Turkish J. Ineq., 3 (1) (2019), Pages 54 –63 .

Turkish Journal of
INEQUALITIES

Available online at www.tjinequality.com

SOME HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES FOR MULTIDIMENSIONAL PREINVEX FUNCTIONS

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Abstract. In this study, we investigated on multidimensional preinvex functions. Particularly, we verified Hermite-Hadamard type integral inequalities for these functions. In the meanwhile, we obtained some important results of related inequalities for these functions.

1. Introduction and Preliminaries

Convex functions are important and provide a base to build literature of mathematical inequalities. A function $f : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$ is called convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

for all $x, y \in I$ and $\lambda \in [0,1]$. A classical inequality for convex functions is Hadamard's inequality, this is given as follows [6]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

where $f : I \to \mathbb{R}$ is a convex function for all $a, b \in I,\ a < b.$ This inequality is used to provide estimations of the mean value of a continuous convex function.

In current years, lot of efforts has been made by many mathematicians to generalize classical convexity. It must be known that preinvexity indicates a generalization of convexity.

Therefore, Hermite-Hadamard type inequalities for preinvex functions were obtained by many researchers. For examples, Hanson [8], Ben-Isreal-Mond [1], Pini [11], Mohan-Neogy [15], Weir-Mond [16], Yang-Li [17], Noor [9, 10], Mishra [12], etc. studied the basic properties of the preinvex functions. Let us recall some known results concerning invexty and preinvexity of functions:

Definition 1.1. A set $I \subseteq \mathbb{R}^n$ is called invex with respect to the continuous function $\eta : I \times I \to \mathbb{R}^n$, if $x + \lambda\eta(y,x) \in I,\ \forall x, y \in I,\ \lambda \in [0,1]$. The invex set I is also called a η-connected set (see [1]).

Key words and phrases. Multidimensional function, preinvexity, Hermite-Hadamard inequality.
2010 *Mathematics Subject Classification.* Primary: 26D15. Secondary: 26B25.
Received: 14/04/2019
Accepted: 21/06/2019.

Definition 1.2. Let $I \subseteq \mathbb{R}^n$ be invex with respect to $\eta : I \times I \to \mathbb{R}^n$. A functions $f : I \to \mathbb{R}$ is called preinvex with respect to η, if

$$f\left(x + \lambda\eta\left(y, x\right)\right) \leq \lambda f\left(y\right) \;\; + \left(1 - \lambda\right) f\left(x\right)$$

for all $x, y \in I$ and $\lambda \in [0, 1]$. If the above inequality is reversed, then f is said to be a pre-concave. Moreover,if we choose $\eta\left(y, x\right) = y - x$, then f is a convex functions (see [15]).

Mohan and Neogy [15] proved that an invex function is also preinvex under following Condition C.

Condition C: Let $I \subseteq \mathbb{R}^n$ be invex with respect to $\eta : I \times I \to \mathbb{R}^n$. It is told that the function η satisfies Condition C, if

$$\eta\left(y, y + \lambda\eta\left(x, y\right)\right) = -\lambda\eta\left(x, y\right);$$

$$\eta\left(x, y + \lambda\eta\left(x, y\right)\right) = (1 - \lambda)\eta\left(x, y\right)$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Theorem 1.1. *Let $f : [a, a + \eta\left(b, a\right)] \to (0, \infty)$ be a preinvex functions and with $\eta\left(b, a\right) > 0$. If $f \in L\left[a, a + \eta\left(b, a\right)\right]$, then the following inequality holds*

$$f\left(\frac{2a + \eta\left(b, a\right)}{2}\right) \leq \frac{1}{\eta\left(b, a\right)} \int_a^{a+\eta(b,a)} f\left(x\right) dx \leq \frac{f\left(a\right) + f\left(b\right)}{2}$$

(see [10]).

The Hermite-Hadamard type integral inequality for convex functions has received renewed attention in recent years and the remarkable varieties of refinements and generalizations have been found in [1–17].

In this context, the Hermite-Hadamard type inequalities for harmonically convex functions were obtained by many researchers in literature (see [4, 5, 7, 9, 12]).

For example, De la Cal et al. [5] obtained multidimensional Hermite-Hadamard inequalities in 2006. Nowadays, Ellahi [4] derived Hadamard's inequality for s-convex function on n-coordinates. Viloria et al. [7] verified Hermite-Hadamard type inequalities for harmonically convex functions on n-coordinates.

In the light of these, we defined preinvexity for multidimensional functions. Our supplemental claim is to obtain Hermite-Hadamard type inequalities for these functions.

2. Main Results

The main goal of this section is to present Hermite-Hadamard type inequalities for multidimensional preinvex. Let a_i, b_i be real numbers such that $a_i < b_i$ for $i = 1, 2, \ldots, n, \;\; n \geq 2$.

