Turkish J. Ineq., 2 (2) (2018), Pages 54 –70 .

Turkish Journal of
INEQUALITIES

Available online at www.tjinequality.com

OSTROWSKI TYPE CONFORMABLE FRACTIONAL INTEGRALS FOR GENERALIZED (g, s, m, φ)-PREINVEX FUNCTIONS

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Abstract. In the present paper, the notion of generalized (g, s, m, φ)-preinvex function is introduced and some new integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula involving generalized (g, s, m, φ)-preinvex functions are given. Moreover, some generalizations of Ostrowski type inequalities for generalized (g, s, m, φ)-preinvex functions via conformable fractional integrals are established. At the end, some applications to special means are given.

1. Introduction

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the interior of I. The nonnegative real numbers are denoted by $\mathbb{R}_\circ = [0, +\infty)$. The set of integrable functions on the interval $[a, b]$ is denoted by $L[a, b]$.

The following result is known in the literature as the Ostrowski inequality, see [9] and the references cited therein, which gives an upper bound for the approximation of the integral average $\frac{1}{b-a}\int_a^b f(t)dt$ by the value $f(x)$ at point $x \in [a, b]$.

Theorem 1.1. *Let $f : I \longrightarrow \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping differentiable in the interior I° of I, and let $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \le M$ for all $x \in [a, b]$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right], \quad \forall x \in [a, b]. \tag{1.1}$$

For other recent results concerning Ostrowski type inequalities, see [9] and the references cited therein, also, see [19] and the references cited therein.

Key words and phrases. Ostrowski type inequality, Hölder inequality, power mean inequality, Riemann-Liouville fractional integral, conformable fractional integral, s-convex function in the second sense, m-invex.

2010 *Mathematics Subject Classification.* Primary: 26A51; Secondary: 26A33, 26D07, 26D10, 26D15.

Received: 14/08/2018

Accepted: 31/12/2018.

Fractional calculus, see [10] and the references cited therein, was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

Definition 1.1. Let $f \in L[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \displaystyle\int_0^{+\infty} e^{-u} u^{\alpha-1} du$. Here $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.
In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Ostrowski type inequalities for functions of different classes, see [10] and the references cited therein.
The following definitions will be used in the sequel.

Definition 1.2. The Euler beta function is defined for $a, b > 0$ as

$$\beta(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Definition 1.3. The incomplete beta function is defined for $a, b > 0$ as

$$\beta_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1} dt, \quad 0 < x \leq 1.$$

For $x = 1$, the incomplete beta function coincides with the complete beta function.

Definition 1.4. Let $g : [0,1] \longrightarrow [0,1]$ be a strictly increasing function. The generalized incomplete beta function is defined for $a, b > 0$ as

$$B_{g(x)}(a,b) = \int_{g(0)}^{g(x)} t^{a-1}(1-t)^{b-1} dt.$$

For $g(x) = x$, the generalized incomplete beta function coincides with the incomplete beta function.

In the following, we give some definitions and properties of conformable fractional integrals which help to obtain main identity and results. Recently, some authors, started to study on conformable fractional integrals. In [7], Khalil et al. defined the fractional integral of order $0 < \alpha \leq 1$ only. In [1], Abdeljawad gave the definition of left and right conformable fractional integrals of any order $\alpha > 0$.

Definition 1.5. Let $\alpha \in (n, n+1]$ and set $\beta = \alpha - n$, then the left conformable fractional integral starting at a is defined by

$$\left(I_\alpha^a f\right)(t) = \frac{1}{n!}\int_a^t (t-x)^n (x-a)^{\beta-1} f(x)dx.$$

Analogously, the right conformable fractional integral is defined by

$$\left({}^b I_\alpha f\right)(t) = \frac{1}{n!}\int_t^b (x-t)^n (b-x)^{\beta-1} f(x)dx.$$

Notice that if $\alpha = n+1$, then $\beta = \alpha - n = n+1-n = 1$, where $n = 0, 1, 2, \ldots,$ and hence $\left(I_\alpha^a f\right)(t) = \left(J_{n+1}^a f\right)(t)$.

In [13], Set et al. established a generalization of Hermite-Hadamard type inequality for s-convex functions and gave some remarks to show the relationships with the classical and Riemann-Liouville fractional integrals inequality by using the given properties of conformable fractional integrals.

Theorem 1.2. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function with $0 \le a < b$, $s \in (0, 1]$, and $f \in L[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for conformable fractional integrals hold*

$$\frac{\Gamma(\alpha-n)}{\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right) \le \frac{1}{2^s(b-a)^\alpha}\left[\left(I_\alpha^a f\right)(b) + \left({}^b I_\alpha f\right)(a)\right]$$
$$\le \left[\frac{\beta(n+s+1, \alpha-n) + \beta(n+1, \alpha-n+s)}{n!}\right]\frac{f(a)+f(b)}{2^s},$$

with $\alpha \in (n, n+1]$, $n \in \mathbb{N}$, $n = 0, 1, 2, \ldots,$ where Γ is Euler gamma function.

Also Set et al. established some results for some kind of inequalities via conformable fractional integrals, see [14–17].
Now, let us recall some definitions of various convex functions.

Definition 1.6. [4] A non-negative function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}_\circ$ is said to be P-function or P-convex, if

$$f(tx + (1-t)y) \le f(x) + f(y), \quad \forall x, y \in I,\ t \in [0, 1].$$

Definition 1.7. [6] A function $f : \mathbb{R}_\circ \longrightarrow \mathbb{R}$ is said to be s-convex in the second sense, if

$$f(\lambda x + (1-\lambda)y) \le \lambda^s f(x) + (1-\lambda)^s f(y) \tag{1.2}$$

for all $x, y \in \mathbb{R}_\circ$, $\lambda \in [0, 1]$ and $s \in (0, 1]$.

