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SOME INEQUALITIES FOR $GG-$CONVEX FUNCTIONS

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Abstract. We establish some new integral inequalities for $GG-$convex functions by using an integral equality which is proved by the authors in [11]. We also give some new estimations for special means of real numbers.

1. Introduction

A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on I, if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

The Hermite-Hadamard inequality gives us upper and lower bounds for the mean-value of a convex function which is given as (See [12]):

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

Some recent results, generalizations and improvements see the papers [13–17].

Anderson *et al.* gave the following definition in [4]:

Definition 1.1. A function $M : (0,\infty) \times (0,\infty) \to (0,\infty)$ is called a Mean function if

(1) $M(x,y) = M(y,x)$,
(2) $M(x,x) = x$,
(3) $x < M(x,y) < y$, whenever $x < y$,
(4) $M(ax,ay) = aM(x,y)$ for all $a > 0$.

Based on the definition of mean function, let us recall special means (See [4])

1. Arithmetic Mean: $M(x,y) = A(x,y) = \frac{x+y}{2}$.
2. Geometric Mean: $M(x,y) = G(x,y) = \sqrt{xy}$.
3. Harmonic Mean: $M(x,y) = H(x,y) = 1/A\left(\frac{1}{x}, \frac{1}{y}\right)$.

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4. Logarithmic Mean: $M(x,y) = L(x,y) = (x-y)/(\log x - \log y)$ for $x \neq y$ nd $L(x,x) = x$.

5. Identric Mean: $M(x,y) = I(x,y) = (1/e)\left(x^x/y^y\right)^{1/(x-y)}$ for $x \neq y$ nd $I(x,x) = x$.

In [4], Anderson *et al.* also defined generalized convexity as follows:

Definition 1.2. Let $f : I \to (0,\infty)$ be continuous, where I is subinterval of $(0,\infty)$. Let M and N be any two Mean functions. We say f is MN-convex (concave) if

$$f(M(x,y)) \leq (\geq) N(f(x), f(y))$$

for all $x, y \in I$.

Recall the definitions of $AG-$convex functions, $GG-$convex functions and $GA-$convex functions that are given in [2] by Niculescu:

The $AG-$convex functions (usually known as $\log -$convex functions) are those functions $f : I \to (0,\infty)$ for which

$$x, y \in I \text{ and } \lambda \in [0,1] \Longrightarrow f(\lambda x + (1-\lambda)y) \leq f(x)^{1-\lambda} f(y)^{\lambda}, \tag{1.1}$$

i.e., for which $\log f$ is convex.

The $GG-$convex functions (called in what follows multiplicatively convex functions) are those functions $f : I \to J$ (acting on subintervals of $(0,\infty)$) such that

$$x, y \in I \text{ and } \lambda \in [0,1] \Longrightarrow f\left(x^{1-\lambda} y^{\lambda}\right) \leq f(x)^{1-\lambda} f(y)^{\lambda}. \tag{1.2}$$

The class of all $GA-$convex functions is constituted by all functions $f : I \to \mathbb{R}$ (defined on subintervals of $(0,\infty)$) for which

$$x, y \in I \text{ and } \lambda \in [0,1] \Longrightarrow f\left(x^{1-\lambda} y^{\lambda}\right) \leq f(x)^{1-\lambda} + f(y)^{\lambda}. \tag{1.3}$$

Besides, recall that the condition of $GG-$convexity is given in the following theorem by Anderson *et al.* in [4].

Theorem 1.1. *Let I be an open interval of $(0,\infty)$ and let $f : I \to (0,\infty)$ be differentiable. In parts $(4)-(9)$, let $I = (0,b)$, $0 < b < \infty$.*

(1) f is $AA-$convex (concave) if and only if $f'(x)$ is increasing (decreasing).

(2) f is $AG-$convex (concave) if and only if $\frac{f'(x)}{f(x)}$ is increasing (decreasing).

(3) f is $AH-$convex (concave) if and only if $\frac{f'(x)}{f(x)^2}$ is increasing (decreasing).

(4) f is $GA-$convex (concave) if and only if $xf'(x)$ is increasing (decreasing).

(5) f is $GG-$convex (concave) if and only if $\frac{xf'(x)}{f(x)}$ is increasing (decreasing).

(6) f is $GH-$convex (concave) if and only if $\frac{xf'(x)}{f(x)^2}$ is increasing (decreasing).

