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INSTABILITY FOR MACKEY-GLASS DIFFERENTIAL EQUATION WITH VARIABLE COEFFICIENTS

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ABSTRACT. This paper considers the Hyers-Ulam instability for Mackey-Glass differential equation with variable coefficients and a nonconstant delay. It also investigates the Hyers -Ulam-Rassias instability for Mackey-Glass equation. The results are illustrated by given examples.

1. INTRODUCTION

In 1940, Ulam [30] posed an important problem before the Mathematics Club of Wisconsin University concerning the stability of group homomorphisms . A significant breakthrough came in 1941, when Hyers [7] gave an answer to Ulam's problem. During the last two decades very important contributions to the stability problems of functional equations were given by many mathematicians [4, 6–10, 14, 15, 18–20, 26]. More than twenty years ago, a generalization of Ulam's problem was proposed by replacing functional equations with differential equations: The differential equation $F(t, y(t), y'(t), ..., y^{(n)}(t)) = 0$ has the Hyers-Ulam stability if for given $\varepsilon > 0$ and a function y such that

$$\left|F(t, y(t), y'(t), ..., y^{(n)}(t))\right| \leq \varepsilon$$

there exists a solution y_0 of the differential equation such that

$$|y(t) - y_0(t)| \leq K(\varepsilon)$$

and $\lim\limits_{\varepsilon\to 0} K(\varepsilon) = 0$.

The first step in the direction of investigating the Hyers-Ulam stability of differential equations was taken by Obloza [16, 17]. Thereafter, Alsina and Ger [1] have studied the Hyers-Ulam stability of the linear differential equation $y'(t) = y(t)$. The Hyers-Ulam stability problems of linear differential equations of first order and second order with constant coefficients were studied in the papers [31] by using the method of integral factors. The

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results given in [11, 27, 28] have been generalized by Popa and Rus [22, 23] for the linear differential equations of nth order with constant coefficients.

In addition to above-mentioned studies, several authors have studied the Hyers-Ulam stability for differential equations of first and second order [5, 12, 13, 29]. In [24] sufficient conditions of Ulam stabilities for Mackey-Glass equation with variable coefficients had been established. In [2] Brillouët-Belluot indicated that there are only few outcomes of which we could say that they concern nonstability of functional equations. However, in [25], Qarawani investigated the Hyers-Ulam instability of linear and nonlinear differential equations of second order.

This paper investigates the Hyers-Ulam-Rassias and Hyers-Ulam instability of Mackey-Glass equation with variable coefficients :

$$x' + \alpha(t)x = \frac{\beta(t)x(g(t))}{1 + x^{\gamma}(g(t))}, \qquad t \geq 0 \tag{1.1}$$

with the initial condition

$$x(0) = 0 \tag{1.2}$$

and the initial function $x(t) = \varphi(t), \forall t \leq 0$, where $\varphi(t)$ is positive and continuous on $(-\infty, 0)$ $, x \in C^1(I)$, $I = [a, b]$, $0 < a < b \leq \infty$ $, \gamma > 0$, $\alpha(t) : I \to [0, \infty)$ is a bounded function and $\beta(t)$ is a positive function on I, and $g(t)$ is a nonnegative continuous function such that $g(0) = 0$, $g(t) \leq t$ and $\lim\limits_{t\to\infty} g(t) = \infty$.

2. Preliminaries

Definition 2.1. [27] We say that equation (1.1) has the Hyers-Ulam (HU) stability with initial conditions (1.2) if there exists a positive constant $K > 0$ with the following property:

For every $\varepsilon > 0, x \in C^1[a, b]$, if

$$|x' + \alpha(t)x - \frac{\beta(t)x(g(t))}{1 + x^{\gamma}(g(t))}| \leq \varepsilon \tag{2.1}$$

and $x(0) = 0$, there exists some $w \in C^1[a, b]$ satisfying the equation (1.1) and $w(0) = 0$, such that $|w(t) - x(t)| \leq K\varepsilon$.

Definition 2.2. [27] We say that equation (1.1) with initial condition (1.2) has the Hyers-Ulam-Rassias (HUR) stability with respect to θ if there exists a positive constant $K > 0$with the following property:

For each $x(t)$ satisfying

$$\left|x' + \alpha(t)x - \frac{\beta(t)x(g(t))}{1 + x^{\gamma}(g(t))}\right| \leq \theta(t) \tag{2.2}$$

there exists some solution $w \in C^1[a, b]$ of the equation (1.1) with (1.2) such that $|x(t) - w(t)| \leq K\theta(t)$.