It is clear that a s-convex function must be convex on $\mathbb{R}_\circ$ as usual. The s-convex functions in the second sense have been investigated, see [6].

Definition 1.8. [2] A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y, x) \in K$ for every $x, y \in K$ and $t \in [0, 1]$.

Notice that every convex set is invex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not necessarily true. For more details, see [2],[20].

Definition 1.9. [12] The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect η, if for every $x, y \in K$ and $t \in [0,1]$, we have that

$$f\left(x + t\eta(y,x)\right) \leq (1-t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not true.

The Gauss-Jacobi type quadrature formula has the following

$$\int_a^b (x-a)^p(b-x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^{\star}|f|, \tag{1.3}$$

for certain $B_{m,k}, \gamma_k$ and rest $R_m^{\star}|f|$, see [18].

Recently, in [8], Liu obtained several integral inequalities for the left-hand side of (1.3) under the Definition 1.6 of P-function.
Also, in [11], Özdemir et al. established several integral inequalities concerning the left-hand side of (1.3) via some kinds of convexity.

Motivated by these results, in Sect. 2, the notion of generalized (g,s,m,φ)-preinvex function will be introduced and some new integral inequalities for the left-hand side of (1.3) involving generalized (g,s,m,φ)-preinvex functions will be given. In Sect. 3, some generalizations of Ostrowski type inequalities for generalized (g,s,m,φ)-preinvex functions via conformable fractional integrals will be obtain. In Sect. 4, some applications to special means will be provided as well.

2. New integral inequalities

Definition 2.1. [5] A set $K \subseteq \mathbb{R}^n$ is said to be m-invex with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ for any fixed $m \in (0,1]$, if $mx + t\eta(y,x,m) \in K$ holds for each $x, y \in K$ and any $t \in [0,1]$.

Remark 2.1. In Definition 2.1, under certain conditions, the mapping $\eta(y,x,m)$ could reduce to $\eta(y,x)$. For example when $m = 1$, then the m-invex set reduced in invex set on K.

We next give new definition, to be referred as generalized (g,s,m,φ)-preinvex function.

Definition 2.2. Let $K \subseteq \mathbb{R}^n$ be an open m-invex set with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$, $g : [0,1] \longrightarrow [0,1]$ be a differentiable function and $\varphi : I \longrightarrow K$ is a continuous function. For $f : K \longrightarrow \mathbb{R}$ and some fixed $s, m \in (0,1]$, if

$$f\left(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m)\right) \leq m(1-g(t))^s f(\varphi(x)) + g^s(t) f(\varphi(y)) \tag{2.1}$$

is valid for all $x, y \in I, t \in [0,1]$, then we say that f is a generalized (g,s,m,φ)-preinvex function with respect to η.

Remark 2.2. In Definition 2.2, it is worthwhile to note that the class of generalized (g, s, m, φ)-preinvex function is a generalization of the class of s-convex in the second sense function given in Definition 1.7. Also, for $g(t) = t$, $t \in [0, 1]$ and $\varphi(x) = x$, $\forall\, x \in I$, we get the notion of generalized (s, m)-preinvex function, see [5].

In this section, in order to prove our main results regarding some new integral inequalities involving generalized (g, s, m, φ)-preinvex functions, we need the following lemma.

Lemma 2.1. *Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0, 1] \longrightarrow [0, 1]$ is a differentiable function. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow \mathbb{R}$ is a continuous function on K° with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$ and $\eta(\varphi(b), \varphi(a), m) > 0$. Then for some fixed $m \in (0, 1]$ and any fixed $p, q > 0$, we have*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$
$$= \eta^{p+q+1}(\varphi(b), \varphi(a), m)$$
$$\times \int_0^1 g^p(t)(1 - g(t))^q f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))dg(t).$$

Proof. It is easy to observe that

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$
$$= \eta(\varphi(b), \varphi(a), m) \int_0^1 (m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m) - m\varphi(a))^p$$
$$\times (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - m\varphi(a) - g(t)\eta(\varphi(b), \varphi(a), m))^q$$
$$\times f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))dg(t)$$
$$= \eta^{p+q+1}(\varphi(b), \varphi(a), m)$$
$$\times \int_0^1 g^p(t)(1 - g(t))^q f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))dg(t).$$

So, the proof of this lemma is completed. $\square$

Theorem 2.1. *Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0, 1] \longrightarrow [0, 1]$ is a strictly increasing function. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow \mathbb{R}$ is a continuous function on K° with $\eta(\varphi(b), \varphi(a), m) > 0$. If $k > 1$ and $|f|^{\frac{k}{k-1}}$ is a generalized (g, s, m, φ)-preinvex function on an open m-invex set K with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$ for some fixed $s, m \in (0, 1]$, then for any fixed $p, q > 0$, we have*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$
$$\le \frac{\eta^{p+q+1}(\varphi(b), \varphi(a), m)}{(s+1)^{\frac{k-1}{k}}} B^{\frac{1}{k}}(g(t); k, p, q)$$
$$\times \Big[m \left((1 - g(0))^{s+1} - (1 - g(1))^{s+1} \right) |f(\varphi(a))|^{\frac{k}{k-1}}$$
$$+ (g^{s+1}(1) - g^{s+1}(0)) |f(\varphi(b))|^{\frac{k}{k-1}} \Big]^{\frac{k-1}{k}},$$

where $B(g(t);k,p,q) := \int_0^1 g^{kp}(t)(1-g(t))^{kq}dg(t)$.

