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ON GENERALIZED WEIGHTED FRACTIONAL INEQUALITIES

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Abstract. Our first aim is to establish a new identity for differentiable function involving Riemann-Liouville fractional integrals. Then, we obtain some new weighted versions of fractional trapezoid and Ostrowski type inequalities. Moreover, we give some weighted inequalities as special cases.

1. Introduction

In recent years, the Hermite-Hadamard inequality, which is the first fundamental result for convex mappings with a natural geometrical interpretation and many applications, has drawn attention much interest in elementary mathematics.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g., [33, p.137], [10]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave.

Over the last twenty years, the numerous studies have focused on to obtain new bound for left hand side and right hand side of the inequality (1.1). For some examples, please refer to ([2, 4, 6, 10, 11, 29, 36, 37, 39, 40, 47])

On the other hand, Ostrowski [34] proved the following classical integral inequality associated with the differentiable mapping.

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b), i.e., $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then, the following*

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inequality holds:

$$\left| f(x) - \frac{1}{b-a} \int\limits_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \left\| f' \right\|_\infty \tag{1.2}$$

for all $x \in [a, b]$. *The constant* $\frac{1}{4}$ *is the best possible.*

The overall structure of the paper takes the form of three sections including introduction. The remainder of this work is organized as follows: we first give weighted version of (1.1) and definitions of Riemann-Liouville fractional integral operators. We also mention fractional Hermite-Hadamard and Ostrowski type inequalities obtained in earlier works. In Section 2, we establish an important weighted equalities for differentiable functions involving fractional integrals. Using this identity given Section 2, we obtain some weighted fractional type inequalities. We also give several weighted Hermite-Hadamard and Ostrowski type inequalities as special cases.

The weighted version of the inequalities (1.1), so-called Hermite-Hadamard-Fejér inequalities, was given by Fejer in [15] as follow:

Theorem 1.2. $f : [a, b] \to \mathbb{R}$, *be a convex function, then the inequalities*

$$f\left(\frac{a+b}{2}\right) \int\limits_a^b g(x)dx \leq \int\limits_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2} \int\limits_a^b g(x)dx \tag{1.3}$$

hold, where $g : [a, b] \to \mathbb{R}$ *is non-negative, integrable, and symmetric about* $x = \frac{a+b}{2}$ *(i.e.* $g(x) = g(a + b - x)$*).*

Tseng et al. give the following Lemma and by using this Lemma they obtain several weighed inequalities in [52].

Lemma 1.1. *Let* $f : [a, b] \to \mathbb{R}$ *be a differentiable mapping on* (a, b) *with* $a < b$ *and let* $g : [a, b] \to \mathbb{R}$. *If* $f', g \in L[a, b]$, *then for all* $x \in [a, b]$ *we have the following equality for fractional integrals*

$$f(a) \int\limits_a^x g(t)dt + f(b) \int\limits_x^b g(t)dt - \int\limits_a^b f(t)g(t)dt = \int\limits_a^b \left(\int\limits_x^t g(s)ds \right) f'(t)dt. \tag{1.4}$$

1.1. **Fractional Calculus and Some Inequalities.** In this subsection, we will give some necessary definitions and mathematical preliminaries of fractional calculus theory.

Definition 1.1. Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$ of order $\alpha > 0$ are defined by

$$J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.

For more information about fraction calculus please refer to ([16,27,30,38].)

In [43], Sarikaya et al. first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1.3. *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0\leq a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then we have the following inequalities for fractional integrals*

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\leq\frac{f(a)+f(b)}{2} \tag{1.5}$$

for $\alpha>0$.

On the other hand, Işcan gave following Lemma and using this Lemma he proved the following Fejer type inequalities for Riemann-Liouville fractional integrals in [19].

Lemma 1.2. *If $g:[a,b]\to\mathbb{R}$ is integrable and symmetric to $(a+b)/2$ with $a<b$, then*

$$J_{a+}^{\alpha}g(b)=J_{b-}^{\alpha}g(a)=\frac{1}{2}\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]$$

with $\alpha>0$.

