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EXTENDED JENSEN'S TYPE INEQUALITIES FOR DIAMOND INTEGRALS VIA TAYLOR'S FORMULA

RABIA BIBI[1], AMMARA NOSHEEN[2], AND JOSIP PEČARIĆ[3]

Abstract. In the paper several extensions to inequalities of Jensen type are given for κ-convex functions, including Jensen's inequality, and converse to it for diamond integrals. It is also mentioned that the obtained results are also true for Jensen–Steffensen inequality and its converse respectively. These extensions are obtained by using Taylor's formula. Finally an improved Hölder type inequality is obtained as an application improved Jensen inequality.

1. Introduction

Inequality proved by Jensen is popular in mathematical analysis. It is used to establish many classical inequalities both for continuous and discrete analysis, so the developements in Jensen's inequality leads to the developments in many other inequalities. There are many versions of Jensen's inequality and its converse, including Jensen–Steffensen inequality, their generalizations and refinements, which can be found in the literature (see [9–11, 13]).

The time scales theory is proposed by Hilger in 1988 and it is well developed in [4, 5] by M. Bohner and A. Peterson. Delta and nabla calculus are intial approaches to study time scales calculus. Q. Sheng et al., [14] introduced alpha diamond derivative and integral as convex combination of delta derivative and integral, which gives more balanced approximation in computations. A. M. C. Brito da Cruz, N. Martins, D. F. M. Torres have defined approximately symmetric derivative and integral (generalized form of alpha diamond derivative and integral), called diamond derivative and integral, in [6, 7] respectively.

The authors of the paper have established Jensen inequality and Jensen Steffensen inequality for diamond integrals in [3, 12] respectively. Now present paper is set to improve respective Jensen's functional, Jensen Steffensen functional, their converses and related bounds with the help of Taylor formula. Finally improved Jensen functional is applied to find Hölder type inequality as an application.

Key words and phrases. Diamond integrals, Jensen's inequality, Jensen–Steffensen inequality, Taylor's formula.

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2. Preliminary Results

A time scale $\mathbb{T}$ is considered to be a closed, non empty subset of real line. For $c_1, c_2 \in \mathbb{T}$ $(c_1 < c_2)$, denote a time scale interval by $[c_1, c_2]^{\mathbb{T}} = [c_1, c_2] \bigcap \mathbb{T}$. To classify the points backward, respectively forward operators $\rho, \sigma : \mathbb{T} \to \mathbb{T}$ are:

$$\rho(t) = \sup\{\hat{\upsilon} \in \mathbb{T} : \ \hat{\upsilon} < \upsilon\} \quad \text{and} \quad \sigma(\upsilon) = \inf\{\hat{\upsilon} \in \mathbb{T} : \ \hat{\upsilon} > \upsilon\}.$$

The point υ is left-scattered if $\rho(\upsilon) < \upsilon$, respectively right-scattered when $\sigma(\upsilon) > t$. Clearly υ is left dense when $\rho(\upsilon) = \upsilon$, respectively right-dense when $\sigma(\upsilon) = \upsilon$. For detailed discussion on delta, nabla and alpha diamond calculus readers are referred to [4,5,14].

The diamond integrals and there properties are given in [7], Some basic facts are as follow:

Definition 2.1. Suppose $\gamma, \psi : \mathbb{T} \to \mathbb{R}$ are such that $\gamma\psi$ and $(1-\gamma)\,\psi$ are delta and nabla integrable respectively on $[c_1, c_2]^{\mathbb{T}}$, and for $s, \tilde{\upsilon} \in [c_1, c_2]^{\mathbb{T}}$

$$\gamma(\tilde{\upsilon}) = \lim_{s \to \tilde{\upsilon}} \frac{\sigma(\tilde{\upsilon}) - s}{\sigma(\tilde{\upsilon}) + 2\tilde{\upsilon} - 2s - \rho(\tilde{\upsilon})} \quad \forall \tilde{\upsilon} \in \mathbb{T}.$$

Then $\Diamond$-integral of ψ from c_1 to c_2 (or on $[c_1, c_2]^{\mathbb{T}}$) is given by

$$\int_{c_1}^{c_2} \psi(\tilde{\upsilon}) \Diamond \tilde{\upsilon} = \int_{c_1}^{c_2} \gamma(\tilde{\upsilon}) \psi(\tilde{\upsilon}) \Delta \tilde{\upsilon} + \int_{c_1}^{c_2} (1 - \gamma(\tilde{\upsilon})) \psi(\tilde{\upsilon}) \nabla \tilde{\upsilon}.$$