Proof. Since $|f|^{\frac{k}{k-1}}$ is a generalized (g,s,m,φ)-preinvex function on K, combining with Lemma 2.1, Hölder's inequality and properties of the modulus, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\left[\int_0^1 g^{kp}(t)(1-g(t))^{kq}dg(t)\right]^{\frac{1}{k}}$$

$$\times\left[\int_0^1 |f(m\varphi(a)+g(t)\eta(\varphi(b),\varphi(a),m))|^{\frac{k}{k-1}}dg(t)\right]^{\frac{k-1}{k}}$$

$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)B^{\frac{1}{k}}(g(t);k,p,q)$$

$$\times\left[\int_0^1 \left(m(1-g(t))^s|f(\varphi(a))|^{\frac{k}{k-1}}+g^s(t)|f(\varphi(b))|^{\frac{k}{k-1}}\right)dg(t)\right]^{\frac{k-1}{k}}$$

$$=\frac{\eta^{p+q+1}(\varphi(b),\varphi(a),m)}{(s+1)^{\frac{k-1}{k}}}B^{\frac{1}{k}}(g(t);k,p,q)$$

$$\times\Big[m\left((1-g(0))^{s+1}-(1-g(1))^{s+1}\right)|f(\varphi(a))|^{\frac{k}{k-1}}$$

$$+(g^{s+1}(1)-g^{s+1}(0))|f(\varphi(b))|^{\frac{k}{k-1}}\Big]^{\frac{k-1}{k}}.$$

So, the proof of this theorem is completed. □

Corollary 2.1. *Under the same conditions as in Theorem 2.1 for $g(t)=t$, we get*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq \frac{\eta^{p+q+1}(\varphi(b),\varphi(a),m)}{(s+1)^{\frac{k-1}{k}}}\Big[\beta(kp+1,kq+1)\Big]^{\frac{1}{k}}\left(m|f(\varphi(a))|^{\frac{k}{k-1}}+|f(\varphi(b))|^{\frac{k}{k-1}}\right)^{\frac{k-1}{k}}.$$

Theorem 2.2. *Let $\varphi: I \longrightarrow K$ be a continuous function and $g:[0,1]\longrightarrow[0,1]$ is a strictly increasing function. Assume that $f: K=[m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]\longrightarrow\mathbb{R}$ is a continuous function on K° with $\eta(\varphi(b),\varphi(a),m)>0$. If $l\geq 1$ and $|f|^l$ is a generalized (g,s,m,φ)-preinvex function on an open m-invex set K with respect to $\eta: K\times K\times(0,1]\longrightarrow\mathbb{R}$ for some fixed $s,m\in(0,1]$, then for any fixed $p,q>0$,*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)B^{\frac{l-1}{l}}(g(t);1,p,q)$$

$$\times\Big[m|f(\varphi(a))|^l B(g(t);1,p,q+s)+|f(\varphi(b))|^l B(g(t);1,p+s,q)\Big]^{\frac{1}{l}}.$$

Proof. Since $|f|^l$ is a generalized (g,s,m,φ)-preinvex function on K, combining with Lemma 2.1 and the well-known power mean inequality and properties of the modulus, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\left[\int_0^1 g^p(t)(1-g(t))^q dg(t)\right]^{\frac{l-1}{l}}$$

$$\times\left[\int_0^1 g^p(t)(1-g(t))^q\left|f(m\varphi(a)+g(t)\eta(\varphi(b),\varphi(a),m))\right|^l dg(t)\right]^{\frac{1}{l}}$$

$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)B^{\frac{l-1}{l}}(g(t);1,p,q)$$

$$\times\left[\int_0^1 g^p(t)(1-g(t))^q\left(m(1-g(t))^s|f(\varphi(a))|^l+g^s(t)|f(\varphi(b))|^l\right)dg(t)\right]^{\frac{1}{l}}$$

$$= \eta^{p+q+1}(\varphi(b),\varphi(a),m)B^{\frac{l-1}{l}}(g(t);1,p,q)$$

$$\times\Big[m|f(\varphi(a))|^l B(g(t);1,p,q+s)+|f(\varphi(b))|^l B(g(t);1,p+s,q)\Big]^{\frac{1}{l}}.$$

So, the proof of this theorem is completed. □

Corollary 2.2. *Under the conditions of Theorem 2.2 for $g(t)=t$, we get*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\Big[\beta(p+1,q+1)\Big]^{\frac{l-1}{l}}$$

$$\times\Big[m|f(\varphi(a))|^l\beta\left(p+1,q+s+1\right)+|f(\varphi(b))|^l\beta\left(p+s+1,q+1\right)\Big]^{\frac{1}{l}}.$$

3. Ostrowski type conformable fractional integrals

In this section, in order to prove our main results regarding some generalizations of Ostrowski type inequalities for generalized (g,s,m,φ)-preinvex functions via conformable fractional integrals, we need the following lemma.

Lemma 3.1. *Let $\varphi: I\longrightarrow K$ be a continuous function and $g:[0,1]\longrightarrow[0,1]$ is a differentiable function. Suppose $K=[m\varphi(a),m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]\subseteq\mathbb{R}$ be an open m-invex subset with respect to $\eta: K\times K\times(0,1]\longrightarrow\mathbb{R}$ for some fixed $m\in(0,1]$ and let $\eta(\varphi(b),\varphi(a),m)>0$. Assume that $f:K\longrightarrow\mathbb{R}$ is a differentiable function on K° and $f'\in L(K)$. Then for $\alpha>0$, we have*

$$\frac{\eta^{\alpha}(\varphi(x),\varphi(a),m)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\Big[B_{g(1)}(n+1,\alpha-n)f(m\varphi(a)+g(1)\eta(\varphi(x),\varphi(a),m))\Big]$$