We consider n-dimensional interval $\Delta^n = \prod_{i=1}^n \left[a_i, b_i\right] \subseteq [0, \infty)^n$. From here on in, it is known that $\mathbb{N}$ $(\mathbb{N}_0)$ is the set of all positive integers (non-negative integers) and
$\mathbb{R}^n_+ := \{a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n : \;\; a_i > 0, \;\;\; i = 1, 2, \ldots, n\}$.

Firstly, we give definition of invexity and preinvexity for multidimensional functions:

Definition 2.1. Let the continuous function $\eta : \Delta^n \times \Delta^n \to \mathbb{R}^n$ be invex on the set Δ^n. Then $\Delta^n = \prod_{i=1}^n [a_i, b_i]$ is called invex with respect to η , if $(\mathbf{x} + \lambda\eta(\mathbf{y}, \mathbf{x})) \in \Delta^n$ for all $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \Delta^n$ and $\lambda \in [0, 1]$.

Definition 2.2. A function $f : \Delta^n \to \mathbb{R}$ is said to be preinvex with respect to η on Δ^n if the following inequality holds

$$f(\mathbf{x} + \lambda\eta(\mathbf{y}, \mathbf{x})) \le \lambda f(\mathbf{y}) + (1 - \lambda) f(\mathbf{x})$$

for all $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \Delta^n$ and $\lambda \in [0, 1]$. If the above inequality is reversed then f is said to be pre-concave with respect to η on Δ^n.

Definition 2.3. A function $f : \Delta^n \to \mathbb{R}$ is called preinvex with respect to η on n-coordinates if the following mappings are preinvex with respect to η:

$$f^i_{x_n}(x) := f(x_1, \ldots, x_{i-1},\ x, x_{i+1}, \ldots, x_n).$$

Lemma 2.1. *Every multidimensional preinvex function with respect to η on Δ^n is preinvex with respect to η on n-coordinates, but converse is not true.*

Proof. Let $f : \Delta^n \to \mathbb{R}$ be a preinvex function with respect to η on Δ^n. Consider $f^i_{x_n} : [a_i, b_i] \to \mathbb{R}$ defined by $f^i_{x_n}(x) := f(x_1, \ldots, x_{i-1},\ x, x_{i+1}, \ldots, x_n)$. Then for $x, y \in [a_i, b_i]$ and $\lambda \in [0, 1]$

$$\begin{aligned} f^i_{x_n}(x + \lambda\eta(y, x)) &:= f(x_1, \ldots, x_{i-1}, (x + \lambda\eta(y, x)), x_{i+1}, \ldots, x_n) \\ &\le \lambda f(x_1, \ldots, x_{i-1},\ y, x_{i+1}, \ldots, x_n) + (1 - \lambda) f(x_1, \ldots, x_{i-1},\ x, x_{i+1}, \ldots, x_n) \\ &= \lambda f^i_{x_n}(y) + (1 - \lambda) f^i_{x_n}(x) \end{aligned}$$

which implies $f^i_{x_n}$ is preinvex with respect to η on $[a_i, b_i]$, that is, f is preinvex with respect to η on n-coordinates. □

For converse we give the following counter example:

Example 2.1. Let us consider a function $f : [0, 1]^n \to [0, \infty)$ defined as

$$f^i_{x_n}(x) := f(x_1, \ldots, x_{i-1},\ x, x_{i+1}, \ldots, x_n) = x_1\ x_2\ \ldots\ x_n$$

for $x \in [a_i, b_i]$. Moreover let be $\eta : [0, 1]^n \to [0, \infty)$, $\eta(\mathbf{y}, \mathbf{x}) = \mathbf{y} - \mathbf{x}$ for all $\mathbf{x},\ \mathbf{y} \in \Delta^n$. Then f is not preinvex with respect to η on $[0, 1]^n$. Indeed, let us assume that f is preinvex with respect to η on $[0, 1]^n$, then we can write

$$f(\mathbf{x} + \lambda\eta(\mathbf{y}, \mathbf{x})) \le \lambda f(\mathbf{y}) + (1 - \lambda) f(\mathbf{x})$$

for all $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \Delta^n$ and $\lambda \in [0, 1]$. But for $\mathbf{x} = (1, 1, , \ldots, 1,\ 0)$, $\mathbf{y} = (0, 1, \ldots, 1) \in [0, 1]^n$ and $\lambda \in [0, 1]$, we have

$$\begin{aligned} f(\mathbf{x} + \lambda\eta(\mathbf{y}, \mathbf{x})) &= f((1, 1, \ldots, 1,\ 0) + \lambda((0, 1, \ldots, 1) - (1, 1, \ldots 1,\ 0))) \\ &= f((1, 1, \ldots, 1,\ 0) + \lambda(-1, 0, \ldots, 0, 1)) = f(1 - \lambda, 1, \ldots, 1, \lambda) = \lambda(1 - \lambda). \end{aligned}$$