The diamond integral reduces to diamond-α integral, if γ is chosen to be constant. Moreover, if $\mathbb{T} = \mathbb{R}$, then

$$\int_{c_1}^{c_2} \psi(\tilde{\upsilon}) \Diamond \tilde{\upsilon} = \int_{c_1}^{c_2} \psi(\tilde{\upsilon}) \mathrm{d}\tilde{\upsilon};$$

if $\mathbb{T} = h\mathbb{Z}$ where $h > 0$, then

$$\int_{c_1}^{c_2} \psi(\tilde{\upsilon}) \Diamond \tilde{\upsilon} = \frac{h}{2} \left(\sum_{\iota = c_1/h}^{c_2/h - 1} \psi(\iota h) + \sum_{\iota = c_1/h + 1}^{c_2/h} \psi(\iota h) \right);$$

if $\mathbb{T} = q^{\mathbb{N}_0}$ where $q > 1$, then

$$\int_{c_1}^{c_2} \psi(\tilde{\upsilon}) \Diamond \tilde{\upsilon} = \frac{q-1}{q+1} \left(\sum_{\iota = \log_q(c_1)}^{\log_q(c_2) - 1} q^{\iota+1} \psi(q^\iota) + \sum_{\iota = \log_q(c_1) + 1}^{\log_q(c_2)} q^{\iota - 1} \psi(q^\iota) \right).$$

Properties of the diamond integrals are analogous to the properties of delta, nabla and diamond-α integrals.

Jensen's inequality and Jensen Steffensen inequality via diamond integrals, which we have to extend are the following:

Theorem 2.1 (Jensen's Inequality [3]). *Let $h \in C([c_1, c_2]^{\mathbb{T}}, \mathbb{R}_+)$ be such that $\int\limits_{c_1}^{c_2} h(\tilde{\upsilon}) \Diamond \tilde{\upsilon} > 0$ and $\psi \in C([c_1, c_2]^{\mathbb{T}}, I)$. If $\Psi \in C(I, \mathbb{R})$ is convex, then*

$$\Psi\left(\frac{\int_{c_1}^{c_2} h(\varsigma) \psi(\varsigma) \Diamond \varsigma}{\int_{c_1}^{c_2} h(\varsigma) \Diamond \varsigma} \right) \le \frac{\int_{c_1}^{c_2} h(\varsigma) \Psi(\psi(\varsigma)) \Diamond \varsigma}{\int_{c_1}^{d_1} h(\varsigma) \Diamond \varsigma}. \tag{2.1}$$

Under the conditions of above theorem, its possible to define the following Jensen's linear functional for diamond integrals:

$$\mathcal{J}_1(\Psi) = \frac{\int\limits_{c_1}^{c_2} h(\varsigma)\Psi(\psi(\varsigma))\Diamond\varsigma}{\int\limits_{c_1}^{c_2} h(\varsigma)\Diamond\varsigma} - \Psi\left(\frac{\int\limits_{c_1}^{c_2} h(\varsigma)\psi(\varsigma)\Diamond\varsigma}{\int\limits_{c_1}^{c_2} h(\varsigma)\Diamond\varsigma}\right). \tag{2.2}$$

Clearly $\mathcal{J}_1(\Psi) \geq 0$.

Definition 2.2 ($\Diamond$-Steffensen-Popoviciu ($\Diamond$-SP) Weight [12])**.** Let $g \in \mathrm{C}([a,b]_{\mathbb{T}}, \mathbb{R})$. Then $w \in \mathrm{C}([a,b]_{\mathbb{T}}, \mathbb{R})$ is an $\Diamond$-integrable Steffensen–Popoviciu ($\Diamond$-integrable SP) weight for g if

$$\int_a^b w(t)\Diamond t > 0 \quad \text{and} \quad \int_a^b \Phi(g(t))^+ w(t)\Diamond t \geq 0,$$

hold for every convex function $\Phi \in \mathrm{C}([m,M],\mathbb{R})$, where

$$m = \inf_{t\in[a,b]_{\mathbb{T}}} g(t) \quad \text{and} \quad M = \sup_{t\in[a,b]_{\mathbb{T}}} g(t).$$

Theorem 2.2 (Jensen–Steffensen inequality [12])**.** *Let $\psi \in \mathrm{C}([c_1,c_2]^{\mathbb{T}}, [d_1,d_2])$ and $w \in \mathrm{C}([c_1,c_2]^{\mathbb{T}}, \mathbb{R})$ satisfies $\int_{c_1}^{c_2} w(\varsigma)\Diamond\varsigma > 0$. Suppose $\Psi \in \mathrm{C}([d_1,d_2],\mathbb{R})$ is convex such that*

$$d_1 = \inf_{\tilde{\upsilon}\in[c_1,c_2]^{\mathbb{T}}} \psi(\varsigma) \quad \textit{and} \quad d_2 = \sup_{\tilde{\upsilon}\in[c_1,c_2]^{\mathbb{T}}} \psi(\varsigma).$$

Then the two statements (i) and (ii) are equivalent:

(i) *w is an $\Diamond$-integrable SP weight for ψ which is $\Diamond$-integrable on $[c_1,c_2]^{\mathbb{T}}$, i.e.,*

$$\int_{c_1}^{c_2} \Psi(\psi(\varsigma))^+ w(\varsigma)\Diamond\varsigma \geq 0.$$

(ii) *The inequality* (2.1) *holds.*

Under the conditions of Theorem 2.2, the Jensen functional (2.2) becomes Jensen–Steffensen functional.