$$-\frac{\eta^{\alpha}(\varphi(x),\varphi(b),m)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\Big[B_{g(1)}(n+1,\alpha-n)f(m\varphi(b)+g(1)\eta(\varphi(x),\varphi(b),m))\Big]$$
$$-\frac{1}{\eta(\varphi(b),\varphi(a),m)}$$
$$\times\Bigg[\int_{m\varphi(a)+g(0)\eta(\varphi(x),\varphi(a),m)}^{m\varphi(a)+g(1)\eta(\varphi(x),\varphi(a),m)}(x-m\varphi(a))^n(m\varphi(a)+\eta(\varphi(x),\varphi(a),m)-x)^{\alpha-n-1}f(x)dx$$
$$-\int_{m\varphi(b)+g(0)\eta(\varphi(x),\varphi(b),m)}^{m\varphi(b)+g(1)\eta(\varphi(x),\varphi(b),m)}(x-m\varphi(b))^n(m\varphi(b)+\eta(\varphi(x),\varphi(b),m)-x)^{\alpha-n-1}f(x)dx\Bigg]$$
$$=\frac{\eta^{\alpha+1}(\varphi(x),\varphi(a),m)}{\eta(\varphi(b),\varphi(a),m)}\int_0^1 B_{g(t)}(n+1,\alpha-n)f'(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))dg(t)$$
$$-\frac{\eta^{\alpha+1}(\varphi(x),\varphi(b),m)}{\eta(\varphi(b),\varphi(a),m)}\int_0^1 B_{g(t)}(n+1,\alpha-n)f'(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))dg(t). \tag{3.1}$$

We denote

$$S_{f,g,\eta,\varphi}(x;\alpha,n,m,a,b)$$
$$=\frac{\eta^{\alpha+1}(\varphi(x),\varphi(a),m)}{\eta(\varphi(b),\varphi(a),m)}\int_0^1 B_{g(t)}(n+1,\alpha-n)f'(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))dg(t)$$
$$-\frac{\eta^{\alpha+1}(\varphi(x),\varphi(b),m)}{\eta(\varphi(b),\varphi(a),m)}\int_0^1 B_{g(t)}(n+1,\alpha-n)f'(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))dg(t).$$

Proof. A simple proof of the equality can be done by performing an integration by parts in the integrals from the right side and changing the variable. The details are left to the interested reader. □

Remark 3.1. If we choose $\alpha=n+1$ where $n=0,1,2,\dots,$ $m=1$, $g(t)=t$, $\eta(\varphi(y),\varphi(x),1)=\varphi(y)-\varphi(x)$ and $\varphi(x)=x$, $\forall\,x,y\in I$ in Lemma 3.1, we get [9, Lemma 1].

Corollary 3.1. *Under the same conditions as in Lemma 3.1, if we choose $\alpha\in(n,n+1]$ where $n=0,1,2,\dots$ and $g(t)=t$, we get the following equality for the right conformable fractional integrals:*

$$\frac{\beta(n+1,\alpha-n)}{\eta(\varphi(b),\varphi(a),m)}$$
$$\times\Big\{\eta^{\alpha}(\varphi(x),\varphi(a),m)f(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))$$
$$-\eta^{\alpha}(\varphi(x),\varphi(b),m)f(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))\Big\}$$
$$-\frac{n!}{\eta(\varphi(b),\varphi(a),m)}$$
$$\times\Big[\Big({}^{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))}I_\alpha f\Big)(m\varphi(a))-\Big({}^{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}I_\alpha f\Big)(m\varphi(b))\Big]$$
$$=\frac{\eta^{\alpha+1}(\varphi(x),\varphi(a),m)}{\eta(\varphi(b),\varphi(a),m)}\int_0^1 \beta_t(n+1,\alpha-n)f'(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))dt$$
$$-\frac{\eta^{\alpha+1}(\varphi(x),\varphi(b),m)}{\eta(\varphi(b),\varphi(a),m)}\int_0^1 \beta_t(n+1,\alpha-n)f'(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))dt. \tag{3.2}$$

By using Lemma 3.1, the following results can be obtained for the corresponding version for power of first derivative.

Theorem 3.1. *Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0,1] \longrightarrow [0,1]$ is a strictly increasing function. Suppose $K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $s, m \in (0,1]$ and let $\eta(\varphi(b), \varphi(a), m) > 0$. Assume that $f : K \longrightarrow \mathbb{R}$ is a differentiable function on K°. If $|f'|^q$ is a generalized (g, s, m, φ)-preinvex function on K, $q > 1$, $p^{-1} + q^{-1} = 1$, then for $\alpha > 0$, we have*

$$|S_{f,g,\eta,\varphi}(x;\alpha,n,m,a,b)| \leq \left(\frac{1}{s+1}\right)^{\frac{1}{q}} \frac{\delta^{\frac{1}{p}}(g(t);p)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+1}\Big[m\left((1-g(0))^{s+1}-(1-g(1))^{s+1}\right)|f'(\varphi(a))|^q$$

$$+(g^{s+1}(1)-g^{s+1}(0))|f'(\varphi(x))|^q\Big]^{\frac{1}{q}}$$

$$+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+1}\Big[m\left((1-g(0))^{s+1}-(1-g(1))^{s+1}\right)|f'(\varphi(b))|^q$$

$$+(g^{s+1}(1)-g^{s+1}(0))|f'(\varphi(x))|^q\Big]^{\frac{1}{q}}\Bigg\}, \tag{3.3}$$

where $\delta(g(t);p) := \int_0^1 B^p_{g(t)}(n+1,\alpha-n)dg(t)$.