3. On Hyers-Ulam and Hyers-Ulam-Rassias instability of solutions

Theorem 3.1. *Assume that $x : I \to \mathbb{R}$ is a continuously differentiable function and satisfies the inequality (2.1), and the initial condition (1.2). Suppose that*

a) $\sup\limits_{t\geq 0}\left[\int\limits_0^t \exp\left(-\int\limits_s^t \alpha(\tau)d\tau\right)ds\right] < \infty,$

b) $\sup\limits_{t\geq 0}\left[\int\limits_0^t \beta(s)\exp\left(-\int\limits_s^t \alpha(\tau)d\tau\right)ds\right] = \infty.$

Then the problem (1.1)-(1.2) is unstable in the sense of HU on a finite interval I.

Proof. Suppose that $\varepsilon > 0$ and $x \in C^1(I)$ satisfies the inequality (2.1) and the initial condition $x(0) = 0$. We will show that zero solution $y_0(t) \equiv 0$ of the equation (1.1) will satisfy the inequality $\sup\limits_{x\geq x_0}|x(t) - y_0(t)| > k\varepsilon$. On the contrary, let us assume that there exists $\varepsilon_0 > 0$ such that $\sup\limits_{x\geq x_0}|x(t)| \leq k\varepsilon_0$. Then we can find a constant $M > 0$ such that $M = \sup\limits_{t\geq t_0}|x(t)|$.

Multiplying inequality (2.1) by integrating factor $\exp\left(\int\limits_0^t \alpha(s)ds\right)$, we get

$$\left|\left(x(t)\exp\left(\int_0^t \alpha(s)ds\right)\right)' - \frac{\beta(t)x(g(t))\exp\left(\int\limits_0^t \alpha(s)ds\right)}{1 + x^\gamma(g(t))}\right| \leq \varepsilon\exp\left(\int_0^t \alpha(s)ds\right). \tag{3.1}$$

By integrating inequality (3.1) from 0 to t we obtain that

$$\begin{aligned} -\varepsilon\int_0^t \exp\left(\int_0^s \alpha(\tau)d\tau\right)ds &\leq x(t)\exp\left(\int_0^t \alpha(s)ds\right) - \int_0^t \frac{\beta(s)x(g(s))\exp\left(\int\limits_0^s \alpha(\tau)d\tau\right)}{1 + x^\gamma(g(s))} \\ &\leq \varepsilon\int_0^t \exp\left(\int_0^s \alpha(\tau)d\tau\right)ds. \end{aligned} \tag{3.2}$$

Multiplying (3.2) by $\exp\left(-\int\limits_0^t \alpha(s)ds\right)$, we get

$$\begin{aligned} -\varepsilon\int_0^t \exp\left(-\int_s^t \alpha(\tau)d\tau\right)ds &\leq x(t) - \int_0^t \frac{\beta(s)x(g(s))\exp\left(-\int\limits_s^t \alpha(\tau)d\tau\right)}{1 + x^\gamma(g(s))}ds \\ &\leq \varepsilon\int_0^t \exp\left(-\int_s^t \alpha(\tau)d\tau\right)ds. \end{aligned} \tag{3.3}$$

Here it should be noted that the solution $x(t)$ is an increasing and positive on $(0,\infty)$ [31].

Now, let us denote by $k = \sup\limits_{t\geq 0}\left[\int\limits_0^t \exp\left(-\int\limits_s^t \alpha(\tau)d\tau\right)ds\right] < \infty$. Then, by virtue of (3.3), we obtain

$$-k\varepsilon \leq x(t) - \int\limits_0^t \frac{\beta(t)x(g(s))\exp\left(-\int\limits_s^t \alpha(\tau)d\tau\right)}{1+x^{\gamma}(g(s)} ds \leq k\varepsilon.$$

Using the triangle inequality and because of $x^{\gamma}(g(s) \geq 0$ we obtain for $x(t)$ the estimation

$$\begin{aligned} x(t) &\geq -k\varepsilon + \int\limits_0^t \frac{\beta(t)x(g(s))\exp\left(-\int\limits_s^t \alpha(\tau)d\tau\right)}{1+x^{\gamma}(g(s))} ds \\ &\geq -k\varepsilon + \int\limits_0^t \beta(t)x(g(s))\exp\left(-\int\limits_s^t \alpha(\tau)d\tau\right) ds. \end{aligned} \tag{3.4}$$

By applying mean value theorem to integral in (3.4), we have

$$x(t) \geq -k\varepsilon + x(g(s^*))\int\limits_0^t \beta(s)\exp\left(-\int\limits_s^t \alpha(\tau)d\tau\right) ds$$

where $s^* \in [0,t]$. By virtue of conditions (a), (b), from (3.4) we get that the solution $x(t)$ will be unbounded and hence we get a contradiction.