On the other hand, $f(\mathbf{x}) = f(\mathbf{y}) = 0$ for $\mathbf{x} = (1, 1, , \ldots, 1,\ 0)$, $\mathbf{y} = (0, 1, \ldots, 1) \in [0, 1]^n$. Then

$$\lambda f(\mathbf{y}) + (1 - \lambda) f(\mathbf{x}) = 0.$$

Thus $f(\mathbf{x}+\lambda\eta(\mathbf{y},\mathbf{x}))>\lambda f(\mathbf{y})+(1-\lambda)f(\mathbf{x})$ for all $\lambda\in[0,1]$, that is, f is not preinvex with respect to η on $[0,1]^n$. That case is contradiction with preinvexity of f.

From here on out, let us assume that $\Omega^n=\prod_{i=1}^n[a_i,a_i+\eta(b_i,a_i)]\subseteq[0,\infty)^n$, a_i,b_i is real numbers such that $a_i<b_i$ for each $i=1,2,\ldots,n,\quad n\geq 2$.

Remark 2.1. If $f:\Omega^n\to\mathbb{R}_+$ is a preinvex function with respect to η on n-coordinates, then $f_{x_n}^i:[a_i,a_i+\eta(b_i,a_i)]\to\mathbb{R}$ is preinvex with respect to η on $[a_i,a_i+\eta(b_i,a_i)]$ for each $i=1,2,\ldots,n$. If $f_{x_n}^i\in L[a_i,a_i+\eta(b_i,a_i)]$ for each $i=1,2,\ldots,n$, then

$$f_{x_n}^i\left(\frac{2a_i+\eta(b_i,a_i)}{2}\right)\leq\frac{1}{\eta(b_i,a_i)}\int_{a_i}^{a_i+\eta(b_i,a_i)}f_{x_n}^i(x_i)\,dx_i\leq\frac{f_{x_n}^i(a_i)+f_{x_n}^i(b_i)}{2}. \tag{2.1}$$

Theorem 2.1. *Let $f:\Omega^n\to\mathbb{R}_+$ be a preinvex function with respect to η on n-coordinates. If $f_{x_n}^i\in L[a_i,a_i+\eta(b_i,a_i)]$ for each $i=1,2,\ldots,n-1$, then the following inequality holds*

$$\begin{aligned}
&\sum_{i=1}^{n-1}f\left(x_1,\ldots,x_{i-1},\frac{2a_i+\eta(b_i,a_i)}{2},\frac{2a_{i+1}+\eta(b_{i+1},a_{i+1})}{2},x_{i+2},\ldots,x_n\right)\\
&\leq\sum_{i=1}^{n-1}\frac{1}{\eta(b_i,a_i)}\int_{a_i}^{a_i+\eta(b_i,a_i)}f_{x_n}^i\left(\frac{2a_i+\eta(b_i,a_i)}{2}\right)dx_i\\
&\leq\sum_{i=1}^{n-1}\frac{1}{\eta(b_i,a_i)\,\eta(b_{i+1},a_{i+1})}\int_{a_i}^{a_i+\eta(b_i,a_i)}\int_{a_{i+1}}^{a_{i+1}+\eta(b_{i+1},a_{i+1})}f_{x_n}^{i+1}(x_{i+1})\,dx_{i+1}dx_i\\
&\leq\sum_{i=1}^{n-1}\frac{1}{2\eta(b_i,a_i)}\int_{a_i}^{a_i+\eta(b_i,a_i)}\left(f_{x_n}^i(a_{i+1})+f_{x_n}^i(b_{i+1})\right)dx_i\\
&\leq\frac{1}{4}\sum_{i=1}^{n-1}\left[\begin{array}{c}f(x_1,\ldots,x_{i-1},a_i,a_{i+1},x_{i+2},\ldots,x_n)\\+f(x_1,\ldots,x_{i-1},b_i,a_{i+1},x_{i+2},\ldots,x_n)\\+f(x_1,\ldots,x_{i-1},a_i,b_{i+1},x_{i+2},\ldots,x_n)\\+f(x_1,\ldots,x_{i-1},b_i,b_{i+1},x_{i+2},\ldots,x_n)\end{array}\right].
\end{aligned} \tag{2.2}$$