3. Improvements of Jensen's functional and related bounds

Let $\kappa \in \mathbb{N}\bigcup\{0\}$ and $\hat{\psi} : [d_1,d_2] \to \mathbb{R}$ be such that $\hat{\psi}^{(\kappa-1)}$ is absolutely continuous. Then

$$\hat{\psi}(\eta) = T_{\kappa-1}(\hat{\psi};c,\eta) + R_{\kappa-1}(\hat{\psi};c,\eta), \qquad \eta, c \in [d_1,d_2], \tag{3.1}$$

where $T_{\kappa-1}(\hat{\psi};c,\eta)$ represents Taylor's polynomial with degree $\kappa-1$, i.e.,

$$T_{\kappa-1}(\hat{\psi};c,\eta) = \sum_{\iota=0}^{(\kappa-1)} \frac{\hat{\psi}^{(\iota)}(c)}{\iota!}(\eta-c)^{\iota}$$

and $R_{\kappa-1}(\hat{\psi};c,\eta)$ is remainder given by

$$R_{\kappa-1}(\hat{\psi};c,\eta) = \frac{1}{(\kappa-1)!}\int_c^{\eta} \hat{\psi}^{(\kappa)}(\tilde{\eta})(\eta-\tilde{\eta})^{(\kappa-1)} d\tilde{\eta}.$$

Apply (3.1) at the points d_1 and d_2. Respective expressions are

$$\hat{\psi}(\eta) = \sum_{\iota=0}^{(\kappa-1)} \frac{\hat{\psi}^{(\iota)}(d_1)}{\iota!}(\eta - d_1)^{\iota} + \frac{1}{(\kappa-1)!}\int_{d_1}^{d_2} \hat{\psi}^{(\kappa)}(\tilde{\eta})(\eta - \tilde{\eta})_+^{(\kappa-1)} d\tilde{\eta}, \tag{3.2}$$

$$\hat{\psi}(\eta) = \sum_{\iota=0}^{(\kappa-1)} \frac{\hat{\psi}^{(\iota)}(d_2)}{\iota!}(d_2 - \eta)^{\iota}(-1)^{\iota} - \frac{1}{(\kappa-1)!}\int_{d_1}^{d_2} (-1)^{(\kappa-1)}\hat{\psi}^{(\kappa)}(\tilde{\eta})(\tilde{\eta} - \eta)_+^{(\kappa-1)} d\tilde{\eta}, \tag{3.3}$$

where,

$$(\eta - \tilde{\eta})_+ := \begin{cases} (\eta - \tilde{\eta}) & \text{if} \quad \tilde{\eta} \leq \eta, \\ 0 & \text{if} \quad \tilde{\eta} > \eta. \end{cases}$$

Identities (3.2) and (3.3) are helpful to construct the following results: Next result is generalization of Jensen's functional for κ-convex functions.

Theorem 3.1. *Let $\kappa \in \mathbb{Z}_+$ and $\Psi^{(\kappa-1)}$ is absolutely continuous for κ-convex function $\Psi : [d_1, d_2] \to \mathbb{R}$. Take $h \in C([c_1, c_2]^{\mathbb{T}}, \mathbb{R}_+)$ which satisfies $\int_{c_1}^{c_2} h(\tilde{\upsilon})\Diamond\tilde{\upsilon} > 0$ and $\psi \in C([c_1, c_2]^{\mathbb{T}}, [d_1, d_2])$.*

If

$$\mathfrak{J}_1\left((\psi - \upsilon)_+^{(\kappa-1)}\right) \geq 0, \qquad \upsilon \in [d_1, d_2], \tag{3.4}$$

then

$$\mathfrak{J}_1(\Psi(\psi)) \geq \sum_{\iota=2}^{(\kappa-1)} \frac{\Psi^{(\iota)}(d_1)}{\iota!}\mathfrak{J}_1\left((\psi - d_1)^{\iota}\right) \tag{3.5}$$

and if

$$(-1)^{(\kappa-1)}\mathfrak{J}_1\left((\upsilon - \psi)_+^{(\kappa-1)}\right) \leq 0, \qquad \upsilon \in [d_1, d_2], \tag{3.6}$$

then

$$\mathfrak{J}_1(\Psi(\psi)) \geq \sum_{\iota=2}^{(\kappa-1)} \frac{(-1)^{\iota}\Psi^{(\iota)}(d_2)}{\iota!}\mathfrak{J}_1\left((d_2 - \psi)^{\iota}\right). \tag{3.7}$$

Proof. Using the identities (3.2) and (3.3) in Jensen's functional (2.2), respectively, we get the following identities:

$$\mathfrak{J}_1(\Psi(\psi)) = \sum_{\iota=2}^{(\kappa-1)} \frac{\Psi^{(\iota)}(d_1)}{\iota!}\mathfrak{J}_1\left((\psi - d_1)^{\iota}\right) + \frac{1}{(\kappa-1)!}\int_{d_1}^{d_2} \Psi^{(\kappa)}(\upsilon)\mathfrak{J}_1\left((\psi - \upsilon)_+^{(\kappa-1)}\right) d\upsilon \tag{3.8}$$

and

$$\mathfrak{J}_1(\Psi(\psi)) = \sum_{\iota=2}^{(\kappa-1)} \frac{(-1)^{\iota}\Psi^{(\iota)}(d_2)}{\iota!}\mathfrak{J}_1\left(d_2-\psi)^{\iota}\right) \\ - \frac{(-1)^{(\kappa-1)}}{(\kappa-1)!}\int_{d_1}^{d_2}\Psi^{(\kappa)}(\upsilon)\mathfrak{J}_1\left((\upsilon-\psi)_+^{(\kappa-1)}\right)d\upsilon. \quad (3.9)$$

