(e_1)\right|^j+(1-\eta)\left|Y'(e_2)\right|^j\right]d\eta\right)^{\frac{1}{j}}+\left(\int_{\alpha_k}^1\theta_{p-1}^{\frac{j}{j-1}}(e_1,e_2,\eta)\,d\eta\right)^{1-\frac{1}{j}}\\ &\times\left(\int_{\alpha_k}^1|\mu-\eta|^j\left[\eta\left|Y'(e_1)\right|^j+(1-\eta)\left|Y'(e_2)\right|^j\right]d\eta\right)^{\frac{1}{j}}\Bigg\}. \quad (2.13)\end{aligned}$$

By applying Lemma 2.3 and Lemma 2.4 and using similar arguments as in proving Theorem 2.2, we get the desired result. □

Theorem 2.4. *If the assumptions A, B, C and F are fulfilled, then the following inequality holds true for all* λ, $\mu\in[0,1]$

$$\begin{aligned}\left|\Phi_k(\lambda,\mu)\right| \le \left(\frac{e_2^p-e_1^p}{p}\right)&\Bigg\{\left[\chi_4\left(1,0;e_1,e_2,\lambda,\alpha_k,\left\lfloor\frac{j-n}{j-1}\right\rfloor,p\right)\right]^{1-\frac{1}{j}}\\ &\times\left(\chi_4(0,1;e_1,e_2,\lambda,\alpha_k,\lfloor n\rfloor,p)\left|Y'(e_1)\right|^j+\chi_4(1,-1;e_1,e_2,\lambda,\alpha_k,\lfloor n\rfloor,p)\left|Y'(e_2)\right|^j\right)^{\frac{1}{j}}\\ &+\left[\chi_4\left(1,0;e_2,e_1,1-\mu,1-\alpha_k,\left\lfloor\frac{j-n}{j-1}\right\rfloor,p\right)\right]^{1-\frac{1}{j}}\\ &\times\Big(\chi_4(1,-1;e_2,e_1,1-\mu,1-\alpha_k,\lfloor n\rfloor,p)\left|Y'(e_1)\right|^j\\ &+\chi_4(0,1,e_2,e_1,1-\mu,1-\alpha_k,\lfloor n\rfloor,p)\left|Y'(e_2)\right|^j\Big)^{\frac{1}{j}}\Bigg\}, \quad (2.14)\end{aligned}$$

where $\lfloor k\rfloor$ *is the floor function.*

Proof. Using Lemma 2.1, applying the Hölder inequality and using the p-convexity of $\left|Y'\right|^j$, $j>1$, we have

$$\begin{aligned}\left|\Phi_k(\lambda,\mu)\right| \le \left(\frac{e_2^p-e_1^p}{p}\right)&\Bigg\{\left(\int_0^{\alpha_k}|\lambda-\eta|^{\frac{j-n}{j-1}}\theta_{p-1}(e_1,e_2;\eta)\,d\eta\right)^{1-\frac{1}{j}}\\ &\times\left(\int_0^{\alpha_k}|\lambda-\eta|^n\theta_{p-1}(e_1,e_2;\eta)\left[\eta\left|Y'(e_1)\right|^j+(1-\eta)\left|Y'(e_2)\right|^j\right]d\eta\right)^{\frac{1}{j}}\\ &+\left(\int_{\alpha_k}^1|\mu-\eta|^{\frac{j-n}{j-1}}\theta_{p-1}(e_1,e_2;\eta)\,d\eta\right)^{1-\frac{1}{j}}\\ &\times\left(\int_{\alpha_k}^1|\mu-\eta|^n\theta_{p-1}(e_1,e_2;\eta)\left[\eta\left|Y'(e_1)\right|^j+(1-\eta)\left|Y'(e_2)\right|^j\right]d\eta\right)^{\frac{1}{j}}. \quad (2.15)\end{aligned}$$

Let

$$\chi_4(\alpha,\beta;e_1,e_2,\xi,c,r,p)=\int_0^c(\alpha+\beta u)\,|\xi-u|^r\,\theta_{p-1}(e_1,e_2;\eta)\,du$$
$$=\begin{cases}\int_0^c(\alpha+\beta u)(\xi-u)^r\,\theta_{p-1}(e_1,e_2;u)\,du, & \xi\geq c,\\ \\ \int_0^c(\alpha+\beta u)(u-\xi)^r\,\theta_{p-1}(e_1,e_2;u)\,du, & 0\leq\xi\leq c,\end{cases}$$

where p, $\alpha,\beta\in\mathbb{R}$, $0<e_1<e_2$, ξ, $c\geq 0$, $p\neq 0$ and $r\in\{0\}\cup\mathbb{N}$. The above integral can be solved numerically by using the software Matlab or Mathematica. By using this integral, we have the following observations

$$\int_0^{\alpha_k}|\lambda-\eta|^{\frac{j-n}{j-1}}\,\theta_{p-1}(e_1,e_2;\eta)\,d\eta\leq\chi_4\left(1,0;e_1,e_2,\lambda,\alpha_k,\left\lfloor\frac{j-n}{j-1}\right\rfloor,p\right),$$

$$\int_{\alpha_k}^1|\mu-\eta|^{\frac{j-n}{j-1}}\,\theta_{p-1}(e_1,e_2;\eta)\,d\eta=\int_0^{1-\alpha_k}|1-\mu-\eta|^{\frac{j-n}{j-1}}\,\theta_{p-1}(e_2,e_1;\eta)\,d\eta$$
$$\leq\chi_4\left(1,0;e_2,e_1,1-\mu,1-\alpha_k,\left\lfloor\frac{j-n}{j-1}\right\rfloor,p\right),$$