(7) f is $HA-$convex (concave) if and only if $x^2 f'(x)$ is increasing (decreasing).

(8) f is $HG-$convex (concave) if and only if $\frac{x^2 f'(x)}{f(x)}$ is increasing (decreasing).

(9) f is $HH-$convex (concave) if and only if $\frac{x^2 f'(x)}{f(x)^2}$ is increasing (decreasing).

In [11], authors proved the following lemma and established new inequalities of Hermite-Hadamard type for $GA-$convex functions:

Lemma 1.1. *Let $f : I \subset \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ be a differentiable mapping on I^o and $a, b \in I^o$ with $a < b$. If $f' \in L[a, b]$, then the following identity holds:*

$$\begin{aligned} & bf(b) - af(a) - \int_a^b f(u)\,du \\ = \ & (\ln x - \ln a) \int_0^1 x^{2t} a^{2(1-t)} f'\left(x^t a^{1-t}\right) dt - (\ln x - \ln b) \int_0^1 x^{2t} b^{2(1-t)} f'\left(x^t b^{1-t}\right) dt \end{aligned}$$

for all $x \in [a, b]$.

For recent results, generalizations, improvements and counterparts see the papers [1–10] and references therein.

The main aim of this paper is to prove some new integral inequalities for $GG-$convex functions by using the above integral identity. Also some applications to special means are given.

2. Main Results

Theorem 2.1. *Let $f : I \subset \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ be a differentiable mapping on I^o, $a, b \in I^o$ with $a < b$ and $f' \in L[a, b]$. If $|f'(x)|$ is $GG-$convex function on $[a, b]$, then one has the following inequality:*

$$\begin{aligned} & \left| bf(b) - af(a) - \int_a^b f(u)\,du \right| \\ \leq \ & (\ln x - \ln a)\, L\left(a^2 |f'(a)|, x^2 |f'(x)|\right) \\ & + (\ln b - \ln x)\, L\left(x^2 |f'(x)|, b^2 |f'(b)|\right) \end{aligned}$$

for all $x \in [a, b]$.

Proof. From Lemma 1.1 and by using the $GG-$convexity of $|f'(x)|$, we have

$$\begin{aligned} & \left| bf(b) - af(a) - \int_a^b f(u)\,du \right| \\ \leq \ & (\ln x - \ln a) \int_0^1 x^{2t} a^{2(1-t)} f'\left(x^t a^{1-t}\right) dt \\ & + (\ln b - \ln x) \int_0^1 x^{2t} b^{2(1-t)} f'\left(x^t b^{1-t}\right) dt \end{aligned}$$

$$\leq (\ln x - \ln a) \int_0^1 x^{2t} a^{2(1-t)} \left[|f'(x)|^t |f'(a)|^{1-t} \right] dt + (\ln b - \ln x) \int_0^1 x^{2t} b^{2(1-t)} \left[|f'(x)|^t |f'(b)|^{1-t} \right] dt.$$

By a simple computation, we deduce

$$\begin{aligned} & \left| bf(b) - af(a) - \int_a^b f(u)\,du \right| \\ \leq\ & (\ln x - \ln a) \left(\frac{x^2 |f'(x)| - a^2 |f'(a)|}{\ln(x^2 |f'(x)|) - \ln(a^2 |f'(a)|)} \right) \\ & + (\ln b - \ln x) \left(\frac{b^2 |f'(b)| - x^2 |f'(x)|}{\ln(b^2 |f'(b)|) - \ln(x^2 |f'(x)|)} \right). \end{aligned}$$

Which completes the proof. $\square$

Theorem 2.2. *$f : I \subset \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ be a differentiable mapping on I^o, $a, b \in I^o$ with $a < b$ and $f' \in L[a, b]$. If $|f'(x)|^q$ is $GG-$convex function on $[a, b]$, then the following inequality holds:*

$$\begin{aligned} & \left| bf(b) - af(a) - \int_a^b f(u)\,du \right| \\ \leq\ & (\ln x - \ln a) L^{1-\frac{1}{q}} \left(a^2, x^2 \right) L^{\frac{1}{q}} \left(a^2 |f'(a)|^q, x^2 |f'(x)|^q \right) \\ & + (\ln b - \ln x) L^{1-\frac{1}{q}} \left(x^2, b^2 \right) L^{\frac{1}{q}} \left(x^2 |f'(x)|^q, b^2 |f'(b)|^q \right) \end{aligned}$$

for all $x \in [a, b]$ and $q \geq 1$.