Absolute continuity of $\Psi^{(\kappa-1)}$ implies almost everywhere existence of $\Psi^{(\kappa)}$. Since Ψ is κ-convex, one has that $\Psi^{(\kappa)}(\upsilon) \geq 0 \ \forall \upsilon \in [d_1, d_2]$. Hence (3.5) and (3.7) are obtained by using (3.8) and (3.9) respectively. □

Let us denote

$$\mathfrak{M}_1(\upsilon) = \mathfrak{J}_1\left((\psi-\upsilon)_+^{(\kappa-1)}\right), \quad \upsilon \in [d_1, d_2],$$
$$\mathfrak{M}_2(\upsilon) = (-1)^{(\kappa-1)}\mathfrak{J}_1\left((\upsilon-\psi)_+^{(\kappa-1)}\right), \quad \upsilon \in [d_1, d_2],$$

and the Čhebyšhev functional $\mathfrak{C}(\mathfrak{M}_\iota, \mathfrak{M}_\iota)$, for $\iota \in \{1, 2\}$.

$$\mathfrak{C}(\mathfrak{M}_\iota, \mathfrak{M}_\iota) = \frac{1}{d_2-d_1}\int_{d_1}^{d_2}\mathfrak{M}_\iota^2(\upsilon)d\upsilon - \left(\frac{1}{d_2-d_1}\int_{d_1}^{d_2}\mathfrak{M}_\iota(\upsilon)d\upsilon\right)^2.$$

For further generalizations, we need the following two results given in [8]:

Theorem 3.2. *Let $g : [d_1, d_2] \to \mathbb{R}$ be Lebesgue integrable function and $h : [d_1, d_2] \to \mathbb{R}$ be an absolutely continuous function with $(\cdot - d_1)(d_2 - \cdot)[h']^2 \in L[d_1, d_2]$, then we have*

$$|\mathfrak{C}(g,h)| \leq \frac{1}{\sqrt{2}}\left[\mathfrak{C}(g,g)\right]^{\frac{1}{2}}\frac{1}{\sqrt{d_2-d_1}} \\ \left(\int_{d_1}^{d_2}(\upsilon-d_1)(d_2-\upsilon)[h'(\upsilon)]^2 d\upsilon\right)^{\frac{1}{2}}, \quad (3.10)$$

where

$$\mathfrak{C}(g,h) = \frac{1}{d_2-d_1}\int_{d_1}^{d_2}g(\upsilon)h(\upsilon)d\upsilon \\ - \frac{1}{d_2-d_1}\int_{d_1}^{d_2}g(\upsilon)d\upsilon\frac{1}{d_2-d_1}\int_{d_1}^{d_2}h(\upsilon)d\upsilon.$$

The constant $\frac{1}{\sqrt{2}}$ in the inequality (3.10) is best possible.

Theorem 3.3. *Assume that $h : [d_1, d_2] \to \mathbb{R}$ is monotonic nondecreasing and $g : [d_1, d_2] \to \mathbb{R}$ is absolutely continuous with $g' \in L_\infty[d_1, d_2]$. Then we have the inequality*

$$|\mathfrak{C}(g,h)| \leq \frac{1}{2(d_2-d_1)} \parallel g' \parallel_\infty \int_{d_1}^{d_2}(\upsilon-d_1)(d_2-\upsilon)d\upsilon,$$

where the constant $\frac{1}{2}$ is the best possible.

In next theorem we obtain another generalization of Jensen's functional for κ-convex functions.

Theorem 3.4. *Let $\kappa \in \mathbb{N}$. Let h and ψ satisfy assumptions of Theorem 3.1. Further suppose $\Psi : [d_1, d_2] \to \mathbb{R}$, where $\Psi^{(\kappa)}$ is absolutely continuous and $(\cdot - d_1)(d_2 - \cdot)[\Psi^{(\kappa+1)}]^2 \in L[d_1, d_2]$. Then:*

(i)

$$\mathfrak{J}_1(\Psi(\psi)) = \sum_{\iota=2}^{(\kappa-1)} \frac{\Psi^{(\iota)}(d_1)}{\iota!} \mathfrak{J}_1\left((\psi - d_1)^{\iota}\right) + \frac{\Psi^{(\kappa-1)}(d_2) - \Psi^{(\kappa-1)}(d_1)}{(d_2 - d_1)(\kappa-3)!} \int_{d_1}^{d_2} \mathfrak{M}_1(\upsilon) d\upsilon + \mathfrak{K}_\kappa^1(d_1, d_2, \Psi), \quad (3.11)$$

where the remainder $\mathfrak{K}_\kappa^1(d_1, d_2, \Psi)$ satisfies the bound

$$\left|\mathfrak{K}_\kappa^1(d_1, d_2, \Psi)\right| \leq \frac{\sqrt{d_2 - d_1}}{\sqrt{2}(\kappa-1)!} \left[\mathfrak{C}(\mathfrak{M}_1, \mathfrak{M}_1)\right]^{\frac{1}{2}} \left|\int_{d_1}^{d_2} (\upsilon - d_1)(d_2 - \upsilon) \left[\Psi^{(\kappa+1)}(\upsilon)\right]^2 d\upsilon\right|^{\frac{1}{2}}, \quad (3.12)$$

