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**INEQUALITIES INVOLVING THE GAMMA AND DIGAMMA
FUNCTIONS**

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ABSTRACT. We improve the upper bounds of the following inequalities proved in [H. Alzer and N. Batir, Monotonicity properties of the gamma function, Appl. Math. Letters, 20(2007), 778-781].

$$\exp\left(-\frac{1}{2}\psi\left(x + \frac{1}{3}\right)\right) < \frac{\Gamma(x)}{\sqrt{2\pi x}e^{-x}} < \exp\left(-\frac{1}{2}\psi(x)\right),$$

and

$$\frac{1}{2}\psi'(x + 1/3) < \log x - \psi(x) < \frac{1}{2}\psi'(x).$$

Here Γ is the classical gamma function and ψ is the digamma function.

1. INTRODUCTION

As it is well known the gamma function is defined by the improper integral

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$$

for all complex numbers z except the non-positive integers, and provides an extension of the factorial function. It is well known that it is one of the most important special functions and has very important applications in probability theory, combinatorics, statistical and quantum mechanics, number theory, and nuclear physics. It satisfies the fundamental functional equation $\Gamma(z + 1) = z\Gamma(z)$. The Weierstrass' product form of it is given by

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}, \quad (1.1)$$

where

$$\gamma = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right) = 0.57721\dots$$

is the Euler-Mascheroni constant, see [9, pg. 346]. The most important function related to the gamma function is the digamma or psi function, which is defined by the logarithmic

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derivative of Γ , that is, $\psi(x) = \Gamma'(x)/\Gamma(x)$, $x > 0$. Furthermore, the derivatives ψ', ψ'', \dots , are called polygamma functions. The digamma and polygamma functions are also very important special functions and they have important applications in mathematics and other disciplines such as physics and statistics. They are also connected to generalized harmonic numbers and many other special functions such as the Riemann zeta, Hurwitz zeta and Clausen functions. Over the decades, many mathematicians studied these functions and they obtained remarkable inequalities, interesting properties including monotonicity and convexity, please see [2, 4, 7, 8, 12–22] and references therein. Taking the logarithm of both sides of identity (1.1), we obtain for $x > -1$

$$\log \Gamma(x+1) = -\gamma x + \sum_{k=1}^{\infty} \left[\frac{x}{k} - \log(x+k) + \log k \right]. \quad (1.2)$$

Differentiation gives

$$\psi(x+1) = -\gamma + \sum_{k=1}^{\infty} \left[\frac{1}{k} - \frac{1}{k+x} \right]. \quad (1.3)$$

In [3] the authors proved that the function

$$G_c(x) = \log \Gamma(x) - x \log x + x - \frac{1}{2} \log(2\pi) + \frac{1}{2} \psi(x+c)$$

is completely monotonic on $(0, \infty)$ if and only if $c \geq 1/3$; while $-G_b(x)$ is completely monotonic on $(0, \infty)$ if and only if $b = 0$. This result has many conclusions. For example, $G_{1/3}(x) > 0$ and $G_0(x) < 0$ lead to

$$\exp\left(-\frac{1}{2}\psi\left(x+\frac{1}{3}\right)\right) < \frac{\Gamma(x)}{\sqrt{2\pi x^x e^{-x}}} < \exp\left(-\frac{1}{2}\psi(x)\right). \quad (1.4)$$

Similarly, $G'_{1/3}(x) < 0$ and $G'_0(x) > 0$ lead to

$$\frac{1}{2}\psi'(x+1/3) < \log x - \psi(x) < \frac{1}{2}\psi'(x). \quad (1.5)$$

In 2011 C. Mortici [11] improved the upper and lower bounds given in (1.4) and proved the following inequalities for $x \geq 2$:

$$\begin{aligned} \exp\left(-\frac{1}{2}\psi\left(x+\frac{1}{3}\right) + \frac{1}{72x^2}\right) &< \frac{\Gamma(x)}{\sqrt{2\pi x^x e^{-x}}} \\ &< \exp\left(-\frac{1}{2}\psi\left(x+\frac{1}{3}\right) + \frac{1}{72x^2} + \frac{11}{3240x^3}\right). \end{aligned} \quad (1.6)$$

The lower bounds in (1.4) and (1.5) are very accurate but the same thing is not valid for the upper bounds. In the new paper [5] the author improved the upper bound given in (1.4) and proved for $x > 0$ that

$$\exp\left(-\frac{1}{2}\psi(\delta_*(x))\right) < \frac{\Gamma(x)}{\sqrt{2\pi x^x e^{-x}}} < \exp\left(-\frac{1}{2}\psi(\delta^*(x))\right), \quad (1.7)$$

where

$$\delta_*(x) = x + \frac{1}{3} \quad \text{and} \quad \delta^*(x) = \frac{1}{2} \frac{1}{(x+1) \log(1+1/x) - 1}.$$