Obviously, $y_0(t) \equiv 0$ is a solution of the equation (1.1) satisfying the initial condition (1.2) and such that $|x(t) - y_0(t)| > K\varepsilon$, which completes the proof.

$\square$

Now we give an example illustrating Theorem 3.1

Example 3.1. Consider the equation

$$x' + \frac{2}{t+1}x = \frac{e^t x\left(t-\frac{t}{2}\right)}{1+x^{10}\left(t-\frac{t}{2}\right)} \qquad ,0 \leq t \leq b \leq \infty \tag{3.5}$$

with the initial condition

$$x(0) = 0.$$

and with $g(t) = \dfrac{t}{2} \leq t$, $\lim\limits_{t\to\infty} g(t) = \infty$. Let the history function $\varphi(t) = -t, t \in [-1,0]$. Consider the inequality

$$\left| x' + \frac{2}{t+1}x - \frac{e^t x\left(\frac{t}{2}\right)}{1 + x^{10}\left(\frac{t}{2}\right)} \right| \leq \varepsilon. \tag{3.6}$$

Suppose that $\varepsilon > 0$ and $x \in C^1(I)$ satisfies the inequality (3.6) and the initial condition $x(0) = 0$. We will show that zero solution $y_0(t) \equiv 0$ of the equation (3.5) will satisfy the inequality $\sup\limits_{x \geq x_0} |x(t) - y_0(t)| > k\varepsilon$. On the contrary, let us assume that there exists $\varepsilon_0 > 0$ such that $\sup\limits_{x \geq x_0} |x(t)| \leq k\varepsilon_0$. Then we can find a constant $M > 0$ such that $M = \sup\limits_{t\ \geq t_0} |x(t)|$. Multiplying (3.6) by

$$\exp\left(\int_0^t \frac{2}{s+1}ds\right) = (t+1)^2,$$

and integrating with respect to t, we obtain the inequality

$$\left| x(t)(t+1)^2 - \int_0^t \frac{(s+1)^2 e^s x\left(\frac{s}{2}\right)}{1 + \left[x\left(\frac{s}{2}\right)\right]^{10}} ds \right| \leq \varepsilon(t+1)^2. \tag{3.7}$$

Now multiply inequality (3.7) by $\exp\left(-\int_0^t \frac{2}{s+1}ds\right) = (t+1)^{-2}$ to get

$$-\varepsilon \leq x(t) - (t+1)^{-2} \int_0^t \frac{(s+1)^2 e^s x\left(\frac{s}{2}\right)}{1 + \left[x\left(\frac{s}{2}\right)\right]^{10}} ds \leq \varepsilon.$$

Since $x\,(t) \geq 0$ and hence $1 + \left[x\left(\frac{t}{2}\right)\right]^{10} \geq 1$, we have the estimate

$$x(t) \geq -\varepsilon + (t+1)^{-2} \int_0^t (s+1)^2 e^s x\left(\frac{s}{2}\right) ds. \tag{3.8}$$

By applying mean value theorem to integral in inequality to (3.8), we have

$$\begin{aligned} x(t) &\geq -\varepsilon + x\left(\frac{s^*}{2}\right)(t+1)^{-2} \int_0^t (s+1)^2 e^s ds \\ &= x\left(\frac{s^*}{2}\right) e^t \left(\frac{t^2 - e^{-t} + 1}{(t+1)^2}\right) - \varepsilon. \end{aligned} \tag{3.9}$$

Since $\lim\limits_{t\to\infty} e^t \left(\frac{t^2 - e^{-t} + 1}{(t+1)^2}\right) = \infty$ and from (3.8) we get $x(t) \to \infty$, as $t \to \infty$. The contradiction proves the instability of Eq. (3.5).