(ii)

$$\mathfrak{J}_1(\Psi(\psi)) = \sum_{\iota=2}^{(\kappa-1)} \frac{(-1)^{\iota}\Psi^{(\iota)}(d_2)}{\iota!} \mathfrak{J}_1\left(d_2 - \psi)^{\iota}\right) + \frac{\Psi^{(\kappa-1)}(d_2) - \Psi^{(\kappa-1)}(d_1)}{(d_2 - d_1)(\kappa-1)!} \int_{d_1}^{d_2} \mathfrak{M}_2(\upsilon) d\upsilon + \mathfrak{K}_\kappa^2(d_1, d_2, \Psi), \quad (3.13)$$

where the remainder $\mathfrak{K}_\kappa^2(d_1, d_2, \Psi)$ satisfies the bound

$$\left|\mathfrak{K}_\kappa^2(d_1, d_2, \Psi)\right| \leq \frac{\sqrt{d_2 - d_1}}{\sqrt{2}(\kappa-3)!} \left[\mathfrak{C}(\mathfrak{M}_2, \mathfrak{M}_2\right]^{\frac{1}{2}} \left|\int_{d_1}^{d_2} (\upsilon - d_1)(d_2 - \upsilon) \left[\Psi^{(\kappa+1)}(\upsilon)\right]^2 d\upsilon\right|^{\frac{1}{2}}.$$

Proof. (i) Replace g by $\mathfrak{M}_1$ and h by $\Phi^{(n)}$ in Theorem 3.2 to get

$$\left|\frac{1}{M-m}\int_m^M \mathfrak{M}_1(x)\Phi^{(n)}(x)dx - \frac{1}{M-m}\int_m^M \mathfrak{M}_1(x)dx \frac{1}{M-m}\int_m^M \Phi^{(n)}(x)dx\right| \leq \frac{1}{\sqrt{2}}\left[\mathfrak{C}(\mathfrak{M}_1, \mathfrak{M}_1)\right]^{\frac{1}{2}} \frac{1}{\sqrt{M-m}} \left|\int_m^M (x-m)(M-x)\left[\Phi^{(n+1)}(x)\right]^2 dx\right|^{\frac{1}{2}},$$

where the remainder satisfies the estimation (3.12) and one gets

$$\frac{1}{(n-1)!}\int_m^M \mathfrak{M}_1(x)\Phi^{(n)}(x)dx = \frac{\Phi^{n-1}(M) - \Phi^{n-1}(m)}{(M-m)(n-1)!}\int_m^M \mathfrak{M}_1(x)dx + \mathfrak{K}_n^1(m, M, \Phi).$$

Hence (3.11) follows from (3.8).

(ii) Similar to the above part.

□

Next theorem gives the Grüss type inequality.

Theorem 3.5. *Let $\kappa \geq 1$ be positive integer. Let h and ψ satisfy the conditions of Theorem 3.1. Suppose $\Psi : [d_1, d_2] \to \mathbb{R}$, where $\Psi^{(\kappa)}$ is absolutely continuous and $\Psi^{(\kappa+1)} \geq 0$. Then (i) and (ii) hold.*

(i) *The equation (3.11) holds with the following bound of the remainder $\mathfrak{K}_\kappa^1(d_1, d_2, \Psi)$*

$$\left|\mathfrak{K}_\kappa^1(d_1, d_2, \Psi)\right| \leq \frac{d_2 - d_1}{(\kappa-1)!}\|\mathfrak{M}_1'\|_\infty \left[\frac{\Psi^{(\kappa-1)}(d_2) + \Psi^{(\kappa-1)}(d_1)}{2} - \frac{\Psi^{\kappa-2}(d_2) - \Psi^{\kappa-2}(d_1)}{d_2 - d_1}\right]. \quad (3.14)$$

(ii) *The equation (3.13) holds with the remainder $\mathfrak{K}_\kappa^2(d_1, d_2, \Psi)$ satisfies the estimation*

$$\left|\mathfrak{K}_\kappa^2(d_1, d_2, \Psi)\right| \leq \frac{d_2 - d_1}{(\kappa-1)!}\|\mathfrak{M}_2'\|_\infty \left[\frac{\Psi^{(\kappa-1)}(d_2) + \Psi^{(\kappa-1)}(d_1)}{2} - \frac{\Psi^{\kappa-2}(d_2) - \Psi^{\kappa-2}(d_1)}{d_2 - d_1}\right].$$

Proof. (i) Replacing g by $\mathfrak{M}_1$ and h by $\Phi^{(n)}$ in Theorem 3.3, we get

$$\begin{aligned}
&\left|\frac{1}{M-m}\int_m^M \mathfrak{M}_1(x)\Phi^{(n)}(x)dx\right. \qquad (3.15)\\
&\qquad \left. - \frac{1}{M-m}\int_m^M \mathfrak{M}_1(x)dx \frac{1}{M-m}\int_m^M \Phi^{(n)}(x)dx\right|\\
&\leq \frac{1}{2(M-m)}\|\mathfrak{M}_1'\|_\infty \int_m^M (x-m)(M-x)\Phi^{n+1}(x)dx.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_m^M (x-m)(M-x)\Phi^{n+1}(x)dx\\
&= \int_m^M [2x - (m+M))]\Phi^{(n)}(x)dx\\
&= (M-m)[\Phi^{n-1}(M) + \Phi^{n-1}(m)] - 2(\Phi^{n-2}(M) - \Phi^{n-2}(m)),
\end{aligned}$$

the inequality (3.14) is deduced by using identity (3.8) and the inequality (3.15).