Proof. Using preinvexity of $f_{x_n}^{i+1}$ for each $i=1,2,\ldots,n-1$, then

$$\begin{aligned}
f_{x_n}^{i+1}\left(\frac{2a_{i+1}+\eta(b_{i+1},a_{i+1})}{2}\right)&\leq\frac{1}{\eta(b_{i+1},a_{i+1})}\int_{a_{i+1}}^{a_{i+1}+\eta(b_{i+1},a_{i+1})}f_{x_n}^{i+1}(x_{i+1})\,dx_{i+1}\\
&\leq\frac{f_{x_n}^{i+1}(a_{i+1})+f_{x_n}^{i+1}(b_{i+1})}{2}.
\end{aligned}$$

Integrating above inequality with respect to x_i on $[a_i,a_i+\eta(b_i,a_i)]$ for each $i=1,2,\ldots,n-1$

$$\begin{aligned}
&\frac{1}{\eta(b_i,a_i)}\int_{a_i}^{a_i+\eta(b_i,a_i)}f_{x_n}^{i+1}\left(\frac{2a_{i+1}+\eta(b_{i+1},a_{i+1})}{2}\right)dx_i\\
&\leq\frac{1}{\eta(b_i,a_i)\,\eta(b_{i+1},a_{i+1})}\int_{a_i}^{a_i+\eta(b_i,a_i)}\int_{a_{i+1}}^{a_{i+1}+\eta(b_{i+1},a_{i+1})}f_{x_n}^{i+1}(x_{i+1})\,dx_{i+1}dx_i\\
&\leq\frac{1}{2\eta(b_i,a_i)}\int_{a_i}^{a_i+\eta(b_i,a_i)}\left(f_{x_n}^{i+1}(a_{i+1})+f_{x_n}^{i+1}(b_{i+1})\right)dx_i.
\end{aligned} \tag{2.3}$$

Using the Hermite-Hadamard inequality

$$f\left(x_1,\ldots,x_{i-1},\frac{2a_i+\eta\left(b_i,a_i\right)}{2},\frac{2a_{i+1}+\eta\left(b_{i+1},a_{i+1}\right)}{2},x_{i+2},\ldots,x_n\right)$$

$$\leq \frac{1}{\eta\left(b_i,a_i\right)}\int_{a_i}^{a_i+\eta(b_i,a_i)} f_{x_n}^{i+1}\left(\frac{2a_{i+1}+\eta\left(b_{i+1},a_{i+1}\right)}{2}\right)dx_i \tag{2.4}$$

for each $i\in\{1,2,\ldots,n-1\}$ and also

$$\frac{1}{2\eta\left(b_i,a_i\right)}\int_{a_i}^{a_i+\eta(b_i,a_i)}\left(f_{x_n}^{i+1}\left(a_{i+1}\right)+f_{x_n}^{i+1}\left(b_{i+1}\right)\right)dx_i$$

$$=\frac{1}{2}\left[\frac{1}{\eta\left(b_i,a_i\right)}\int_{a_i}^{a_i+\eta(b_i,a_i)} f_{x_n}^{i+1}\left(a_{i+1}\right)dx_i+\frac{1}{\eta\left(b_i,a_i\right)}\int_{a_i}^{a_i+\eta(b_i,a_i)} f_{x_n}^{i+1}\left(b_{i+1}\right)dx_i\right]$$

$$\leq\frac{1}{2}\left[\begin{array}{c}\frac{f(x_1,\ldots,x_{i-1},a_i,a_{i+1},x_{i+2},\ldots,x_n)+f(x_1,\ldots,x_{i-1},b_i,a_{i+1},x_{i+2},\ldots,x_n)}{2}\\+\frac{f(x_1,\ldots,x_{i-1},a_i,b_{i+1},x_{i+2},\ldots,x_n)+f(x_1,\ldots,x_{i-1},b_i,b_{i+1},x_{i+2},\ldots,x_n)}{2}\end{array}\right]$$

$$=\frac{1}{4}\left[\begin{array}{l} f\left(x_1,\ldots,x_{i-1},a_i,a_{i+1},x_{i+2},\ldots,x_n\right)\\ +f\left(x_1,\ldots,x_{i-1},b_i,a_{i+1},x_{i+2},\ldots,x_n\right)\\ +f\left(x_1,\ldots,x_{i-1},a_i,b_{i+1},x_{i+2},\ldots,x_n\right)\\ +f\left(x_1,\ldots,x_{i-1},b_i,b_{i+1},x_{i+2},\ldots,x_n\right)\end{array}\right] \tag{2.5}$$

for each $i\in\{1,2,\ldots,n-1\}$. Using the inequalities (2.4) and (2.5) in (2.3) and taking summation from 1 to $n-1$, we have the inequality (2.2). □