Now we will establish HUR instability for Eq. (1.1) in an infinite interval $0 \le a \le t \le b,\ b = \infty$.

Theorem 3.2. *Let $x : I \to \mathbb{R}$ be a continuously differentiable function and satisfy Eq. (2.2) with initial condition (1.2). Assume there exists a continuous function $\theta(t) : [0, \infty) \to (0, \infty)$ such that*

$$\sup_{t\ \ge 0} \int_0^t \theta(s) \exp\left(- \int_s^t \alpha(\tau) d\tau \right) ds < \infty,$$

and the following condition is satisfied

$$\sup_{t\ \ge 0} \int_0^t \beta(s) \exp\left(- \int_s^t \alpha(\tau) d\tau \right) ds = \infty.$$

Then the problem (1.1)-(1.2) is unstable in the sense of HUR as $t \to \infty$.

Proof. Suppose that $\theta(t) > 0$ and $x \in C^1(I)$ satisfies the inequality (3.14) and the initial condition $x(0) = 0$. We will show that zero solution $y_0(t) \equiv 0$ of the equation (3.13) will satisfy the inequality $\sup\limits_{x \ge x_0} |x(t) - y_0(t)| > k\theta(t)$. On the contrary, let us assume that there exists $\theta_0(t) > 0$ such that $\sup\limits_{x \ge x_0} |x(t)| \le k\theta_0(t)$. Then we can find a constant $M > 0$ such that $M = \sup\limits_{t\ \ge t_0} |x(t)|$.

Now multiplying (2.2) by integrating factor $\exp\left(\int\limits_0^t \alpha(s) ds \right)$, we get

$$\left| \left(x(t) \exp\left(\int_0^t \alpha(s) ds \right) \right)' - \frac{\beta(t) x(g(t)) \exp\left(\int\limits_0^t \alpha(s) ds \right)}{1 + x^\gamma(g(t))} \right| \le \theta(t) \exp\left(\int_0^t \alpha(s) ds \right). \tag{3.10}$$

By integrating inequality (3.10) from 0 to t we obtain

$$\begin{aligned} - \int_0^t \theta(s) \exp\left(\int_0^s \alpha(\tau) d\tau \right) ds \le x(t) \exp\left(\int_0^t \alpha(s) ds \right) - \int_0^t \frac{\beta(s) x(g(s)) \exp\left(\int\limits_0^s \alpha(\tau) d\tau \right)}{1 + x^\gamma(g(s))} \\ \le \int_0^t \theta(s) \exp\left(\int_0^s \alpha(\tau) d\tau \right) ds. \end{aligned} \tag{3.11}$$

Multiplying (3.11) by $\exp\left(-\int\limits_0^t \alpha(s)ds\right)$ we get

$$-\int\limits_0^t \theta(s)\exp\left(-\int\limits_s^t \alpha(\tau)d\tau\right)ds \le x(t) - \int\limits_0^t \frac{\beta(s)x(g(s))\exp\left(-\int\limits_s^t \alpha(\tau)d\tau\right)}{1+x^\gamma(g(s))}ds$$
$$\le \int\limits_0^t \theta(s)\exp\left(-\int\limits_s^t \alpha(\tau)d\tau\right)ds.$$

Since $x^\gamma(g(s) \ge 0$ and using the triangle inequality, for $x(t)$ we obtain the estimation

$$x(t) \ge -\int\limits_0^t \theta(s)\exp\left(-\int\limits_s^t \alpha(\tau)d\tau\right)ds + \int\limits_0^t \frac{\beta(t)x(g(s))\exp\left(-\int\limits_s^t \alpha(\tau)d\tau\right)}{1+x^\gamma(g(s))}ds$$
$$\ge \int\limits_0^t \beta(t)x(g(s))\exp\left(-\int\limits_s^t \alpha(\tau)d\tau\right)ds - \int\limits_0^t \theta(s)\exp\left(-\int\limits_s^t \alpha(\tau)d\tau\right)ds. \tag{3.12}$$

By applying mean value theorem to first integral in (3.12), we have

$$x(t) \ge x(g(s^*))\int\limits_0^t \beta(s)\exp\left(-\int\limits_s^t \alpha(\tau)d\tau\right)ds - \int\limits_0^t \theta(s)\exp\left(-\int\limits_s^t \alpha(\tau)d\tau\right)ds,$$

where $s^* \in [0,t]$. By boundedness assumption on the solution $x(t)$, $|x(g(s^*))|$ will be a constant.