Proof. From Lemma 1.1, by using the $GG-$convexity of $|f'(x)|$ and by Hölder integral inequality, we have

$$\begin{aligned} & \left| bf(b) - af(a) - \int_a^b f(u)\,du \right| \\ \leq\ & (\ln x - \ln a) \int_0^1 x^{2t} a^{2(1-t)} \left| f'\left(x^t a^{1-t} \right) \right| dt \\ & + (\ln b - \ln x) \int_0^1 x^{2t} b^{2(1-t)} \left| f'\left(x^t b^{1-t} \right) \right| dt \end{aligned}$$

$$\leq (\ln x-\ln a)\left(\int_0^1 x^{2t}a^{2(1-t)}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 x^{2t}a^{2(1-t)}\left[\left|f'(x)\right|^{qt}\left|f'(a)\right|^{q(1-t)}\right]dt\right)^{\frac{1}{q}}$$
$$+(\ln b-\ln x)\left(\int_0^1 x^{2t}b^{2(1-t)}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 x^{2t}b^{2(1-t)}\left[\left|f'(x)\right|^{qt}\left|f'(b)\right|^{q(1-t)}\right]dt\right)^{\frac{1}{q}}.$$

By making use of the necessary computation, we get

$$\left|bf(b)-af(a)-\int_a^b f(u)\,du\right|$$
$$\leq (\ln x-\ln a)^{\frac{1}{q}}\left(\frac{x^2-a^2}{2}\right)^{1-\frac{1}{q}}\left(\frac{x^2\left|f'(x)\right|^q-a^2\left|f'(a)\right|^q}{\ln\left(x^2\left|f'(x)\right|^q\right)-\ln\left(a^2\left|f'(a)\right|^q\right)}\right)^{\frac{1}{q}}$$
$$+(\ln b-\ln x)^{\frac{1}{q}}\left(\frac{b^2-x^2}{2}\right)^{1-\frac{1}{q}}\left(\frac{b^2\left|f'(b)\right|^q-x^2\left|f'(x)\right|^q}{\ln\left(b^2\left|f'(b)\right|^q\right)-\ln\left(x^2\left|f'(x)\right|^q\right)}\right)^{\frac{1}{q}}.$$

This completes the proof. □

Theorem 2.3. *Let $f: I\subset\mathbb{R}_+=(0,\infty)\to\mathbb{R}$ be a differentiable mapping on I^o, $a,b\in I^o$ with $a<b$ and $f'\in L[a,b]$. If $\left|f'(x)\right|^q$ is $GG-$convex function on $[a,b]$, then the following inequality holds:*

$$\left|bf(b)-af(a)-\int_a^b f(u)\,du\right|$$
$$\leq (\ln x-\ln a)\,L^{1-\frac{1}{q}}\left(a^{\frac{2q}{q-1}},x^{\frac{2q}{q-1}}\right)L^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(x)\right|^q\right)$$
$$+(\ln b-\ln x)\,L^{1-\frac{1}{q}}\left(x^{\frac{2q}{q-1}},b^{\frac{2q}{q-1}}\right)L^{\frac{1}{q}}\left(\left|f'(x)\right|^q,\left|f'(b)\right|^q\right)$$

for all $x\in[a,b]$ and $q>1$.

Proof. Since $\left|f'(x)\right|$ is $GG-$convex function on $[a,b]$, from Lemma 1.1 and by using Hölder integral inequality, we can write

$$\left|bf(b)-af(a)-\int_a^b f(u)\,du\right|$$
$$\leq (\ln x-\ln a)\int_0^1 x^{2t}a^{2(1-t)}\left|f'\left(x^ta^{1-t}\right)\right|dt$$
$$+(\ln b-\ln x)\int_0^1 x^{2t}b^{2(1-t)}\left|f'\left(x^tb^{1-t}\right)\right|dt$$

$$\leq a^2(\ln x-\ln a)\left(\int_0^1\left(\frac{x}{a}\right)^{\frac{2qt}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left[\left|f'(x)\right|^{qt}\left|f'(a)\right|^{q(1-t)}\right]dt\right)^{\frac{1}{q}}$$
$$+b^2(\ln b-\ln x)\left(\int_0^1\left(\frac{x}{b}\right)^{\frac{2qt}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left[\left|f'(x)\right|^{qt}\left|f'(b)\right|^{q(1-t)}\right]dt\right)^{\frac{1}{q}}.$$