Theorem 2.2. *Let $f:\Omega^n\to\mathbb{R}_+$ be a preinvex function with respect to η on n-coordinates. If $f_{x_n}^i\in L\left[a_i,a_i+\eta\left(b_i,a_i\right)\right]$ for each $i=1,2,\ldots,n$, then we obtain for $a,b\in\Delta^n$*

$$\sum_{i=1}^{n}\frac{1}{2\eta\left(b_i,a_i\right)}\int_{a_i}^{a_i+\eta(b_i,a_i)}\left(f_{a_n}^i\left(x_i\right)+f_{b_n}^i\left(x_i\right)\right)dx_i$$

$$\leq\frac{n}{2}\left[f\left(\boldsymbol{a}\right)+f\left(\boldsymbol{b}\right)\right]+\frac{1}{2}\sum_{i=1}^{n}\left[f_{a_n}^i\left(b_i\right)+f_{b_n}^i\left(a_i\right)\right]. \tag{2.6}$$

Proof. Using preinvexity of $f_{x_n}^i$ for each $i=1,2,\ldots,n$, then

$$\frac{1}{\eta\left(b_i,a_i\right)}\int_{a_i}^{a_i+\eta(b_i,a_i)} f_{a_n}^i\left(x_i\right)dx_i\leq\frac{f\left(\mathbf{a}\right)+f_{a_n}^i\left(b_i\right)}{2},$$

$$\frac{1}{\eta\left(b_i,a_i\right)}\int_{a_i}^{a_i+\eta(b_i,a_i)} f_{b_n}^i\left(x_i\right)dx_i\leq\frac{f_{b_n}^i\left(a_i\right)+f\left(\mathbf{b}\right)}{2},$$

Thus

$$\frac{1}{\eta\left(a_i,b_i\right)}\int_{a_i}^{a_i+\eta(b_i,a_i)}\left[f_{a_n}^i\left(x_i\right)+f_{b_n}^i\left(x_i\right)\right]dx_i\leq\frac{f\left(\mathbf{a}\right)+f\left(\mathbf{b}\right)+f_{a_n}^i\left(b_i\right)+f_{b_n}^i\left(a_i\right)}{2}.$$

Taking summation from 1 to n the above inequalities we get the inequality (2.6). □

Theorem 2.3. *Let $f:\Omega^n\to\mathbb{R}_+$ be a preinvex function with respect to η on n-coordinates. If $f_{x_n}^i\in L\left[a_i,a_i+\eta\left(b_i,a_i\right)\right]$ for each $i=1,2,\ldots,n$, then the following inequality holds*

$$\begin{aligned}
&f\left(\frac{2a_1+\eta\left(b_1,a_1\right)}{2},\ldots,\frac{2a_{n-1}+\eta\left(b_{n-1},a_{n-1}\right)}{2},\frac{2a_n+\eta\left(b_n,a_n\right)}{2}\right)\\
&\le\left(\prod_{i=1}^{n}\frac{1}{\eta\left(b_i,a_i\right)}\right)\int_{a_1}^{a_1+\eta(b_1,a_1)}\ldots\int_{a_n}^{a_n+\eta(b_n,a_n)}f(x_n)dx_n\ldots dx_1\\
&\le\frac{1}{2^n}\sum_{\delta\in l_i(n)}f\left(\delta\boldsymbol{a}+\left(1-\delta\right)\boldsymbol{b}\right),
\end{aligned}\tag{2.7}$$

where

$$l_i\left(n\right):=\left\{\delta\in\mathbb{N}_0^n:\ \delta_i\le1,\ \left|\delta\right|=n+1-i,\ i=1,\ldots,n+1\right\},$$
$$\left|\delta\right|:=\delta_1+\cdots+\delta_n\in\mathbb{N};\ \delta\boldsymbol{a}:=\left(\delta_1a_1,\ldots,\delta_na_n\right)\in\mathbb{N}_0^n$$

for $\boldsymbol{a},\boldsymbol{b}\in\Delta^n$.

Proof. Let $\omega_n:=a_n+\eta\left(b_n,a_n\right)$. Then using (2.1), we get the following inequality for $f_{x_n}^n$

$$f_{x_n}^n\left(\frac{a_n+\omega_n}{2}\right)\le\frac{1}{\omega_n-a_n}\int_{a_n}^{\omega_n}f_{x_n}^n\left(x_n\right)dx_n\le\frac{f_{x_n}^n\left(a_n\right)+f_{x_n}^n\left(b_n\right)}{2}.\tag{2.8}$$

