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A DIFFERENTIAL INEQUALITY INVOLVING RUSCHEWEYH OPERATOR

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Abstract. In the present paper, we study certain differential inequalities involving Ruscheweyh operator. As particular cases to our main result, we derive certain results for starlike and convex functions.

1. Introduction

Let $\mathcal{H}$ denote the class of functions f, analytic in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$. Let $\mathcal{A}_n$ be the subclass of $\mathcal{H}$, consisting functions f of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \text{ for } n \in \mathbb{N} = \{1,\ 2,\ 3,\ \ldots\},$$

in $\mathbb{E}$. A function $f \in \mathcal{A}_n$ is said to be starlike of order α if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ 0 \leq \alpha < 1,\ z \in \mathbb{E}.$$

The class of such functions is denoted by $\mathcal{S}_n^*(\alpha)$. A function $f \in \mathcal{A}_n$ is said to be convex of order α in $\mathbb{E}$, if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,\ 0 \leq \alpha < 1,\ z \in \mathbb{E}.$$

Let $\mathcal{K}_n(\alpha)$, denote the class of all functions $f \in \mathcal{A}_n$ that are convex of order α in $\mathbb{E}$. Note that $\mathcal{S}_1^*(\alpha) = \mathcal{S}^*(\alpha)$ and $\mathcal{K}_1(\alpha) = \mathcal{K}(\alpha)$, $0 \leq \alpha < 1$ are the usual classes of univalent starlike functions and univalent convex functions. Also note that $\mathcal{A}_1 = \mathcal{A}$. Let f and g be two analytic functions in open unit disk $\mathbb{E}$. Then we say f is subordinate to g in $\mathbb{E}$, denoted by $f \prec g$ if there exist a Schwarz function w analytic in $\mathbb{E}$ with $w(0) = 0$ and $|w(z)| < 1,\ z \in \mathbb{E}$ such that $f(z) = g(w(z)),\ z \in \mathbb{E}$. In case the function g is univalent, the

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above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.
The Taylor's series expansions of $f,\ g \in \mathcal{A}$ are given as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \text{ and } g(z) = z + \sum_{k=2}^{\infty} b_k z^k.$$

Then the convolution/Hadamard product of f and g is denoted by $f * g$, and defined as

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

Ruscheweyh [4] introduced a differential operator R^{λ}, (Known as Ruscheweyh differential operator) for $f \in \mathcal{A}$ is defined as follows

$$R^{\lambda} f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z),\ \lambda \geq -1,\ z \in \mathbb{E}. \tag{1.1}$$

For $\lambda = n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$

$$R^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!},\ z \in \mathbb{E}.$$

Lecko et al. [3] observed that for $\lambda \geq -1$, the expression given in (1.1) becomes

$$R^{\lambda} f(z) = z + \sum_{k=2}^{\infty} \frac{(\lambda+1)(\lambda+2)\ldots(\lambda+k-1)}{(k-1)!} a_k z^k,\ z \in \mathbb{E},$$

and for every $\lambda > -1$

$$\begin{aligned} R^1 R^{\lambda} f(z) &= z(R^{\lambda} f)'(z) = z\left(\frac{z}{(1-z)^{\lambda+1}} * f(z)\right)' \\ &= \frac{z}{(1-z)^{\lambda+1}} * (zf'(z)) = R^{\lambda}(zf'(z)) = R^{\lambda} R^1 f(z),\ z \in \mathbb{E}. \end{aligned}$$

We notice that

$$R^{-1} f(z) = z,\ R^0 f(z) = f(z),\ R^1 f(z) = zf'(z) \text{ and } R^2 f(z) = zf'(z) + \frac{z^2}{2} f''(z),$$

and so on. For $\lambda \geq -1$ and for $z \in \mathbb{E}$, we have

$$z(R^{\lambda} f)'(z) = (\lambda+1) R^{\lambda+1} f(z) - \lambda R^{\lambda} f(z). \tag{1.2}$$