Proof. Since $|f'|^q$ is a generalized (g, s, m, φ)-preinvex function, combining with Lemma 3.1, Hölder inequality and taking the modulus, we have

$$|S_{f,g,\eta,\varphi}(x;\alpha,n,m,a,b)|$$

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+1}}{|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 B_{g(t)}(n+1,\alpha-n)|f'(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))|dg(t)$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+1}}{|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 B_{g(t)}(n+1,\alpha-n)|f'(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))|dg(t)$$

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+1}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 B^p_{g(t)}(n+1,\alpha-n)dg(t)\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 |f'(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))|^q dg(t)\right)^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+1}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 B^p_{g(t)}(n+1,\alpha-n)dg(t)\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 |f'(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))|^q dg(t)\right)^{\frac{1}{q}}$$

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+1}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 B^p_{g(t)}(n+1,\alpha-n)dg(t)\right)^{\frac{1}{p}}$$

$$\times\left[\int_0^1 \left(m(1-g(t))^s|f'(\varphi(a))|^q + g^s(t)|f'(\varphi(x))|^q\right)dg(t)\right]^{\frac{1}{q}}$$
$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+1}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 B^p_{g(t)}(n+1,\alpha-n)dg(t)\right)^{\frac{1}{p}}$$
$$\times\left[\int_0^1 \left(m(1-g(t))^s|f'(\varphi(b))|^q + g^s(t)|f'(\varphi(x))|^q\right)dg(t)\right]^{\frac{1}{q}}$$
$$=\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\frac{\delta^{\frac{1}{p}}(g(t);p)}{\eta(\varphi(b),\varphi(a),m)}$$
$$\times\Big\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+1}\Big[m\left((1-g(0))^{s+1}-(1-g(1))^{s+1}\right)|f'(\varphi(a))|^q$$
$$+(g^{s+1}(1)-g^{s+1}(0))|f'(\varphi(x))|^q\Big]^{\frac{1}{q}}$$
$$+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+1}\Big[m\left((1-g(0))^{s+1}-(1-g(1))^{s+1}\right)|f'(\varphi(b))|^q$$
$$+(g^{s+1}(1)-g^{s+1}(0))|f'(\varphi(x))|^q\Big]^{\frac{1}{q}}\Big\}.$$

So, the proof of this theorem is completed. □

Corollary 3.2. *Under the same conditions as in Theorem 3.1, if we choose $\alpha \in (n, n+1]$ where $n = 0, 1, 2, \ldots,$ and $g(t) = t$ we get the following inequality for the right conformable fractional integrals:*

$$\Bigg|\frac{\beta(n+1,\alpha-n)}{\eta(\varphi(b),\varphi(a),m)}$$
$$\times\Big\{\eta^\alpha(\varphi(x),\varphi(a),m)f(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))$$
$$-\eta^\alpha(\varphi(x),\varphi(b),m)f(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))\Big\}$$
$$-\frac{n!}{\eta(\varphi(b),\varphi(a),m)}$$
$$\times\left[\left({}^{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))}I_\alpha f\right)(m\varphi(a)) - \left({}^{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}I_\alpha f\right)(m\varphi(b))\right]\Bigg|$$
$$\le\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\frac{\delta^{\frac{1}{p}}}{\eta(\varphi(b),\varphi(a),m)}$$
$$\times\Big\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+1}\left[m|f'(\varphi(a))|^q+|f'(\varphi(x))|^q\right]^{\frac{1}{q}}$$
$$+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+1}\left[m|f'(\varphi(b))|^q+|f'(\varphi(x))|^q\right]^{\frac{1}{q}}\Big\}, \tag{3.4}$$

where $\delta := \int_0^1 \beta_t^p(n+1,\alpha-n)dt$.

Corollary 3.3. *Under the same conditions as in Corollary 3.2, if we choose* $\alpha = n+1$, $n = 0, 1, 2, \dots,$ *and* $|f'| \leq K$*, we get the following inequality for the right Riemann-Liouville fractional integrals:*

$$\left|\frac{1}{\eta(\varphi(b),\varphi(a),m)}\right.$$
$$\times\Big\{\eta^{\alpha}(\varphi(x),\varphi(a),m)f(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))$$
$$-\eta^{\alpha}(\varphi(x),\varphi(b),m)f(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))\Big\}$$
$$-\frac{\Gamma(\alpha+1)}{\eta(\varphi(b),\varphi(a),m)}$$
$$\left.\times\Big[J^{\alpha}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))-}f(m\varphi(a))-J^{\alpha}_{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))-}f(m\varphi(b))\Big]\right|$$
$$\leq \frac{K}{(p\alpha+1)^{1/p}}\left(\frac{m+1}{s+1}\right)^{\frac{1}{q}}\left[\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+1}+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+1}}{\eta(\varphi(b),\varphi(a),m)}\right]. \tag{3.5}$$

Theorem 3.2. *Let* $\varphi : I \longrightarrow K$ *be a continuous function and* $g : [0,1] \longrightarrow [0,1]$ *is a strictly increasing function. Suppose* $K = [m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)] \subseteq \mathbb{R}$ *be an open m-invex subset with respect to* $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ *for some fixed* $s, m \in (0,1]$ *and let* $\eta(\varphi(b),\varphi(a),m) > 0$*. Assume that* $f : K \longrightarrow \mathbb{R}$ *is a differentiable function on* K°*. If* $|f'|^q$ *is a generalized* (g,s,m,φ)*-preinvex function on* K, $q \geq 1$*, then for* $\alpha > 0$*, we have*

$$|S_{f,g,\eta,\varphi}(x;\alpha,n,m,a,b)|$$
$$\leq \left(\frac{1}{s+1}\right)^{\frac{1}{q}}\frac{\Big[g(1)B_{g(1)}(n+1,\alpha-n)-B_{g(1)}(n+2,\alpha-n)\Big]^{1-\frac{1}{q}}}{\eta(\varphi(b),\varphi(a),m)}$$
$$\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+1}$$
$$\times\Big[m\left(B_{g(1)}(n+1,\alpha-n+s+1)-(1-g(1))^{s+1}B_{g(1)}(n+1,\alpha-n)\right)|f'(\varphi(a))|^q$$
$$+\left(g^{s+1}(1)B_{g(1)}(n+1,\alpha-n)-B_{g(1)}(n+s+2,\alpha-n)\right)|f'(\varphi(x))|^q\Big]^{\frac{1}{q}}$$
$$+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+1}$$
$$\times\Big[m\left(B_{g(1)}(n+1,\alpha-n+s+1)-(1-g(1))^{s+1}B_{g(1)}(n+1,\alpha-n)\right)|f'(\varphi(b))|^q$$
$$+\left(g^{s+1}(1)B_{g(1)}(n+1,\alpha-n)-B_{g(1)}(n+s+2,\alpha-n)\right)|f'(\varphi(x))|^q\Big]^{\frac{1}{q}}\Bigg\}. \tag{3.6}$$