By integrating on $[a_{n-1},\omega_{n-1}]$, we get

$$\begin{aligned}
&\frac{1}{\omega_{n-1}-a_{n-1}}\int_{a_{n-1}}^{\omega_{n-1}}f_{x_n}^n\left(\frac{a_n+\omega_n}{2}\right)dx_{n-1}\\
&\le\frac{1}{\left(\omega_{n-1}-a_{n-1}\right)\left(\omega_n-a_n\right)}\int_{a_{n-1}}^{\omega_{n-1}}\int_{a_n}^{\omega_n}f_{x_n}^n\left(x_n\right)dx_ndx_{n-1}\\
&\le\frac{1}{\omega_{n-1}-a_{n-1}}\int_{a_{n-1}}^{\omega_{n-1}}\frac{f_{x_n}^n\left(a_n\right)+f_{x_n}^n\left(b_n\right)}{2}dx_{n-1}.
\end{aligned}\tag{2.9}$$

From (2.3), (2.4), respectively

$$f\left(x_1,\ldots,x_{n-2},\frac{a_{n-1}+\omega_{n-1}}{2},\frac{a_n+\omega_n}{2}\right)\le\frac{1}{\omega_{n-1}-a_{n-1}}\int_{a_{n-1}}^{\omega_{n-1}}f_{x_n}^n\left(\frac{a_n+\omega_n}{2}\right)dx_{n-1};\tag{2.10}$$

$$\begin{aligned}
&\frac{1}{\omega_{n-1}-a_{n-1}}\int_{a_{n-1}}^{\omega_{n-1}}\frac{f_{x_n}^n\left(a_n\right)+f_{x_n}^n\left(b_n\right)}{2}dx_{n-1}\\
&=\frac{1}{2\left(\omega_{n-1}-a_{n-1}\right)}\int_{a_{n-1}}^{\omega_{n-1}}f_{x_n}^n\left(a_n\right)dx_{n-1}+\frac{1}{2\left(\omega_{n-1}-a_{n-1}\right)}\int_{a_{n-1}}^{\omega_{n-1}}f_{x_n}^n\left(b_n\right)dx_{n-1}\\
&\le\frac{1}{2^2}\left[\begin{array}{c}f\left(x_1,\ldots,x_{n-2},a_{n-1},a_n\right)+f\left(x_1,\ldots,x_{n-2},b_{n-1},a_n\right)\\+f\left(x_1,\ldots,x_{n-2},a_{n-1},b_n\right)+f\left(x_1,\ldots,x_{n-2},b_{n-1},b_n\right)\end{array}\right].
\end{aligned}\tag{2.11}$$

From (2.9)-(2.11)

$$f\left(x_1,\ldots,x_{n-2},\frac{a_{n-1}+\omega_{n-1}}{2},\frac{a_n+\omega_n}{2}\right)$$

$$\leq \frac{1}{(\omega_{n-1}-a_{n-1})(\omega_n-a_n)}\int_{a_{n-1}}^{\omega_{n-1}}\int_{a_n}^{\omega_n} f_{x_n}^n(x_n)\,dx_n dx_{n-1}$$

$$\leq \frac{1}{2^2}\begin{bmatrix} f(x_1,\ldots,x_{n-2},a_{n-1},a_n)+f(x_1,\ldots,x_{n-2},b_{n-1},a_n) \\ +f(x_1,\ldots,x_{n-2},a_{n-1},b_n)+f(x_1,\ldots,x_{n-2},b_{n-1},b_n)\end{bmatrix}. \tag{2.12}$$

Integrating on $[a_{n-2},\omega_{n-2}]$ by the inequality (2.12)

$$\frac{1}{\omega_{n-1}-a_{n-1}}\int_{a_{n-2}}^{\omega_{n-2}} f\left(x_1,\ldots,x_{n-2},\frac{a_{n-1}+\omega_{n-1}}{2},\frac{a_n+\omega_n}{2}\right)dx_{n-2}$$

$$\leq \left(\prod_{i=n-2}^{n}\frac{1}{\omega_i-a_i}\right)\int_{a_{n-2}}^{\omega_{n-2}}\int_{a_{n-1}}^{\omega_{n-1}}\int_{a_n}^{\omega_n} f_{x_n}^n(x_n)\,dx_n dx_{n-1}dx_{n-2}$$

$$\leq \frac{1}{(\omega_{n-2}-a_{n-2})}\int_{a_{n-2}}^{\omega_{n-2}}\frac{1}{2^2}\begin{bmatrix} f(x_1,\ldots,x_{n-2},a_{n-1},a_n) \\ +f(x_1,\ldots,x_{n-2},b_{n-1},a_n) \\ +f(x_1,\ldots,x_{n-2},a_{n-1},b_n) \\ +f(x_1,\ldots,x_{n-2},b_{n-1},b_n)\end{bmatrix}dx_{n-2}. \tag{2.13}$$