Wang et al.[7] introduced and studied the following class $N(\lambda,\ \alpha,\ A,\ B)$ of non-Bazilevič functions, defined as follows:

$$N(\lambda,\ \alpha,\ A,\ B) = \left\{ f \in \mathcal{A} : (1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha} - \lambda \frac{zf'(z)}{f(z)} \left(\frac{z}{f(z)}\right)^{\alpha} \prec \frac{1+Az}{1+Bz} \right\},$$

where $0 < \alpha < 1,\ \lambda \in \mathbb{C},\ -1 \leq B \leq 1,\ A \neq B,\ A \in \mathbb{R}$. They made some estimates on $\left(\frac{z}{f(z)}\right)^{\alpha}$. Shanmugam et al. [6] studied the differential operator $(1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha} - \lambda \frac{zf'(z)}{f(z)} \left(\frac{z}{f(z)}\right)^{\alpha}$ using differential subordination to obtain best dominant for $\left(\frac{z}{f(z)}\right)^{\alpha}$.

Recently, Shams et al. [5] studied the Ruscheweyh derivative operator for $f \in \mathcal{A}_n$ that satisfies the following condition:

$$\left| \left(1-\alpha+\alpha(\lambda+2)\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)}\right)\left(\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)^{\mu} - \alpha(\lambda+1)\left(\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)^{\mu+1} - 1\right| < M,$$

and obtained the values of $M,\ \alpha,\ \delta$ and μ for which the function had become starlike of order δ.

The results of above nature motivated us for the work of present paper. We, here, study the following differential inequality

$$\left|\left(\frac{R^{\lambda}f(z)}{R^{\lambda+1}f(z)}\right)^{\beta}\left[1+\alpha-\alpha\left((\lambda+2)\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)}-(\lambda+1)\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)\right]-1\right| < M,$$

where $\alpha > 0,\ \beta > 0$ and $\lambda \geq -1$ and obtain certain results for starlike and convex functions in particular cases.

2. Preliminary

To prove our main result, we shall make use of the following lemma due to Hallenbeck and Ruscheweyh [2].

Lemma 2.1. *Let G be a convex function in $\mathbb{E}$, with $G(0) = a$ and let γ be a complex number with $\Re(\gamma) > 0$. If $F(z) = a + a_n z^n + a_{n+1}z^{n+1} + \ldots$ is analytic in $\mathbb{E}$ and $F \prec G$, then*

$$\frac{1}{z^{\gamma}}\int_0^z F(w)w^{\gamma-1}dw \prec \frac{1}{nz^{\gamma/n}}\int_0^z G(w)w^{\frac{\gamma}{n}-1}dw.$$

3. Main Results

Theorem 3.1. *Let $\alpha,\ \beta$ and δ be real numbers such that $\alpha > 0,\ \beta > 0,\ 0 \leq \delta < 1$ and*

$$M = M(\alpha,\ \beta,\ \delta,\ n) = \frac{\alpha(1-\delta)[n\alpha+\beta]}{\alpha[n+\beta(1-\delta)]+2\beta} > 0,\ \text{where } n \in \mathbb{N}. \tag{3.1}$$

If $f \in \mathcal{A}_n$ satisfies

$$\left|\left(\frac{R^{\lambda}f(z)}{R^{\lambda+1}f(z)}\right)^{\beta}\left[1+\alpha-\alpha\left((\lambda+2)\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)}-(\lambda+1)\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)\right]-1\right| < M \tag{3.2}$$

then

$$\Re\left((\lambda+2)\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)}-(\lambda+1)\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right) > \delta,\ \lambda \geq -1,\ z \in \mathbb{E}.$$

Proof. Define $\left(\frac{R^{\lambda}f(z)}{R^{\lambda+1}f(z)}\right)^{\beta} = u(z),\ z \in \mathbb{E}.$

On differentiating logarithmically, we get

$$\beta\left[\frac{z(R^{\lambda}f(z))'}{R^{\lambda}f(z)} - \frac{z(R^{\lambda+1}f(z))'}{R^{\lambda+1}f(z)}\right] = \frac{zu'(z)}{u(z)} \tag{3.3}$$

Using the equality (1.2), the above equation reduces to

$$(\lambda+2)\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)}-(\lambda+1)\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}=1-\frac{zu'(z)}{\beta u(z)}.$$