(ii) Proof is similar to the above part.

□

Ostrowski type inequalities are obtained in next theorem

Theorem 3.6. *Consider all hypothesis of Theorem 3.1 hold. Moreover, take (p, q) a pair of conjugate exponents, which satisfies $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Assume $|\Psi^{(\kappa)}|^p : [d_1, d_2] \to \mathbb{R}$ is Riemann integrable with $\kappa \geq 3$.*

Then the following two statements hold:

(i)

$$\left|\mathfrak{J}_1(\Psi(\psi)) - \sum_{\iota=2}^{(\kappa-1)} \frac{\Psi^{(\iota)}(d_1)}{\iota!}\mathfrak{J}_1\left((\psi-d_1)^{\iota}\right)\right| \leq \frac{1}{(\kappa-1)!}\|\Psi^{(\kappa)}\|_p \left(\int_{d_1}^{d_2} |\mathfrak{M}_1(\upsilon)|^q \, d\upsilon\right)^{\frac{1}{q}}, \quad (3.16)$$

(ii)

$$\left|\mathfrak{J}_1(\Psi(\psi)) - \sum_{\iota=2}^{\kappa-1} \frac{(-1)^{\iota}\Psi^{(\iota)}(d_2)}{\iota!}\mathfrak{J}_1\left((d_2-\psi)^{\iota}\right)\right| \leq \frac{1}{(\kappa-1)!}\|\Psi^{(\kappa)}\|_p \left(\int_{d_1}^{d_2} |\mathfrak{M}_2(\upsilon)|^q \, d\upsilon\right)^{\frac{1}{q}}, \quad (3.17)$$

where the constants on the right hand side of (3.16) *and* (3.17) *are optimal for* $p = 1$ *and sharp for* $1 < p \leq \infty$.

Proof. (i) Let us denote

$$\Psi_1(x) = \frac{1}{(n-1)!}\left(\mathfrak{M}_1(x)\right).$$

By using the identity (3.8) and Hölder's inequality we obtain

$$\left|\mathfrak{J}_1(\Phi(f)) - \sum_{k=2}^{n-1} \frac{\Phi^{(k)}(m)}{k!}\mathfrak{J}_1\left((f-m)^k\right)\right| = \left|\int_m^M \Psi_1(x)\Phi^{(n)}(x)dx\right| \leq \|\Phi^{(n)}\|_p \left(\int_m^M |\Psi_1(x)|^q \, dx\right)^{\frac{1}{q}}.$$

For the proof of the sharpness of the constant $\left(\int_m^M |\Psi_1(x)|^q \, dx\right)^{\frac{1}{q}}$, look at [2, Theorem 11].

(ii) Proof is similar to the above part.

□

Remark 3.1. Let $\kappa \in \mathbb{Z}_+$ and $\Psi : [d_1, d_2] \to \mathbb{R}$ be such that $\Psi^{(\kappa-1)}$ is absolutely continuous. Suppose $h \in C([c_1, c_2]^{\mathbb{T}}, \mathbb{R})$ satisfies $\int\limits_{c_1}^{c_2} h(\tilde{\upsilon})\Diamond\tilde{\upsilon} > 0$. If h is an $\Diamond$-integrable SP weight for ψ on $[c_1, c_2]^{\mathbb{T}}$, then the identities (3.8) and (3.9) hold, respectively.

If Ψ is κ-convex function, where $\Psi^{(\kappa-1)}$ is absolutely continuous, then (3.5) holds under (3.4) and (3.7) holds under (3.6).

Further, Theorem 3.5 and Theorem 3.6 also hold for Jensen–Steffensen functional.

4. Improvements of the functional by converse of Jensen's inequality

In present section, we construct results for the functionals concerning with converse of Jensen's inequality, analogues to results given in Section 3. For this purpose first of all we define the converse of Jensen's inequality for diamond integrals in the following theorem.