Similarly

$$f\left(x_1,\ldots,x_{n-3},\frac{a_{n-2}+\omega_{n-2}}{2},\frac{a_{n-1}+\omega_{n-1}}{2},\frac{a_n+\omega_n}{2}\right)$$

$$\leq \frac{1}{\omega_{n-2}-a_{n-2}}\int_{a_{n-2}}^{\omega_{n-2}} f\left(x_1,\ldots,x_{n-2},\frac{a_{n-1}+\omega_{n-1}}{2},\frac{a_n+\omega_n}{2}\right)dx_{n-2}; \tag{2.14}$$

$$\frac{1}{\omega_{n-2}-a_{n-2}}\int_{a_{n-2}}^{\omega_{n-2}}\frac{1}{2^2}\begin{bmatrix} f(x_1,\ldots,x_{n-2},a_{n-1},a_n)+f(x_1,\ldots,x_{n-2},b_{n-1},a_n) \\ +f(x_1,\ldots,x_{n-2},a_{n-1},b_n)+f(x_1,\ldots,x_{n-2},b_{n-1},b_n)\end{bmatrix}dx_{n-2}$$

$$\leq \frac{1}{2^3}\begin{bmatrix} f(x_1,\ldots,x_{n-3},a_{n-2},a_{n-1},a_n)+f(x_1,\ldots,x_{n-3},b_{n-2},a_{n-1},a_n) \\ +f(x_1,\ldots,x_{n-3},a_{n-2},b_{n-1},a_n)+f(x_1,\ldots,x_{n-3},b_{n-2},b_{n-1},a_n) \\ +f(x_1,\ldots,x_{n-3},a_{n-2},a_{n-1},b_n)+f(x_1,\ldots,x_{n-3},b_{n-2},a_{n-1},b_n) \\ +f(x_1,\ldots,x_{n-3},a_{n-2},b_{n-1},b_n)+f(x_1,\ldots,x_{n-3},b_{n-2},b_{n-1},b_n)\end{bmatrix}. \tag{2.15}$$

From (2.13)-(2.15)

$$\frac{1}{\omega_{n-2}-a_{n-2}}\int_{a_{n-2}}^{\omega_{n-2}} f\left(x_1,\ldots,x_{n-2},\frac{a_{n-1}+\omega_{n-1}}{2},\frac{a_n+\omega_n}{2}\right)dx_{n-2}$$

$$\leq \left(\prod_{i=n-2}^{n}\frac{1}{\omega_i-a_i}\right)\int_{a_{n-2}}^{\omega_{n-2}}\int_{a_{n-1}}^{\omega_{n-1}}\int_{a_n}^{\omega_n} f_{x_n}^n(x_n)\,dx_n dx_{n-1}dx_{n-2}$$

$$\leq \frac{1}{2^3}\begin{bmatrix} f(x_1,\ldots,x_{n-3},a_{n-2},a_{n-1},a_n)+f(x_1,\ldots,x_{n-3},b_{n-2},a_{n-1},a_n) \\ +f(x_1,\ldots,x_{n-3},a_{n-2},b_{n-1},a_n)+f(x_1,\ldots,x_{n-3},b_{n-2},b_{n-1},a_n) \\ +f(x_1,\ldots,x_{n-3},a_{n-2},a_{n-1},b_n)+f(x_1,\ldots,x_{n-3},b_{n-2},a_{n-1},b_n) \\ +f(x_1,\ldots,x_{n-3},a_{n-2},b_{n-1},b_n)+f(x_1,\ldots,x_{n-3},b_{n-2},b_{n-1},b_n)\end{bmatrix}.$$

Using the above procedure with inductive method, we obtain the inequality (2.7). □

Corollary 2.1. *Under the assumptions Theorem 2.3 for $n = 2$, then*

$$f\left(\frac{2a_1+\eta(b_1,a_1)}{2},\frac{2a_2+\eta(b_2,a_2)}{2}\right)$$
$$\leq \frac{1}{2\eta(b_1,a_1)}\int_{a_1}^{a_1+\eta(b_1,a_1)} f\left(x_1,\frac{2a_2+\eta(b_2,a_2)}{2}\right)dx_1$$
$$+\frac{1}{2\eta(b_2,a_2)}\int_{a_2}^{a_2+\eta(b_2,a_2)} f\left(\frac{2a_1+\eta(b_1,a_1)}{2},x_2\right)dx_2$$
$$\leq \frac{1}{\eta(b_1,a_1)\eta(b_2,a_2)}\int_{a_1}^{a_1+\eta(b_1,a_1)}\int_{a_2}^{a_2+\eta(b_2,a_2)} f(x_1,x_2)\,dx_2dx_1$$
$$\leq \frac{1}{4\eta(b_1,a_1)}\int_{a_1}^{a_1+\eta(b_1,a_1)}[f(x_1,a_2)+f(x_1,b_2)]\,dx_1$$
$$+\frac{1}{4\eta(b_2,a_2)}\int_{a_2}^{a_2+\eta(b_2,a_2)}[f(a_1,x_2)+f(b_1,x_2)]\,dx_2$$
$$\leq \frac{f(a_1,a_2)+f(a_1,b_2)+f(b_1,a_2)+f(b_1,b_2)}{4}.$$