Now, from (3.2), we obtain

$$u(z)+\frac{\alpha}{\beta}zu'(z)\prec 1+Mz.$$

Taking $\gamma=\dfrac{\beta}{\alpha}$ and using Lemma 2.1, we get

$$u(z)\prec 1+\frac{\beta Mz}{n\alpha+\beta},$$

or

$$|u(z)-1|<\frac{\beta M}{n\alpha+\beta}<1,$$

therefore

$$|u(z)|>1-\frac{\beta M}{n\alpha+\beta}. \tag{3.4}$$

On writing $(\lambda+2)\dfrac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)}-(\lambda+1)\dfrac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}=(1-\delta)w(z)+\delta,\ 0\leq\delta<1,$ (3.2) becomes

$$|u(z)\{1+\alpha-\alpha[(1-\delta)w(z)+\delta]\}-1|<M.$$

We need to show that $\Re(w(z))>0,\ \ z\in\mathbb{E}$. If possible, suppose that $\Re(w(z))\not>0,\ \ z\in\mathbb{E}$, then there must exist a point $z_0\in\mathbb{E}$ such that $w(z_0)=ix,\ \ x\in\mathbb{R}$. So, it is sufficient to prove that

$$|u(z_0)\{1+\alpha-\alpha[(1-\delta)ix+\delta]\}-1|\geq M. \tag{3.5}$$

In view of (3.4), we obtain

$$\begin{aligned}
&|u(z_0)\{1+\alpha-\alpha[(1-\delta)ix+\delta]\}-1]|\\
&\geq|u(z_0)[1+\alpha(1-\delta)-\alpha(1-\delta)ix]|-1\\
&=\sqrt{[1+\alpha(1-\delta)]^2+\alpha^2(1-\delta)^2x^2}|u(z_0)|-1\\
&\geq|1+\alpha(1-\delta)||u(z_0)|-1\\
&\geq|1+\alpha(1-\delta)|\left(1-\frac{\beta M}{n\alpha+\beta}\right)-1\geq M.
\end{aligned} \tag{3.6}$$

Thus in view of (3.1), (3.6) is true and hence (3.5) holds. Therefore $\Re(w(z))>0$ *i.e.*

$$\Re\left((\lambda+2)\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)}-(\lambda+1)\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)>\delta,\ \ z\in\mathbb{E}.$$

□

Remark 3.1. From Theorem 3.1, it follows that for $\alpha > 0,\ \beta > 0$ and $0 \le \delta < 1$, if $f \in \mathcal{A}_n$ satisfies

$$\left|\left(\frac{R^\lambda f(z)}{R^{\lambda+1} f(z)}\right)^\beta \left[\frac{1}{\alpha} + 1 - \left((\lambda+2)\frac{R^{\lambda+2} f(z)}{R^{\lambda+1} f(z)} - (\lambda+1)\frac{R^{\lambda+1} f(z)}{R^\lambda f(z)}\right)\right] - \frac{1}{\alpha}\right| < \frac{(1-\delta)(n\alpha+\beta)}{\alpha[n+\beta(1-\delta)]+2\beta}$$

then

$$\Re\left((\lambda+2)\frac{R^{\lambda+2} f(z)}{R^{\lambda+1} f(z)} - (\lambda+1)\frac{R^{\lambda+1} f(z)}{R^\lambda f(z)}\right) > \delta,\ \ z \in \mathbb{E}.$$

Taking $\alpha \to \infty$ in the above remark, we obtain:

Theorem 3.2. *Let $\beta,\ \delta$ be real numbers such that $\beta > 0,\ \ 0 \le \delta < 1$. If $f \in \mathcal{A}_n$ satisfies*

$$\left|\left(\frac{R^\lambda f(z)}{R^{\lambda+1} f(z)}\right)^\beta \left[1 - (\lambda+2)\frac{R^{\lambda+2} f(z)}{R^{\lambda+1} f(z)} + (\lambda+1)\frac{R^{\lambda+1} f(z)}{R^\lambda f(z)}\right]\right| < \frac{n(1-\delta)}{n+\beta(1-\delta)}$$

then

$$\Re\left((\lambda+2)\frac{R^{\lambda+2} f(z)}{R^{\lambda+1} f(z)} - (\lambda+1)\frac{R^{\lambda+1} f(z)}{R^\lambda f(z)}\right) > \delta,\ \ z \in \mathbb{E}.$$

Setting $\lambda = -1$ in Theorem 3.1, we obtain:

Corollary 3.1. *Let $\alpha,\ \beta,\ \delta$ be real numbers such that $\alpha > 0,\ \beta > 0$ and $0 \le \delta < 1$. If $f \in \mathcal{A}_n$ satisfies*

$$\left|(1+\alpha)\left(\frac{z}{f(z)}\right)^\beta - \alpha\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^\beta - 1\right| < \frac{\alpha(1-\delta)(n\alpha+\beta)}{\alpha[n+\beta(1-\delta)]+2\beta},$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \delta,\ \ z \in \mathbb{E}.$$

Hence $f \in \mathcal{S}_n^*(\delta)$.