By a simple computation, we have

$$\left|bf(b)-af(a)-\int_a^b f(u)\,du\right|$$
$$\leq (\ln x-\ln a)\left(\frac{x^{\frac{2q}{q-1}}-a^{\frac{2q}{q-1}}}{\ln x^{\frac{2q}{q-1}}-\ln a^{\frac{2q}{q-1}}}\right)^{1-\frac{1}{q}}\left(\frac{\left|f'(x)\right|^q-\left|f'(a)\right|^q}{\ln\left|f'(x)\right|^q-\ln\left|f'(a)\right|^q}\right)^{\frac{1}{q}}$$
$$+(\ln b-\ln x)\left(\frac{b^{\frac{2q}{q-1}}-x^{\frac{2q}{q-1}}}{\ln b^{\frac{2q}{q-1}}-\ln x^{\frac{2q}{q-1}}}\right)^{1-\frac{1}{q}}\left(\frac{\left|f'(b)\right|^q-\left|f'(x)\right|^q}{\ln\left|f'(b)\right|^q-\ln\left|f'(x)\right|^q}\right)^{\frac{1}{q}}.$$

This last inequality completes the proof. □

Theorem 2.4. *Let $f : I \subset \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I^o, $a, b \in I^o$ with $a < b$ and $f' \in L[a,b]$. If $|f'(x)|^q$ is $GG-$convex function on $[a,b]$, then the following inequality holds:*

$$\left|bf(b)-af(a)-\int_a^b f(u)\,du\right|$$
$$\leq (\ln x-\ln a)\,L^{\frac{1}{q}}\left(\left(a^2\left|f'(a)\right|\right)^q,\left(x^2\left|f'(x)\right|\right)^q\right)$$
$$+(\ln b-\ln x)\,L^{\frac{1}{q}}\left(\left(x^2\left|f'(x)\right|\right)^q,\left(b^2\left|f'(b)\right|\right)^q\right)$$

for all $x \in [a,b]$ and $q \geq 1$.

Proof. By a similar argument to the proof of previous theorem, since $|f'(x)|$ is $GG-$convex function on $[a,b]$, from Lemma 1.1 and by using a version of Hölder integral inequality, we have

$$\left|bf(b)-af(a)-\int_a^b f(u)\,du\right|$$
$$\leq (\ln x-\ln a)\int_0^1 x^{2t}a^{2(1-t)}\left|f'\left(x^t a^{1-t}\right)\right|dt$$
$$+(\ln b-\ln x)\int_0^1 x^{2t}b^{2(1-t)}\left|f'\left(x^t b^{1-t}\right)\right|dt$$

$$\leq a^2(\ln x-\ln a)\left(\int_0^1 dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{x}{a}\right)^{2qt}\left[\left|f'(x)\right|^{qt}\left|f'(a)\right|^{q(1-t)}\right]dt\right)^{\frac{1}{q}}$$

$$+b^2(\ln b-\ln x)\left(\int_0^1 dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{x}{b}\right)^{2qt}\left[\left|f'(x)\right|^{qt}\left|f'(b)\right|^{q(1-t)}\right]dt\right)^{\frac{1}{q}}.$$

By computing the above integrals, we deduce

$$\left|bf(b)-af(a)-\int_a^b f(u)\,du\right|$$

$$\leq (\ln x-\ln a)\left[\frac{(x^2|f'(x)|)^q-(a^2|f'(a)|)^q}{\ln(x^2|f'(x)|)^q-\ln(a^2|f'(a)|)^q}\right]^{\frac{1}{q}}$$

$$+(\ln b-\ln x)\left[\frac{(b^2|f'(b)|)^q-(x^2|f'(x)|)^q}{\ln(b^2|f'(b)|)^q-\ln(x^2|f'(x)|)^q}\right]^{\frac{1}{q}}.$$

This last inequality completes the proof. □

3. Applications to Special Means

Let us recall the special means of two nonnegative real numbers a,b with $a<b$:

a) The arithmetic mean:

$$A=A(a,b):=\frac{a+b}{2},\quad a,b\geq 0,$$

b) The geometric mean:

$$G=G(a,b):=\sqrt{ab},\quad a,b\geq 0,$$

c) The harmonic mean:

$$H=H(a,b):=\frac{2ab}{a+b},\quad a,b\geq 0,$$

d) The logarithmic mean:

$$L=L(a,b):=\begin{cases} a & \text{if } \quad a=b \\ \frac{b-a}{\ln b-\ln a} & \text{if } \quad a\neq b \end{cases},\quad a,b\geq 0,$$

e) The Identric mean.