Example 2.2. Let $f:\Omega^3\to\mathbb{R}_+$ be a preinvex function with respect to η on n-coordinates. If $f^i_{x_n}\in L[a_i,a_i+\eta(b_i,a_i)]$ for each $i=1,2,3$, then the following inequality holds

$$f\left(\frac{2a_1+\eta(b_1,a_1)}{2},\frac{2a_2+\eta(b_2,a_2)}{2},\frac{2a_3+\eta(b_3,a_3)}{2}\right)$$
$$\leq \frac{1}{\eta(b_1,a_1)\eta(b_2,a_2)\eta(b_3,a_3)}\int_{a_1}^{a_1+\eta(b_1,a_1)}\int_{a_2}^{a_2+\eta(b_2,a_2)}\int_{a_3}^{a_3+\eta(b_3,a_3)} f(x_1,x_2,x_3)\,dx_3dx_2dx_1$$
$$\leq \frac{1}{2^3}\left[\begin{array}{c} f(a_1,a_2,a_3)+f(b_1,a_2,a_3)+f(a_1,b_2,a_3)+f(b_1,b_2,a_3)\\ +f(a_1,a_2,b_3)+f(b_1,a_2,b_3)+f(a_1,b_2,b_3)+f(b_1,b_2,b_3)\end{array}\right].$$

Indeed according to Theorem 2.3 for n = 3, we get $f^3_{x_3}(x_3):=f(x_1,x_2,x_3)$ and $l_i(3):=\{\delta\in\mathbb{N}_0^3:\ \delta_i\leq 1,\ |\delta|=4-i\},\ i=1,2,3,4$. Then

$$l_1(3)=\{(1,1,1)\};\ l_2(3)=\{(0,1,1),(1,0,1),\ (1,1,0)\},$$
$$l_3(3)=\{(0,0,1),(0,1,0),\ (1,0,0)\};\ l_4(3)=\{(0,0,0)\},$$

and for $\mathbf{a}=(a_1,a_2,a_3)$, $\mathbf{b}=(b_1,b_2,b_3)\in\Delta^3$

$$\sum_{\delta\in l_1(3)} f(\delta\mathbf{a}+(1-\delta)\mathbf{b})$$
$$= f((1,1,1)(a_1,a_2,a_3)+[(1,1,1)-(1,1,1)](b_1,b_2,b_3)) = f(a_1,a_2,a_3);$$
$$\sum_{\delta\in l_2(3)} f(\delta\mathbf{a}+(1-\delta)\mathbf{b}) = f((0,1,1)(a_1,a_2,a_3)+[(1,1,1)-(0,1,1)](b_1,b_2,b_3))$$
$$+f((1,0,1)(a_1,a_2,a_3)+[(1,1,1)-(1,0,1)](b_1,b_2,b_3))$$
$$+f((1,1,0)(a_1,a_2,a_3)+[(1,1,1)-(1,1,0)](b_1,b_2,b_3))$$
$$= f(a_1,b_2,b_3)+f(a_1,b_2,a_3)+f(a_1,a_2,b_3);$$

So,

$$\sum_{\delta\in l_3(3)} f(\delta\mathbf{a}+(1-\delta)\mathbf{b}) = f(b_1,b_2,a_3)+f(b_1,a_2,b_3)+f(a_1,b_2,b_3);$$

$$\sum_{\delta\in l_4(3)} f\left(\delta\mathbf{a}+(1-\delta)\,\mathbf{b}\right)=f\left(b_1,b_2,b_3\right).$$

Thus

$$\sum_{\delta\in l_i(3)} f\left(\delta\mathbf{a}+(1-\delta)\,\mathbf{b}\right)=f\left(a_1,a_2,a_3\right)$$

$$+f\left(a_1,b_2,b_3\right)+f\left(a_1,b_2,a_3\right)+f\left(a_1,a_2,b_3\right)$$

$$+f\left(b_1,b_2,a_3\right)+f\left(b_1,a_2,b_3\right)+f\left(a_1,b_2,b_3\right)+f\left(b_1,b_2,b_3\right).$$

Using all of the above equalities in (2.12), we obtain the desired result in this example.

Acknowledgements. The authors would like to thank to Assoc. Prof. İmdat İŞCAN from Giresun University owing to enlightening ideas.

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