Theorem 4.1. *Let $h \in C([c_1, c_2]^{\mathbb{T}}, \mathbb{R}_+)$ satisfies $\int\limits_{c_1}^{c_2} h(\tilde{\upsilon})\Diamond\tilde{\upsilon} > 0$ and $\psi \in C([c_1, c_2]^{\mathbb{T}}, [d_1, d_2])$. If $\Psi \in C([d_1, d_2], \mathbb{R})$ is a convex function, then*

$$\frac{\int_{c_1}^{c_2} \Psi(\psi(\tilde{\upsilon}))h(\tilde{\upsilon})\Diamond\tilde{\upsilon}}{\int_{c_1}^{c_2} h(\tilde{\upsilon})\Diamond\tilde{\upsilon}} \le \frac{d_2 - \overline{\psi}}{d_2 - d_1}\Psi(d_1) + \frac{\overline{\psi} - d_1}{d_2 - d_1}\Psi(d_2), \tag{4.1}$$

where

$$\overline{\psi} = \frac{\int_{c_1}^{c_2} \psi(\tilde{\upsilon})h(\tilde{\upsilon})\Diamond\tilde{\upsilon}}{\int_{c_1}^{c_2} h(\tilde{\upsilon})\Diamond\tilde{\upsilon}}.$$

Proof. From the properties of diamond integrals, it is clear that diamond integrals are isotonic linear functionals. Hence the inequality (4.1) follows from [13, Theorem 3.37] (see also [1, Theorem 5.2]). □

Under the assumptions of Theorem 4.1, the following functional is defined which is linear.

$$\mathfrak{J}_2(\Psi) = \frac{d_2 - \overline{\psi}}{d_2 - d_1}\Psi(d_1) + \frac{\overline{\psi} - d_1}{d_2 - d_1}\Psi(d_2) - \frac{\int_{c_1}^{c_2} \Psi(\psi(\tilde{\upsilon}))h(\tilde{\upsilon})\Diamond\tilde{\upsilon}}{\int_{c_1}^{c_2} h(\tilde{\upsilon})\Diamond\tilde{\upsilon}}. \tag{4.2}$$

Clearly $\mathfrak{J}_2(\Psi) \ge 0$. By using the identities (3.2) and (3.3) in the converse of Jensen's functional (4.2), respectively, we get the following identities:

$$\mathfrak{J}_2(\Psi(\psi)) = \sum_{\iota=2}^{\kappa-1} \frac{\Psi^{(\iota)}(d_1)}{\iota!}\mathfrak{J}_2\left((\psi - d_1)^{\iota}\right) + \frac{1}{(\kappa-1)!}\int_{d_1}^{d_2} \Psi^{(\kappa)}(\upsilon)\mathfrak{J}_2\left((\psi - \upsilon)_+^{\kappa-1}\right)d\upsilon$$

and

$$\mathfrak{J}_2(\Psi(\psi)) = \sum_{\iota=2}^{\kappa-1} \frac{(-1)^{\iota}\Psi^{(\iota)}(d_2)}{\iota!}\mathfrak{J}_2\left((d_2 - \psi)^{\iota}\right) - \frac{(-1)^{\kappa-1}}{(\kappa-1)!}\int_{d_1}^{d_2} \Psi^{(\kappa)}(\upsilon)\mathfrak{J}_2\left((\upsilon - \psi)_+^{\kappa-1}\right)d\upsilon.$$

Following theorem can be obtained in a similar way as Theorem 3.1.

Theorem 4.2. *Let $\kappa \in \mathbb{N}$ and $\Psi^{(\kappa-1)}$ is absolutely continuous where $\Psi : [d_1, d_2] \to \mathbb{R}$. Suppose $h \in C([c_1, c_2]^{\mathbb{T}}, \mathbb{R}_+)$ satisfies $\int\limits_{c_1}^{c_2} h(\tilde{\upsilon})\Diamond\tilde{\upsilon} > 0$ and $\psi \in C([c_1, c_2]^{\mathbb{T}}, [d_1, d_2])$. If Ψ to be κ-convex function such that $\Psi^{(\kappa-1)}$ is absolutely continuous. One has that*

(i) *If*

$$\mathfrak{J}_2\left((\psi - \upsilon)_+^{\kappa-1}\right) \ge 0, \qquad \upsilon \in [d_1, d_2],$$

then

$$\mathcal{J}_2(\Psi(\psi)) \geq \sum_{\iota=2}^{\kappa-1} \frac{\Psi^{(\iota)}(d_1)}{\iota!} \mathcal{J}_2\left((\psi-d_1)^{\iota}\right).$$

(ii) *If*

$$(-1)^{\kappa-1}\mathcal{J}_2\left((\upsilon-\psi)_+^{\kappa-1}\right) \leq 0, \qquad \upsilon \in [d_1,d_2],$$

then

$$\mathcal{J}_2(\Psi(\psi)) \geq \sum_{\iota=2}^{\kappa-1} \frac{(-1)^{\iota}\Psi^{(\iota)}(d_2)}{\iota!} \mathcal{J}_2\left((d_2-\psi)^{\iota}\right).$$

Denote

$$\mathfrak{M}_3(\upsilon) = \mathcal{J}_2\left((\psi)-\upsilon)_+^{\kappa-1}\right),$$
$$\mathfrak{M}_4(\upsilon) = (-1)^{\kappa-1}\mathcal{J}_2\left((\upsilon-\psi)_+^{\kappa-1}\right).$$

Remark 4.1. By replacing $\mathcal{J}_1, \mathfrak{M}_1, \mathfrak{M}_2$ with $\mathcal{J}_2, \mathfrak{M}_3, \mathfrak{M}_4$ in Theorem 3.4 – Theorem 3.6, we obtain bounds related to improved functional for converse of Jensen inequality given in this section.

Remark 4.2. All results of this section also hold under the condition of the converse of Jensen–Steffensen inequality as given by the authors in [12].

Remark 4.3. Results given in last two sections of [2] can be constructed for the improved functionals in the paper.

