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A GRÜNBAUM TYPE INEQUALITY FOR k-GAMMA FUNCTION

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Abstract. In this paper, we mainly prove a Grünbaum type inequality for k-gamma function, and also show a characterization of generalized Euler-Mascheroni constant γ_k. This extends and generalizes the main results of Alzer and Kwong.

1. Introduction

The Euler gamma function is defined all positive real numbers x by

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt.$$

The logarithmic derivative of $\Gamma(x)$ is called the psi or digamma function. That is

$$\psi(x)=\frac{d}{dx}\ln\Gamma(x)=\frac{\Gamma'(x)}{\Gamma(x)}=-\gamma-\frac{1}{x}+\sum_{n=1}^{\infty}\frac{x}{n(n+x)},$$

where $\gamma=0.5772\ldots$ is the Euler-Mascheroni constant. The polygamma functions $\psi^{(m)}(x)$ for $m\in\mathbb{N}$ are defined by

$$\psi^{(m)}(x)=\frac{d^m}{dx^m}\psi(x)=(-1)^m m!\sum_{n=0}^{\infty}\frac{1}{(n+x)^{m+1}},x>0.$$

The gamma, digamma and polygamma functions play an important role in the theory of special functions, and are closely related to factorial, fractional differential equations, mathematical physics and crops up in many unexpected place in analysis. The reader may see references ([8–11, 16–21, 34, 35]). Some of the work about origin, history, the complete monotonicity, and inequalities of these special functions may refer to ([1–4, 6, 7, 12–15, 22–31, 36, 37]) and the references therein.

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In 2007, Díaz and Pariguan [10] defined the $k-$analogue of the gamma function for $k>0$ and $x>0$ as

$$\Gamma_k(x)=\int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}dt=\lim_{n\to\infty}\frac{n!k^n(nk)^{\frac{x}{k}-1}}{x(x+k)\cdots(x+(n-1)k)},$$

where $\lim_{k\to 1}\Gamma_k(x)=\Gamma(x)$. Similarly, we may define the $k-$analogue of the digamma and polygamma functions as

$$\psi_k(x)=\frac{d}{dx}\ln\Gamma_k(x)\quad\text{and}\quad\psi_k^{(m)}(x)=\frac{d^m}{dx^m}\psi_k(x).$$

It is well known that the $k-$analogues of the digamma and polygamma functions satisfy the following identities (See [10, 32, 33, 40, 41])

$$\Gamma_k(k)=1, \tag{1.1}$$

$$\Gamma_k(x+k)=x\Gamma_k(x),\quad x>0, \tag{1.2}$$

$$\begin{aligned}\psi_k(x)&=\tfrac{\ln k-\gamma}{k}-\tfrac{1}{x}+\textstyle\sum_{n=1}^\infty\tfrac{x}{nk(nk+x)}\\&=\tfrac{\ln k-\gamma}{k}-\textstyle\int_0^\infty\tfrac{e^{-kt}-e^{-xt}}{1-e^{-kt}}dt,\end{aligned} \tag{1.3}$$

and

$$\begin{aligned}\psi_k^{(m)}(x)&=(-1)^{m+1}m!\textstyle\sum_{n=0}^\infty\tfrac{1}{(nk+x)^{m+1}}\\&=(-1)^{m+1}\textstyle\int_0^\infty\tfrac{1}{1-e^{-kt}}t^me^{-xt}dt.\end{aligned} \tag{1.4}$$

At present, these functions have been extensively studied. In [38], L. Yin, L.-G. Huang, X.-L. Lin and Y.-L. Wang established a concave theorem and some inequalities for k- digamma function. Furthermore, L. Yin, L.-G. Huang, Zh.-M. Song and X.-K. Dou [39] showed several monotonic and concave results related to the generalized digamma and polygamma functions. Their results extends and generalizes the main results of Qi and Guo [12]. In addition, it is worth noting that Krasniqi, Mansour, and Shabani presented some inequalities for q-polygamma functions and q-Riemann Zeta functions by using a q-analogue of Hölder type inequality in [20].