Proof. Since $|f'|^q$ is a generalized (g,s,m,φ)-preinvex function, combining with Lemma 3.1, the well-known power mean inequality and taking the modulus, we have

$$
\begin{aligned}
&\left|S_{f,g,\eta,\varphi}(x;\alpha,n,m,a,b)\right| \\
&\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+1}}{|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 B_{g(t)}(n+1,\alpha-n)|f'(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))|dg(t) \\
&+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+1}}{|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 B_{g(t)}(n+1,\alpha-n)|f'(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))|dg(t) \\
&\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+1}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 B_{g(t)}(n+1,\alpha-n)dg(t)\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 B_{g(t)}(n+1,\alpha-n)|f'(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))|^q dg(t)\right)^{\frac{1}{q}} \\
&+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+1}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 B_{g(t)}(n+1,\alpha-n)dg(t)\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 B_{g(t)}(n+1,\alpha-n)|f'(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))|^q dg(t)\right)^{\frac{1}{q}} \\
&\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+1}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 B_{g(t)}(n+1,\alpha-n)dg(t)\right)^{1-\frac{1}{q}} \\
&\times\left[\int_0^1 B_{g(t)}(n+1,\alpha-n)\left(m(1-g(t))^s|f'(\varphi(a))|^q+g^s(t)|f'(\varphi(x))|^q\right)dg(t)\right]^{\frac{1}{q}} \\
&+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+1}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 B_{g(t)}(n+1,\alpha-n)dg(t)\right)^{1-\frac{1}{q}} \\
&\times\left[\int_0^1 B_{g(t)}(n+1,\alpha-n)\left(m(1-g(t))^s|f'(\varphi(b))|^q+g^s(t)|f'(\varphi(x))|^q\right)dg(t)\right]^{\frac{1}{q}} \\
&=\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\frac{\left[g(1)B_{g(1)}(n+1,\alpha-n)-B_{g(1)}(n+2,\alpha-n)\right]^{1-\frac{1}{q}}}{\eta(\varphi(b),\varphi(a),m)} \\
&\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+1} \\
&\times\Big[m\left(B_{g(1)}(n+1,\alpha-n+s+1)-(1-g(1))^{s+1}B_{g(1)}(n+1,\alpha-n)\right)|f'(\varphi(a))|^q \\
&+\left(g^{s+1}(1)B_{g(1)}(n+1,\alpha-n)-B_{g(1)}(n+s+2,\alpha-n)\right)|f'(\varphi(x))|^q\Big]^{\frac{1}{q}} \\
&+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+1} \\
&\times\Big[m\left(B_{g(1)}(n+1,\alpha-n+s+1)-(1-g(1))^{s+1}B_{g(1)}(n+1,\alpha-n)\right)|f'(\varphi(b))|^q \\
&+\left(g^{s+1}(1)B_{g(1)}(n+1,\alpha-n)-B_{g(1)}(n+s+2,\alpha-n)\right)|f'(\varphi(x))|^q\Big]^{\frac{1}{q}}\Bigg\}.
\end{aligned}
$$

So, the proof of this theorem is completed. □

Corollary 3.4. *Under the same conditions as in Theorem 3.2, if we choose $\alpha \in (n, n+1]$ where $n = 0, 1, 2, \ldots,$ and $g(t) = t$ we get the following inequality for the right conformable fractional integrals:*

$$\begin{aligned}
&\Bigg|\frac{\beta(n+1,\alpha-n)}{\eta(\varphi(b),\varphi(a),m)} \\
&\times\Big\{\eta^{\alpha}(\varphi(x),\varphi(a),m)f(m\varphi(a)+\eta(\varphi(x),\varphi(a),m)) \\
&-\eta^{\alpha}(\varphi(x),\varphi(b),m)f(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))\Big\} \\
&-\frac{n!}{\eta(\varphi(b),\varphi(a),m)} \\
&\times\left[\left({}^{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))}I_{\alpha}f\right)(m\varphi(a))-\left({}^{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}I_{\alpha}f\right)(m\varphi(b))\right]\Bigg| \\
&\leq\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\frac{(\beta(n+1,\alpha-n)-\beta(n+2,\alpha-n))^{1-\frac{1}{q}}}{\eta(\varphi(b),\varphi(a),m)} \\
&\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+1}\Big[m|f'(\varphi(a))|^{q}\beta(n+1,\alpha-n+s+1) \\
&+|f'(\varphi(x))|^{q}\left(\beta(n+1,\alpha-n)-\beta(n+s+2,\alpha-n)\right)\Big]^{\frac{1}{q}} \\
&+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+1}\Big[m|f'(\varphi(b))|^{q}\beta(n+1,\alpha-n+s+1) \\
&+|f'(\varphi(x))|^{q}\left(\beta(n+1,\alpha-n)-\beta(n+s+2,\alpha-n)\right)\Big]^{\frac{1}{q}}\Bigg\}.
\end{aligned} \tag{3.7}$$