Now, since $\int\limits_0^t \theta(s)\exp\left(-\int\limits_s^t \alpha(\tau)d\tau\right)ds < \infty$ and $\sup\limits_{t\ge 0}\int\limits_0^t \beta(s)\exp\left(-\int\limits_s^t \alpha(\tau)d\tau\right)ds = \infty$, the solution $x(t)$ will be unbounded, which is a contradiction.

□

Example 3.2. Consider the equation

$$x' + \frac{1}{t^2+1}x = \frac{x\left(\frac{t}{2}\right)te^{-\tan^{-1}t}}{1+x^{20}\left(\frac{t}{2}\right)} \quad ,0 \le t \le b \le \infty \tag{3.13}$$

with the initial condition

$$x(0) = 0$$

and history function $\varphi(t) = -t, t \in [-2,0]$.

Suppose that $\theta(t) > 0$ and $x \in C^1(I)$ satisfies the inequality (3.14) and the initial condition $x(0) = 0$. We will show that zero solution $y_0(t) \equiv 0$ of the equation (3.13) will satisfy the inequality $\sup\limits_{x\ge x_0}|x(t) - y_0(t)| > k\theta(t)$. On the contrary, let us assume that there exists $\theta_0(t) > 0$ such that $\sup\limits_{x \geq x_0} |x(t)| \leq k\theta_0(t)$. Then we can find a constant $M > 0$ such that $M = \sup\limits_{t\ \geq t_0} |x(t)|$.

Suppose that $x(t)$ satisfies the following inequality

$$\left| x(t)' + \frac{1}{t^2+1}x - \frac{x\left(\frac{t}{2}\right)te^{-\tan^{-1}t}}{1+x^{20}\left(\frac{t}{2}\right)} \right| \leq \theta(t). \tag{3.14}$$

Multiplying (3.14) by $\exp\left(\int\limits_0^t \frac{1}{s^2+1}ds\right) = e^{\tan^{-1}t}$, taking $\dfrac{1}{1+t^2}$ instead of $\theta(t)$ and then integrating from 0 to t, we obtain

$$\begin{aligned} -\int\limits_0^t \frac{1}{1+s^2}e^{\tan^{-1}s}ds &\leq x(t)e^{\tan^{-1}t} - \int\limits_0^t \frac{sx\left(\frac{s}{2}\right)}{1+\left[x\left(\frac{s}{2}\right)\right]^{20}}ds \\ &\leq \int\limits_0^t \frac{1}{1+s^2}e^{\tan^{-1}s}ds. \end{aligned} \tag{3.15}$$

Now multiply (3.15) by $\exp\left(-\int\limits_0^t \frac{1}{s^2+1}ds\right) = e^{-\tan^{-1}t}$ to get

$$\begin{aligned} x(t) &\geq e^{-\tan^{-1}t}\int\limits_0^t \frac{sx\left(\frac{s}{2}\right)}{1+x^{20}\left(\frac{s}{2}\right)}ds - e^{-\tan^{-1}t}\int\limits_0^t \frac{1}{1+s^2}e^{\tan^{-1}s}ds \\ &\geq x\left(\frac{s^*}{2}\right)e^{-\tan^{-1}t}\int\limits_0^t sds - e^{-\tan^{-1}t}\int\limits_0^t \frac{1}{1+s^2}e^{\tan^{-1}s}ds \\ &\geq \frac{1}{2}x\left(\frac{s^*}{2}\right)t^2e^{-\tan^{-1}t} - \left(1-e^{-\tan^{-1}t}\right) \end{aligned} \tag{3.16}$$

where $s^* \in [0,t]$. By false boundedness assumption imposed on the solution $x(t)$, $x\left(\dfrac{s^*}{2}\right)$ will be a constant. Since $\lim\limits_{t\to\infty}\left(1-e^{-\tan^{-1}t}\right) = 1$ and $\lim\limits_{t\to\infty} t^2e^{-\tan^{-1}t} = \infty$, from (3.16) $x(t)$ will be unbounded and we have a contradiction.

4. Conclusion

Here we have established the Hyers-Ulam instability and Hyers-Ulam-Rassias instability of Mackey-Glass differential equation with initial conditions. The results are achieved by integrating the differential equations using method of integrating factor and then estimating the supremum of solutions. To illustrate our theoretical results we have given two examples.

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