5. Application on Hölder's inequality

Theorem 5.1. *Let $\kappa \in \mathbb{Z}_+$ and $w, g, \psi \in C([c_1,c_2]^{\mathbb{T}}, [d_1,d_2])$, where $d_1, d_2 > 0$. Let*

$$\mathcal{J}_{11}\left((\psi-\upsilon)_+^{\kappa-1}\right) \geq 0, \qquad \upsilon \in [d_1,d_2],$$

where

$$\mathcal{J}_{11}\left((\psi-\upsilon)^{\iota}\right) = \frac{\int_{c_1}^{c_2} w(\varsigma)g^q(\varsigma))(\psi(\varsigma)g^{-\frac{q}{p}}(\varsigma)-\upsilon)^{\iota}\Diamond\varsigma}{\int_{c_1}^{c_2} w(\varsigma)g^q(\varsigma)\Diamond\varsigma} - \left(\frac{\int_{c_1}^{c_2} w(\varsigma)\psi(\varsigma)g(\varsigma)\Diamond\varsigma}{\int_{c_1}^{c_2} w(\varsigma)g^q(\varsigma)\Diamond\varsigma} - d_1\right)^{\iota}.$$

Then

$$\left(\int_{c_1}^{c_2} w(\varsigma)\psi^p(\varsigma))\Diamond\varsigma\right)^{\frac{1}{p}} \left(\int_{c_1}^{c_2} w(\varsigma)g^q(\varsigma))\Diamond\varsigma\right)^{\frac{1}{q}} \geq \left(\left(\int_{c_1}^{c_2} w(\varsigma)\psi(\varsigma)g(\varsigma)\Diamond\varsigma\right)^{p}\right.$$
$$\left.+\sum_{\iota=2}^{\kappa-1} \frac{(p(p-1)\dots(p-(\iota-1))(d_1)^{(p-\iota)})}{\iota!}\,\mathcal{J}_{11}\left((\psi-\upsilon)^{\iota}\right)\right)^{\frac{1}{p}}.$$

Proof. By replacing the functions h with wg^q, ψ with $\psi g^{-q/p}$, and $\Psi(y) = y^p$, where $y \geq 0$ and $p \geq 1$ in (3.5), we get

$$\frac{\int_{c_1}^{c_2} w(\varsigma)\psi^p(\varsigma))\Diamond\varsigma}{\int_{c_1}^{c_2} w(\varsigma)g^q(\varsigma))\Diamond\varsigma} - \left(\frac{\int_{c_1}^{c_2} w(\varsigma)\psi(\varsigma)g(\varsigma)\Diamond\varsigma}{\int_{c_1}^{c_2} w(\varsigma)g^q(\varsigma))\Diamond\varsigma}\right)^p \geq \sum_{\iota=2}^{\kappa-1} \frac{(p(p-1)\ldots(p-(\iota-1))(d_1)^{(p-\iota)})}{\iota!} \mathcal{J}_{11}\left((\psi-\upsilon)^\iota\right).$$

$$\frac{\int_{c_1}^{c_2} w(\varsigma)\psi^p(\varsigma))\Diamond\varsigma}{\int_{c_1}^{c_2} w(\varsigma)g^q(\varsigma))\Diamond\varsigma} \geq \left(\frac{\int_{c_1}^{c_2} w(\varsigma)\psi(\varsigma)g(\varsigma)\Diamond\varsigma}{\int_{c_1}^{c_2} w(\varsigma)g^q(\varsigma))\Diamond\varsigma}\right)^p + \sum_{\iota=2}^{\kappa-1} \frac{(p(p-1)\ldots(p-(\iota-1))(d_1)^{(p-\iota)})}{\iota!} \mathcal{J}_{11}\left((\psi-\upsilon)^\iota\right).$$

$$\int_{c_1}^{c_2} w(\varsigma)\psi^p(\varsigma))\Diamond\varsigma \left(\int_{c_1}^{c_2} w(\varsigma)g^q(\varsigma))\Diamond\varsigma\right)^{p-1} \geq \left(\int_{c_1}^{c_2} w(\varsigma)\psi(\varsigma)g(\varsigma)\Diamond\varsigma\right)^p + \sum_{\iota=2}^{\kappa-1} \frac{(p(p-1)\ldots(p-(\iota-1))(d_1)^{(p-\iota)})}{\iota!} \mathcal{J}_{11}\left((\psi-\upsilon)^\iota\right).$$

□

Remark 5.1. If we use $\mathbb{T} = \mathbb{R}$, $\mathbb{Z}$, $q^{\mathbb{N}_0}$, $q > 1$, in the obtained results, then we get respective improved Jensen type inequalities as well as improved Hölder inequality in differential, difference, and quantum calculus as well.

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[1]Department of Mathematics,
Abbottabad University of Science and Technology,
Havelian, Abbottabad, Pakistan
E-mail address: emaorr@gmail.com

[2]Department of Mathematics,
University of Lahore,
Sargodha Campus, Sargodha, Pakistan
E-mail address: hammaran@gmail.com

[3]RUDN University
6 Miklukho-Maklay St, Moscow, 117198
E-mail address: pecaric@mahazu.hazu.hr