Corollary 3.5. *Under the same conditions as in Corollary 3.4, if we choose $\alpha = n+1$, $n = 0, 1, 2, \ldots,$ and $|f'| \leq K$, we get the following inequality for the right Riemann-Liouville fractional integrals:*

$$\begin{aligned}
&\Bigg|\frac{1}{\eta(\varphi(b),\varphi(a),m)} \\
&\times\Big\{\eta^{\alpha}(\varphi(x),\varphi(a),m)f(m\varphi(a)+\eta(\varphi(x),\varphi(a),m)) \\
&-\eta^{\alpha}(\varphi(x),\varphi(b),m)f(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))\Big\} \\
&-\frac{\Gamma(\alpha+1)}{\eta(\varphi(b),\varphi(a),m)} \\
&\times\left[J^{\alpha}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))-}f(m\varphi(a))-J^{\alpha}_{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))-}f(m\varphi(b))\right]\Bigg| \\
&\leq\frac{K}{(\alpha+1)^{1-\frac{1}{q}}}\left(m\beta(\alpha+1,s+1)+\frac{1}{\alpha+s+1}\right)^{\frac{1}{q}} \\
&\times\left[\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+1}+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+1}}{\eta(\varphi(b),\varphi(a),m)}\right].
\end{aligned} \tag{3.8}$$

4. Applications to special means

Definition 4.1. [3] A function $M : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$, is called a Mean function if it has the following properties:

(a) Homogeneity: $M(ax, ay) = aM(x, y)$, for all $a > 0$,
(b) Symmetry: $M(x, y) = M(y, x)$,
(c) Reflexivity: $M(x, x) = x$,
(d) Monotonicity: If $x \leq x'$ and $y \leq y'$, then $M(x, y) \leq M(x', y')$,
(e) Internality: $\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$.

We consider some means for different positive real numbers α and β.

(a) The arithmetic mean:
$$A := A(\alpha, \beta) = \frac{\alpha + \beta}{2}.$$
(b) The geometric mean:
$$G := G(\alpha, \beta) = \sqrt{\alpha\beta}.$$
(c) The harmonic mean:
$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}.$$
(d) The power mean:
$$P_r := P_r(\alpha, \beta) = \left(\frac{\alpha^r + \beta^r}{2}\right)^{\frac{1}{r}}, \quad r \geq 1.$$
(e) The identric mean:
$$I := I(\alpha, \beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right), & \alpha \neq \beta; \\ \alpha, & \alpha = \beta. \end{cases}$$
(f) The logarithmic mean:
$$L := L(\alpha, \beta) = \frac{\beta - \alpha}{\ln(\beta) - \ln(\alpha)}.$$
(g) The generalized log-mean:
$$L_p := L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)}\right]^{\frac{1}{p}}; \quad p \in \mathbb{R} \setminus \{-1, 0\}.$$
(h) The weighted p-power mean:
$$M_p\begin{pmatrix} \alpha_1, & \alpha_2, & \cdots & , \alpha_n \\ u_1, & u_2, & \cdots & , u_n \end{pmatrix} = \left(\sum_{i=1}^{n} \alpha_i u_i^p\right)^{\frac{1}{p}}$$
where $0 \leq \alpha_i \leq 1$, $u_i > 0$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} \alpha_i = 1$.

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$. Now, let a and b be positive real numbers such that $a < b$ and $\alpha > 0$. Consider the function $M := M(\varphi(a), \varphi(b)) : [\varphi(a), \varphi(a) + \eta(\varphi(b), \varphi(a))] \times [\varphi(a), \varphi(a) + \eta(\varphi(b), \varphi(a))] \longrightarrow \mathbb{R}_+$, which is one of the above mentioned means and $\varphi : I \longrightarrow K$ be a continuous function and $g : [0,1] \longrightarrow [0,1]$ is a strictly increasing function. Therefore one can obtain various inequalities using the results of Sect. 3 for these means as follows: Replace $\eta(\varphi(y),\varphi(x),m)$ with $\eta(\varphi(y),\varphi(x))$ and setting $\eta(\varphi(y),\varphi(x)) = M(\varphi(x),\varphi(y))$, $\forall\, x,y \in I$ for value $m=1$ in (3.3) and (3.6), one can obtain the following interesting inequalities involving means:

$$
\begin{gathered}
|S_{f,g,M(\cdot,\cdot),\varphi}(x;\alpha,n,1,a,b)| = \Bigg| \frac{M^{\alpha}(\varphi(a),\varphi(x))}{M(\varphi(a),\varphi(b))} \\
\times \Big[B_{g(1)}(n+1,\alpha-n) f(\varphi(a)+g(1)M(\varphi(a),\varphi(x))) \Big] \\
- \frac{M^{\alpha}(\varphi(b),\varphi(x))}{M(\varphi(a),\varphi(b))} \\
\times \Big[B_{g(1)}(n+1,\alpha-n) f(\varphi(b)+g(1)M(\varphi(b),\varphi(x))) \Big] \\
- \frac{1}{M(\varphi(a),\varphi(b))} \\
\times \Bigg[\int_{\varphi(a)+g(0)M(\varphi(a),\varphi(x))}^{\varphi(a)+g(1)M(\varphi(a),\varphi(x))} (x-\varphi(a))^n (\varphi(a)+M(\varphi(a),\varphi(x))-x)^{\alpha-n-1} f(x)dx \\
- \int_{\varphi(b)+g(0)M(\varphi(b),\varphi(x))}^{\varphi(b)+g(1)M(\varphi(b),\varphi(x))} (x-\varphi(b))^n (\varphi(b)+M(\varphi(b),\varphi(x))-x)^{\alpha-n-1} f(x)dx \Bigg] \Bigg| \\
\leq \left(\frac{1}{s+1} \right)^{\frac{1}{q}} \frac{\delta^{\frac{1}{p}}(g(t);p)}{M(\varphi(a),\varphi(b))} \\
\times \Bigg\{ M^{\alpha+1}(\varphi(a),\varphi(x)) \Big[\Big((1-g(0))^{s+1} - (1-g(1))^{s+1} \Big) |f'(\varphi(a))|^q \\
+ (g^{s+1}(1) - g^{s+1}(0)) |f'(\varphi(x))|^q \Big]^{\frac{1}{q}} \\
+ M^{\alpha+1}(\varphi(b),\varphi(x)) \Big[\Big((1-g(0))^{s+1} - (1-g(1))^{s+1} \Big) |f'(\varphi(b))|^q \\
+ (g^{s+1}(1) - g^{s+1}(0)) |f'(\varphi(x))|^q \Big]^{\frac{1}{q}} \Bigg\}, \qquad (4.1)
\end{gathered}
$$