Taking $\lambda = -1$ in Theorem 3.2, we get the following result:

Corollary 3.2. *Let $\beta,\ \delta$ be real numbers such that $\beta > 0,\ \ 0 \le \delta < 1$. If $f \in \mathcal{A}$ satisfies*

$$\left|\left(\frac{z}{f(z)}\right)^\beta - f'(z)\left(\frac{z}{f(z)}\right)^{\beta+1}\right| < \frac{n(1-\delta)}{n+\beta(1-\delta)}$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \delta,\ \ z \in \mathbb{E}.$$

Hence $f \in \mathcal{S}_n^*(\delta)$.

Remark 3.2. For $n = 1$ in Corollaries 3.1 and 3.2, we get the results of Billing [1] for $p = 1$.

Setting $\lambda = 0$ in Theorem 3.1, we obtain:

Corollary 3.3. *Let α, β and δ be real numbers such that $\alpha > 0$, $0 \leq \delta < 1$, $\beta > 0$. If $f \in \mathcal{A}_n$ satisfies*

$$\left|(1-\alpha)\left(\frac{f(z)}{zf'(z)}\right)^{\beta} - \alpha\left(\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}\right)\left(\frac{f(z)}{zf'(z)}\right)^{\beta} - 1\right| < \frac{\alpha(1-\delta)[n\alpha+\beta]}{\alpha[n+\beta(1-\delta)]+2\beta},$$

then

$$\Re\left(2 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) > \delta,\ z \in \mathbb{E}.$$

For $\lambda = -1$ and replacing $f(z)$ with $zf'(z)$ in Theorem 3.1, we get the result as below:

Corollary 3.4. *Let α, β and δ be real numbers such that $\alpha > 0$, $\beta > 0$ and $0 \leq \delta < 1$. If $f \in \mathcal{A}$ satisfies*

$$\left|\left(\frac{1}{f'(z)}\right)^{\beta}\left[1 + \alpha - \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] - 1\right| < \frac{\alpha(1-\delta)(n\alpha+\beta)}{\alpha[n+\beta(1-\delta)]+2\beta},$$

then

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \delta,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{K}_n(\delta)$.

Putting $\lambda = -1$ and replacing $f(z)$ with $zf'(z)$ in Theorem 3.2, we obtain:

Corollary 3.5. *Let β and δ be real numbers such that $\beta > 0$ and $0 \leq \delta < 1$. If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{zf''(z)}{(f'(z))^{\beta+1}}\right| < \frac{n(1-\delta)}{n+\beta(1-\delta)},$$

then

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \delta,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{K}_n(\delta)$.

Remark 3.3. Setting $n = 1$ in Corollaries 3.4 and 3.5, we get the results of Billing [1] for $p = 1$.

Selecting $\lambda = 0 = \delta$ and $\beta = 1$ in Theorem 3.1, we have:

Corollary 3.6. *Let real number $\alpha > 0$. If $f \in \mathcal{A}_n$ satisfies the following differential inequality*

$$\left|(1-\alpha)\left(\frac{f(z)}{zf'(z)} - 1\right) - \alpha\frac{zf''(z)}{f'(z)}\right| < \frac{n\alpha^2}{(n+1)\alpha+2}$$

then

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \Re\left(\frac{zf'(z)}{f(z)}\right) - 1,\ z \in \mathbb{E}.$$

For $\lambda = 0 = \delta$ and $\beta = 1$ in Theorem 3.2, we obtain:

Corollary 3.7. *If $f \in \mathcal{A}_n$ satisfies the following differential inequality*

$$\left|1 - \left(1 + \frac{zf''(z)}{f'(z)}\right)\frac{f(z)}{zf'(z)}\right| < \frac{n}{n+1}$$

then

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \Re\left(\frac{zf'(z)}{f(z)}\right) - 1,\ z \in \mathbb{E}.$$

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