In the same paper the author established the following inequalities

$$\exp\left(x\psi\left(\frac{x}{\log(x+1)}\right)\right) < \Gamma(x+1) < \exp\left(x\psi\left(\frac{x}{2}+1\right)\right). \tag{1.8}$$

The function $x \rightarrow \log(x) - \psi(x)$ in (1.5) has attracted the attentions of many mathematicians and they have offered bounds of different forms for it. For example, in [10] the authors developed the following inequality for $x > 1$:

$$\frac{1}{2x} < \log x - \psi(x) < \frac{1}{x}. \tag{1.9}$$

In [1] the author extended it to $x > 0$. Refinements of (1.9) were given in [6, Thm. 5] as following:

$$\frac{1}{2x} + \frac{1}{12(x+1/4)^2} < \log x - \psi(x) < \frac{1}{2x} + \frac{1}{12x^2}$$

and

$$\frac{1}{2x} + \frac{1}{12x^2} - \frac{1}{12x^4} < \log x - \psi(x) < \frac{1}{2x} + \frac{1}{12x^2} - \frac{1}{120(x+1/8)^4}.$$

Our aim in this work is to improve the upper bounds given in (1.4) and (1.5). We also offer simpler lower bound in (1.8).

In the proofs we need the following identities. Taking logarithm of both sides of identity (1.1), we obtain for $x > -1$

$$\log \Gamma(x+1) = -\gamma x + \sum_{k=1}^{\infty} \left[\frac{x}{k} - \log(x+k) + \log k \right]. \tag{1.10}$$

Differentiation gives

$$\psi(x+1) = -\gamma + \sum_{k=1}^{\infty} \left[\frac{1}{k} - \frac{1}{k+x} \right] \quad x > -1. \tag{1.11}$$

Differentiation successively gives for $n \in \mathbb{N}$ and $z \in \mathbb{C}$, which is not a negative integer,

$$\psi^{(n)}(z) = (-1)^{n-1} n! \sum_{k=0}^{\infty} \frac{1}{(z+k)^{n+1}}.$$

2. MAIN RESULTS

Theorem 2.1. *For $x > 0$ the following inequalities hold:*

$$\frac{1}{2}\psi'(\alpha(x)) < \log x - \psi(x) < \frac{1}{2}\psi'(\beta(x)), \tag{2.1}$$

where

$$\alpha(x) = x + \frac{1}{3} \quad \text{and} \quad \beta(x) = \frac{1}{\sqrt{2/x - 2\log(1+1/x)}}.$$

Proof. If we use the functional equation $\Gamma(x+1) = x\Gamma(x)$, we get

$$\log x = \log \Gamma(x+1) - \log \Gamma(x).$$

Hence, if we employ (1.2), we obtain

$$\log x = -\gamma + \sum_{k=1}^{\infty} \left[\frac{1}{k} - \log(x+k) + \log(x+k-1) \right]. \tag{2.2}$$

From Taylor's theorem it follows that

$$\log(x+k) - \log(x+k-1) = \frac{1}{x+k-1} - \frac{1}{2(k+\tau(k))^2}, \quad (2.3)$$

where $x-1 < \tau(k) < x$. Thus by the help of (2.3), (2.2) can be written as following:

$$\log x = -\gamma + \sum_{k=1}^{\infty} \left[\frac{1}{k} - \frac{1}{x+k-1} + \frac{1}{2(k+\tau(k))^2} \right]. \quad (2.4)$$

Taking into account (1.11), (2.4) becomes

$$\log x - \psi(x) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{(k+\tau(k))^2}. \quad (2.5)$$

From (2.3) we get

$$\tau(k) = \left[\frac{2}{x+k-1} - 2 \log \left(1 + \frac{1}{x+k-1} \right) \right]^{-\frac{1}{2}} - k. \quad (2.6)$$

We shall show that τ is strictly increasing on $(1, \infty)$. For this purpose we define

$$f(u) = \left[2 \left(\frac{1}{u} - \log \left(1 + \frac{1}{u} \right) \right) \right]^{-\frac{1}{2}} - u.$$

Differentiating f , after replacing u by $1/t$, we get

$$-\frac{1}{t^2} f'(t) = -\frac{t}{t+1} 2^{-3/2} (h(t))^{-3/2} + \frac{1}{t^2},$$

where $h(t) = t - \log(t+1)$. Therefore in order to show that τ is strictly increasing, we only need to see

$$\theta(t) := h(t) - \frac{t^2}{2(t+1)^{2/3}} < 0.$$

Differentiation gives

$$\begin{aligned} \theta'(t) &= \frac{t}{t+1} - \frac{t(t+1)^{2/3} - t^2(t+1)^{-1/3}/3}{(t+1)^{4/3}} \\ &= \frac{t}{(t+1)^{5/3}} \left[(t+1)^{2/3} - 1 - \frac{2t}{3} \right] < 0 \end{aligned}$$

by the well known Bernoulli inequality $(1+x)^\delta < 1 + \delta x$, $0 < \delta < 1$ and $t > 0$. Thus, we have $\theta(t) > \theta(0) = 0$, that is, f is strictly increasing on $(0, \infty)$. Hence we conclude from (2.6) that

$$\frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{(k+\tau(\infty))^2} < \log x - \psi(x) < \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{(k+\tau(1))^2}. \quad (2.7)$$