Very recently, Alzer and Kwong [5] presented an elegant Grünbaum type inequality for gamma function: For $x,y,z\geq 0$ with $x^2+y^2=z^2$, then the inequality

$$\frac{1}{(1+x)^a\Gamma(1+x)}+\frac{1}{(1+y)^a\Gamma(1+y)}\leq 1+\frac{1}{(1+z)^a\Gamma(1+z)} \tag{1.5}$$

holds true if and only if $a\geq\gamma$. It is natural how to extend inequality (1.5) to k-gamma function. In this paper, we shall give a k-analogue of inequality (1.5).

Definition 1.1. Let $\gamma_k=-k\psi_k(k)=\gamma-\ln k$ be the k-analogue of the Euler-Mascheroni's constant. γ_k is the so-called generalized Euler-Mascheroni's constant. It is easily obtained that $\gamma_k\to\gamma$ as $k\to 1$.

Remark 1.1. It is noting that Nantomah defined a new (p,k)- analogue of the Euler-Mascheroni's constant by $C_{p,k}=-\psi_{p,k}(1)$. So $C_k=-\psi_k(1)$. This is another definition of k-analogue of γ.

2. Lemmas

Lemma 2.1. *For $k > 0$, we have*

$$\psi_k(x) = \frac{\ln k}{k} + \frac{\psi(x/k)}{k}. \tag{2.1}$$

Proof. Taking logarithms and differentiating on both sides of the following formula (2.2) (see [10])

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right), \tag{2.2}$$

we can easily obtain the proof. □

Lemma 2.2. *Let $k > 0$ and $G : [1,\infty) \to \mathbb{R}$ be differentiable such that $t \to \frac{G'(k+t)}{t}$ is strictly increasing on $(0,\infty)$. Then, for all x, y, x_1, y_1 with $0 < x_1 \le x \le y \le y_1$ and $x_1^2 + y_1^2 = x^2 + y^2$, we have*

$$G(k+x) + G(k+y) \le G(k+x_1) + G(k+y_1). \tag{2.3}$$

Proof. The proof is completely similar to Lemma 2.1 in [5]. So we omit it for the sake of simplicity. □

Lemma 2.3. *For $k > e^{\gamma-1} = 0.655\ldots$, the function*

$$W_k(x) = \frac{\gamma_k}{x(k+x)} + \frac{\psi_k(k+x)}{x},$$

is positive and decreasing on $(0,\infty)$.

Proof. Direct calculation yields

$$x^2 W_k'(x) = x\psi_k(k+x) - \psi_k(k+x) - \frac{\gamma_k(k+2x)}{(k+x)^2} = R_k(x), \tag{2.4}$$

and

$$\frac{R_k'(x)}{x} = \psi_k''(k+x) + \frac{2\gamma_k}{(k+x)^3}. \tag{2.5}$$

Using formula (1.4) and

$$\frac{1}{x^{\alpha+1}} = \frac{1}{\Gamma(\alpha+1)}\int_0^\infty t^\alpha e^{-xt}dt, \tag{2.6}$$

we have

$$\frac{R_k'(x)}{x} = -\int_0^\infty t^2 e^{-(x+k)t}\frac{1-\gamma_k+\gamma_k e^{-kt}}{1-e^{-kt}}dt < 0.$$

In fact, put

$$A(t) = 1-\gamma_k+\gamma_k e^{-kt}, k > 0.$$

Direct computation yields

$$A'(t) = -k\gamma_k e^{-kt} < 0.$$

This gives

$$1-\gamma_k+\gamma_k e^{-kt} \ge A(0) = 1-\gamma_k > 0.$$

This implies that the function R_k is strictly decreasing on $(0,\infty)$. Thus $R_k(x) \le R_k(0) = 0$. This leads to $W_k'(x) < 0$. It means that W_k is strictly decreasing on $(0,\infty)$. Therefore, we have $W_k(x) > W_k(\infty) = 0$. The proof is complete. □

3. Main results

Theorem 3.1. *Let $k > e^{\gamma-1}$, then the inequality*

$$\frac{1}{(k+x)^\alpha\Gamma_k(k+x)}+\frac{1}{(k+y)^\alpha\Gamma_k(k+y)}\leq\frac{1}{(k+z)^\alpha\Gamma_k(k+z)}+\frac{1}{k^\alpha} \tag{3.1}$$

holds true for all $x, y, z \geq 0$ with $x^2+y^2=z^2$ if and only if $\alpha \geq \gamma_k$.