$$
\begin{gathered}
|S_{f,g,M(\cdot,\cdot),\varphi}(x;\alpha,n,1,a,b)| \\
\leq \left(\frac{1}{s+1} \right)^{\frac{1}{q}} \frac{\Big[g(1)B_{g(1)}(n+1,\alpha-n) - B_{g(1)}(n+2,\alpha-n) \Big]^{1-\frac{1}{q}}}{M(\varphi(a),\varphi(b))} \\
\times \Bigg\{ M^{\alpha+1}(\varphi(a),\varphi(x)) \\
\times \Big[\Big(B_{g(1)}(n+1,\alpha-n+s+1) - (1-g(1))^{s+1} B_{g(1)}(n+1,\alpha-n) \Big) |f'(\varphi(a))|^q \\
+ \Big(g^{s+1}(1) B_{g(1)}(n+1,\alpha-n) - B_{g(1)}(n+s+2,\alpha-n) \Big) |f'(\varphi(x))|^q \Big]^{\frac{1}{q}} \\
+ M^{\alpha+1}(\varphi(b),\varphi(x))
\end{gathered}
$$

$$\times \Big[\Big(B_{g(1)}(n+1, \alpha-n+s+1) - (1-g(1))^{s+1} B_{g(1)}(n+1, \alpha-n) \Big) |f'(\varphi(b))|^q$$
$$+ \Big(g^{s+1}(1) B_{g(1)}(n+1, \alpha-n) - B_{g(1)}(n+s+2, \alpha-n) \Big) |f'(\varphi(x))|^q \Big]^{\frac{1}{q}} \Bigg\}. \tag{4.2}$$

Letting $M(\varphi(x), \varphi(y)) := A, G, H, P_r, I, L, L_p, M_p, \forall\, x, y \in I$ in (4.1) and (4.2), we get the inequalities involving means for a particular choices of a differentiable generalized $(g, s, 1, \varphi)$-preinvex function $|f'|^q$. The details are left to the interested reader.

Acknowledgements. We thank anonymous referee for his/her valuable suggestion regarding the manuscript.

References

[1] T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math., **279** (2015), 57–66.
[2] T. Antczak, *Mean value in invexity analysis*, Nonlinear Anal., **60** (2005), 1473–1484.
[3] P. S. Bullen, *Handbook of Means and Their Inequalities*, Kluwer Academic Publishers, Dordrecht, 2003.
[4] S. S. Dragomir, J. Pečarić and L. E. Persson, *Some inequalities of Hadamard type*, Soochow J. Math., **21** (1995), 335–341.
[5] T. S. Du, J. G. Liao and Y. J. Li, *Properties and integral inequalities of Hadamard-Simpson type for the generalized (s, m)-preinvex functions*, J. Nonlinear Sci. Appl., **9** (2016), 3112–3126.
[6] H. Hudzik and L. Maligranda, *Some remarks on s-convex functions*, Aequationes Math., **48** (1994), 100–111.
[7] R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, *A new definition of fractional derivative*, J. Comput. Appl. Math., **264** (2014), 65–70.
[8] W. Liu, *New integral inequalities involving beta function via P-convexity*, Miskolc Math. Notes, **15**(2) (2014), 585–591.
[9] W. Liu, W. Wen and J. Park, *Ostrowski type fractional integral inequalities for MT-convex functions*, Miskolc Math. Notes, **16**(1) (2015), 249–256.
[10] W. Liu, W. Wen and J. Park, *Hermite-Hadamard type inequalities for MT-convex functions via classical integrals and fractional integrals*, J. Nonlinear Sci. Appl., **9** (2016), 766–777.
[11] M. E. Özdemir, E. Set and M. Alomari, *Integral inequalities via several kinds of convexity*, Creat. Math. Inform., **20**(1) (2011), 62–73.
[12] R. Pini, *Invexity and generalized convexity*, Optimization, **22** (1991), 513–525.
[13] E. Set and A. Gözpinar, *A study on Hermite-Hadamard type inequalities for s-convex functions via conformable fractional integrals*, Studia Universitatis Babes-Bolyai Matematica, **62**(3) (2017), 309–323.
[14] E. Set, A. O. Akdemir and I. Mumcu, *Ostrowski type inequalities for functions whose derivatives are convex via conformable fractional integrals*, Submitted.
[15] E. Set, A. O. Akdemir and I. Mumcu, *Chebyshev type inequalities for conformable fractional integrals*, Submitted.
[16] E. Set, M. Z. Sarikaya and A. Gözpinar, *Some Hermite-Hadamard type inequalities for convex functions via conformable fractional integrals and related inequalities*, Creat. Math. Inform., **26**(2) (2017), 221–229.
[17] E. Set and I. Mumcu, *Hermite-Hadamard-Fejér type inequalies for conformable fractional integrals*, Submitted.
[18] D. D. Stancu, G. Coman and P. Blaga, *Analiză numerică şi teoria aproximării*, Cluj-Napoca: Presa Universitară Clujeană., **2** (2002).
[19] M. Tunç *Ostrowski type inequalities for functions whose derivatives are MT-convex*, J. Comput. Anal. Appl., **17**(4) (2014), 691–696.
[20] X. M. Yang, X. Q. Yang and K. L. Teo, *Generalized invexity and generalized invariant monotonicity*, J. Optim. Theory Appl., **117** (2003), 607–625.

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