From (2.3) we have

$$\tau(1) = \left[\frac{2}{x} - 2 \log(1+1/x) \right]^{-\frac{1}{2}} - 1. \quad (2.8)$$

Applying L'Hospital rule it is not difficult to see that

$$\tau(\infty) = \lim_{k \rightarrow \infty} \tau(k) = \lim_{t \rightarrow 0} f(1/t) + x - 1 = x + \frac{1}{3} - 1. \quad (2.9)$$

Using (2.8) and (2.9) the proof follows from (2.7). \square

The upper bound in (1.7) is very accurate but it is not very useful in practise because of its structure. The following theorem gives much simpler upper bound, which clearly improves the upper bound of (2.1), and has the advantages of simplicity.

Theorem 2.2. *For $x \geq 0$ we have*

$$\frac{1}{2}\psi' \left(x + \frac{1}{3} \right) < \log x - \psi(x) < \frac{1}{2}\psi' \left(x + \frac{1}{3} - \frac{1}{12x+3} \right). \quad (2.10)$$

Proof. Since ψ' is strictly decreasing on $(0, \infty)$, in the light of Theorem 2.1, it suffices to show for $x > 0$ that

$$\frac{1}{\sqrt{\frac{2}{x} - 2 \log(1 + 1/x)}} > x + \frac{1}{3} - \frac{1}{12x+3} \quad (2.11)$$

or simplifying

$$H(x) := \log(1 + 1/x) - \frac{1}{x} + \frac{1}{2 \left(x + \frac{1}{3} - \frac{1}{12x+3} \right)^2} > 0.$$

Differentiation gives

$$H'(x) = -\frac{63 + 188x}{x^3(x+1)(12x+7)^3},$$

so that H is strictly decreasing on $(0, \infty)$. This concludes with $H(x) > \lim_{x \rightarrow \infty} H(x) = 0$. \square

Our next theorem improves the upper bound given in (1.4).

Theorem 2.3. *For $x > 0$ we have*

$$\exp \left(-\frac{1}{2}\psi \left(x + \frac{1}{3} \right) \right) < \frac{\Gamma(x)}{\sqrt{2\pi x^x e^{-x}}} < \exp \left(-\frac{1}{2}\psi \left(x + \frac{1}{3} - \frac{1}{18x+3} \right) \right).$$

Proof. Since the function $e^{-\psi(x)}$ is strictly decreasing on $(0, \infty)$, in the light of (1.7), the only thing we need to show for $x > 0$ is the following inequality:

$$\frac{1}{2} \frac{1}{(x+1) \log(1 + 1/x) - 1} > x + \frac{1}{3} - \frac{1}{18x+3}$$

or equivalently

$$P(x) := \log(1 + 1/x) - \frac{1 + 12x + 12x^2}{6x(x+1)(2x+1)} < 0.$$

Differentiation gives

$$P'(x) = \frac{1}{6x^2(1+x)^2(1+2x)^2} > 0.$$

Thus, P is strictly increasing on $(0, \infty)$, which implies that $P(x) < \lim_{x \rightarrow \infty} P(x) = 0$. \square

Theorem 2.4. *For all $x > 0$, we have*

$$\exp \left(x\psi \left(\frac{x}{2} + 1 - \frac{x^2}{12+2x} \right) \right) < \Gamma(x+1) < \exp \left(x\psi \left(\frac{x}{2} + 1 \right) \right). \quad (2.12)$$

Proof. Since the function ψ is an increasing function, taking into account (1.8), it suffices to see

$$\frac{x}{2} + 1 - \frac{x^2}{12 + 2x} < \frac{x}{\log(x + 1)} \quad (2.13)$$

or after a little simplification,

$$p(x) := \log(x + 1) - \frac{x^2 + 6x}{4x + 6} > 0. \quad (2.14)$$

Differentiating, we get

$$p'(x) = -\frac{x^2}{(x + 1)(3 + 2x)^2},$$

so that p is strictly decreasing on $(0, \infty)$. Thus we have $p(x) > \lim_{x \rightarrow \infty} p(x) = 0$. \square

Remark 2.1. The upper bounds in (1.6) and (1.9) are slightly better than those of (2.3) and (2.5), but the upper bounds in (2.3) and (2.5) have the advantage of simplicity.

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