$$\int_0^{\alpha_k}(1-\eta)\,|\lambda-\eta|^n\,\theta_{p-1}(e_1,e_2;\eta)\,d\eta\leq\chi_4(1,-1;e_1,e_2,\lambda,\alpha_k,\lfloor n\rfloor,p),$$

$$\int_0^{\alpha_k}\eta\,|\lambda-\eta|^n\,\theta_{p-1}(e_1,e_2;\eta)\,d\eta\leq\chi_4(0,1;e_1,e_2,\lambda,\alpha_k,\lfloor n\rfloor,p),$$

$$\int_{\alpha_k}^1\eta\,|\mu-\eta|^n\,\theta_{p-1}(e_1,e_2;\eta)\,d\eta\leq\chi_4(1,-1;e_2,e_1,1-\mu,1-\alpha_k,\lfloor n\rfloor,p)$$

and

$$\int_{\alpha_k}^1(1-\eta)\,|\mu-\eta|^p\,\theta_{p-1}(e_1,e_2;\eta)\,d\eta\leq\chi_4(0,1;e_2,e_1,1-\mu,1-\alpha_k,\lfloor n\rfloor,p),$$

where $\lfloor k\rfloor$ is the floor function. By using the above inequalities in (2.15), we get the inequality (2.14). □

Theorem 2.5. *If the assumptions A, B, C and E are fulfilled, then the following inequality holds true for all λ, $\mu\in[0,1]$*

$$|\Phi_k(\lambda,\mu)|\leq\left(\frac{e_2^p-e_1^p}{p}\right)\left\{\left[\chi_4\left(1,0;e_1,e_2,\lambda,\alpha_k,\left\lfloor\frac{j}{j-1}\right\rfloor,\frac{(j-1)p}{j-p}\right)\right]^{1-\frac{1}{j}}\right.$$
$$\times\left(\chi_3(0,1,e_1,e_2;\alpha_k,p,j)\left|Y'(e_1)\right|^j+\chi_3(1,-1,e_1,e_2,p,\alpha_k,j)\left|Y'(e_2)\right|^j\right)^{\frac{1}{j}}$$
$$+\left[\chi_4\left(1,0;e_2,e_1,1-\mu,1-\alpha_k,\left\lfloor\frac{j}{j-1}\right\rfloor,\frac{(j-1)p}{j-p}\right)\right]^{1-\frac{1}{j}}$$
$$\left.\times\left(\chi_3(1,-1,e_2,e_1;1-\alpha_k,p,j)\left|Y'(e_1)\right|^j+\chi_3(0,1,e_2,e_1;1-\alpha_k,p,j)\left|Y'(e_2)\right|^j\right)^{\frac{1}{j}}\right\},\tag{2.16}$$

where $\lfloor k\rfloor$ is the floor function.

Proof. From Lemma 2.1, applying the Hölder's inequality and using the p-convexity of $\left|Y'\right|^j$, $j > 1$, we have

$$\begin{aligned}
\left|\Phi_k(\lambda,\mu)\right| &\leq \left(\frac{e_2^p - e_1^p}{p}\right)\left\{\int_0^{\alpha_k} |\lambda-\eta|\,\theta_{p-1}(e_1,e_2,\eta)\left|Y'(\theta_p(e_1,e_2;\eta))\right| d\eta \right.\\
&\qquad \left. + \int_{\alpha_k}^1 |\mu-\eta|\,\theta_{p-1}(e_1,e_2;)\left|Y'(\theta_p(e_1,e_2;\eta))\right| d\eta\right\}\\
&\leq \left(\frac{e_2^p - e_1^p}{p}\right)\left\{\left(\int_0^{\alpha_k} |\lambda-\eta|^{\frac{j}{j-1}}\,\theta_{p-1}^{\frac{j}{j-1}}(e_1,e_2;\eta)\,d\eta\right)^{1-\frac{1}{j}}\right.\\
&\times \left(\int_0^{\alpha_k} \theta_{p-1}^j(e_1,e_2;\eta)\left[\eta\left|Y'(e_1)\right|^j + (1-\eta)\left|Y'(e_2)\right|^j\right] d\eta\right)^{\frac{1}{j}}\\
&\qquad + \left(\int_{\alpha_k}^1 |\mu-\eta|^{\frac{j}{j-1}}\,\theta_{p-1}^{\frac{j}{j-1}}(e_1,e_2;\eta)\,d\eta\right)^{1-\frac{1}{j}}\\
&\qquad \left.\left(\int_{\alpha_k}^1 \theta_{p-1}^j(e_1,e_2;\eta)\left[\eta\left|Y'(e_1)\right|^j + (1-\eta)\left|Y'(e_2)\right|^j\right] d\eta\right)^{\frac{1}{j}}\right\}. \qquad (2.17)
\end{aligned}$$

By using Lemma 2.4 and similar arguments as in proving Theorem 2.4, we get the result given in (2.16). □

Remark 2.2. If $0 \leq \lambda \leq \frac{1}{2} \leq \mu \leq 1$, $j = 1$ and $k = \frac{e_1+e_2}{2}$, one can derive some interesting Hermite-Hadamard type inequalities form Theorem 2.2-Theorem 2.5.

3. Conclusions

In this manuscript, we have established a new generalized identity for differentiable mappings. Some auxiliary results are also obtained. By using the p-convexity of the powers of the absolute value of the differentiable mappings and mathematical analysis, we have presented some new Hermite-Hadamard and Simpson's type inequalities. The obtained results are new in the field of inequalities and we expect that the results are useful for the researchers working in the field of mathematical inequalities, numerical analysis and other areas of pure and applied mathematics.

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