$$I=I(a,b):=\begin{cases} a & \text{if } \quad a=b \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } \quad a\neq b \end{cases},\quad a,b\geq 0,$$

f) The $p-$logarithmic mean:

$$L_p=L_p(a,b):=\begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p} & \text{if } \quad a\neq b \\ a & \text{if } \quad a=b \end{cases},\quad p\in\mathbb{R}\backslash\{-1,0\};\quad a,b>0.$$

The following inequality is well known in the literature:

$$H \leq G \leq L \leq I \leq A$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$ (See [12]).

The following propositions hold for our main results:

Proposition 3.1. *Suppose that $a, b \in \mathbb{R}_+$ and $s > 0$. Then, we have*

$$\begin{aligned} &\left|(b-a) L_{s+1}^{s+1}(a,b)\right| \\ \leq\; &\frac{(x-a)}{L(a,x)} L\left(a^{s+2}, x^{s+2}\right) + \frac{(b-x)}{L(x,b)} L\left(x^{s+2}, b^{s+2}\right) \end{aligned} \tag{3.1}$$

for all $x \in [a,b]$.

Proof. The proof is follows from Theorem 2.1 by applying $f(x) = \frac{x^{s+1}}{s+1}$, $x \in \mathbb{R}_+$, $s > 0$, where $|f'(x)|$ is $GG-$convex function. □

Proposition 3.2. *Suppose that $a, b \in \mathbb{R}_+$ and $s > 0$. Then for all $q \geq 1$, one has the inequality*

$$\begin{aligned} &\left|(b-a) L_{s+1}^{s+1}(a,b)\right| \\ \leq\; &\frac{(x-a)}{L(a,x)} L^{1-\frac{1}{q}}\left(a^2, x^2\right) L^{\frac{1}{q}}\left(a^{sq+2}, x^{sq+2}\right) \\ &+\frac{(b-x)}{L(x,b)} L^{1-\frac{1}{q}}\left(x^2, b^2\right) L^{\frac{1}{q}}\left(x^{sq+2}, b^{sq+2}\right) \end{aligned} \tag{3.2}$$

for all $x \in [a,b]$.

Proof. The proof is immediate from Theorem 2.2 applied for $f(x) = \frac{x^{s+1}}{s+1}$, $x \in \mathbb{R}_+$, $s > 0$ where $|f'(x)|^q$ is $GG-$convex function. □

Proposition 3.3. *Suppose that $a, b \in \mathbb{R}_+$ and $s > 0$. Then for all $q > 1$, we have*

$$\begin{aligned} &\left|(b-a) L_{s+1}^{s+1}(a,b)\right| \\ \leq\; &\frac{(x-a)}{L(a,x)} L^{1-\frac{1}{q}}\left(a^{\frac{2q}{q-1}}, x^{\frac{2q}{q-1}}\right) L^{\frac{1}{q}}\left(a^{sq}, x^{sq}\right) \\ &+\frac{(b-x)}{L(x,b)} L^{1-\frac{1}{q}}\left(x^{\frac{2q}{q-1}}, b^{\frac{2q}{q-1}}\right) L^{\frac{1}{q}}\left(x^{sq}, b^{sq}\right) \end{aligned}$$

Proof. The proof is immediate from Theorem 2.3 applied for $f(x) = \frac{x^{s+1}}{s+1}$, $x \in \mathbb{R}_+$, $s > 0$ where $|f'(x)|^q$ is $GG-$convex function. □

Proposition 3.4. *Suppose that $a, b \in \mathbb{R}_+$ and $s > 0$. Then for all $q \geq 1$, we have*

$$\left|(b-a) L_{s+1}^{s+1}(a,b)\right| \leq \frac{(x-a)}{L(a,x)} L^{\frac{1}{q}}\left(a^{sq+2}, x^{sq+2}\right) + \frac{(b-x)}{L(x,b)} L^{\frac{1}{q}}\left(x^{sq+2}, bx^{sq+2}\right) \tag{3.3}$$

Proof. It is easy to see that by applying $f(x) = \frac{x^{s+1}}{s+1}$ to Theorem 2.4, $x \in \mathbb{R}_+$, $s > 0$, where $|f'(x)|^q$ is $GG-$convex function. □

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