Proof. Let $\alpha \geq \gamma_k$. Apply Lemma 2.2 with $x_1 = 0, y_1 = z$ and $\lambda_k(x) = \frac{1}{x^\alpha\Gamma_k(x)}$, it remains to show that $\mu_k(x) = \frac{\lambda_k'(k+x)}{x}$ is strictly increasing on $(0,\infty)$. Simple computation results in

$$-\mu_k(x) = p_k(x)[q_k(x) + W_k(x)]$$

with $p_k(x) = \frac{1}{(k+x)^\alpha\Gamma_k(k+x)}$ and $q_k(x) = \frac{\alpha-\gamma_k}{x(k+x)}$. Lemma 2.3 implies that $q_k(x) + W_k(x)$ is positive and decreasing on $(0,\infty)$. Next, we have

$$(x+k)p_k'(x) = -p_k(x)\left[\frac{\alpha}{x(k+x)}+\frac{\psi_k(k+x)}{x}\right] < 0.$$

This implies that the function p_k is also strictly decreasing on $(0,\infty)$. It follows that the function μ_k is also strictly increasing on $(0,\infty)$.

On the other hand, let

$$\Omega_k(x) = \frac{1}{(k+z)^\alpha\Gamma_k(k+z)}+\frac{1}{k^\alpha}-\frac{1}{(k+x)^\alpha\Gamma_k(k+x)}-\frac{1}{(k+y)^\alpha\Gamma_k(k+y)}.$$

Then

$$\Omega_k(x) \geq \Omega_k(0) = 0.$$

It follows that

$$\Omega_k'(0) = \frac{\alpha + k\Gamma_k'(k)}{k^{\alpha+1}} > 0.$$

Furthermore, we have

$$\alpha \geq -k\Gamma_k'(k) = -k\frac{\Gamma_k'(k)}{\Gamma_k(k)} = -k\psi_k(k) = \gamma_k$$

since $\Gamma_k(k) = 1$. The proof is complete. □

4. Additional comments

If we consider generalized Euler-Mascheroni constant C_k, we also may obtain similar Gaünbaum type inequality.

Proposition 4.1. *For $k > 0$, the following inequality*

$$\frac{1}{2} < C_k < \frac{1}{2}+\frac{k}{12} \tag{4.1}$$

holds true.

Proof. Using Lemma 2.1 and the estimation of digamma function[13]

$$\frac{1}{2x} - \frac{1}{12x^2} + \ln x < \psi(x+1) < \frac{1}{2x} + \ln x, \tag{4.2}$$

we have

$$\frac{1}{2} + \frac{\ln k}{k} < -\frac{1}{k}\psi\left(\frac{1}{k}\right) < \frac{1}{2} + \frac{\ln k}{k} + \frac{k}{12}.$$

The proof is complete. □

Proposition 4.2. *For* $0 < k \le 6$, *the function*

$$S_k(x) = \frac{C_k}{x(k+x)} + \frac{\psi_k(k+x)}{x},$$

is positive and decreasing on $(0, \infty)$.

Proof. The proof is similar to Lemma 2.3. Here, we omit the details. □

Considering the Proposition 4.2 and the proof of Theorem 3.1, we can obtain the following inequality.

Proposition 4.3. *Let* $0 < k \le 6$ *and* $\alpha \ge C_k$. *Then the inequality* (3.1) *holds true for all* $x, y, z \ge 0$ *with* $x^2 + y^2 = z^2$.

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