Theorem 1.4. *Let $f:[a,b]\to\mathbb{R}$ be convex function with $0\leq a<b$ and $f\in L_1[a,b]$. If $g:[a,b]\to\mathbb{R}$ is non-negative, integrable and symmetric to $(a+b)/2$, then the following inequalities for fractional integrals hold*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right] &\leq \left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right] \\ &\leq \frac{f(a)+f(b)}{2}\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right] \end{aligned} \tag{1.6}$$

with $\alpha>0$

Whereupon Sarikaya et al. obtain the Hermite-Hadamard inequality for Riemann-Lioville fractional integrals, many authors have studied to generalize this inequality and establish Hermite-Hadamard inequality other fractional integrals such as k-fractional integral, Hadamard fractional integrals, Katugampola fractional integrals, Conformable fractional integrals, etc. For some of them, please see ([3,7–9,14,18–26,28,31,35,44–46,49,50,53,54,57]).

In [48], Set obtain the following Ostrowski inequality for fractional integrals.

Theorem 1.5. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ and $f'\in L[a,b]$. If $|f'|$ is convex on $[a,b]$ and $|f'(x)|\leq M$, $x\in[a,b]$ then the following inequality for fractional integrals with $\alpha>0$ holds*

$$\begin{aligned} &\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[J_{x+}^{\alpha}f(b)+J_{x-}^{\alpha}f(a)\right]\right| \\ &\leq \frac{M}{b-a}\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{\alpha+1}\right]. \end{aligned}$$

In recent years, several papers have devoted to Ostrowski type inequalities for several type fractional integrals, for some of them please see [1,5,12,17,32,41,42,51,55,56].

2. Some Weighted Fractional Inequalities

Throughout this section, we use the following notation:

$$\begin{aligned}\Lambda_\lambda(f,g) &= \lambda\left[f(a)J^\alpha_{a+}g\left(x\right)+f(b)J^\alpha_{b-}g\left(x\right)\right]+(1-\lambda)\left[J^\alpha_{a+}g\left(x\right)+J^\alpha_{b-}g\left(x\right)\right]f(x)\\ &\quad -\left[J^\alpha_{a+}\left(fg\right)\left(x\right)+J^\alpha_{b-}\left(fg\right)\left(x\right)\right].\end{aligned}$$

for $\lambda\in[0,1]$.

Lemma 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f',g\in L[a,b]$, then for all $x\in[a,b]$ and $\alpha>0$ we have the following equality for fractional integrals*

$$\Lambda_\lambda(f,g) = \frac{1}{\Gamma(\alpha)}\int_a^b K_\lambda(x,t)f'(t)dt \tag{2.1}$$

where $K_\lambda(x,t):[a,b]\times[a,b]\to\mathbb{R}$ is defined by

$$K_\lambda(x,t)=\begin{cases}\lambda\int_x^t(x-s)^{\alpha-1}g(s)ds+(1-\lambda)\int_a^t(x-s)^{\alpha-1}g(s)ds, & a\le t<x\\ \lambda\int_x^t(s-x)^{\alpha-1}g(s)ds+(1-\lambda)\int_b^t(s-x)^{\alpha-1}g(s)ds, & x\le t\le b.\end{cases}$$

Proof. From the definition of $K_\lambda(t,x)$, we have

$$\begin{aligned}\int_a^b K_\lambda(x,t)f'(t)dt &= \lambda\int_a^x\left(\int_x^t(x-s)^{\alpha-1}g(s)ds\right)f'(t)dt \\ &\quad +(1-\lambda)\int_a^x\left(\int_a^t(x-s)^{\alpha-1}g(s)ds\right)f'(t)dt\\ &\quad +\lambda\int_x^b\left(\int_x^t(s-x)^{\alpha-1}g(s)ds\right)f'(t)dt\\ &\quad +(1-\lambda)\int_x^b\left(\int_b^t(s-x)^{\alpha-1}g(s)ds\right)f'(t)dt.\end{aligned} \tag{2.2}$$

Using the integration by parts, we get

$$\begin{aligned}\int_a^t\left(\int_x^t (x-s)^{\alpha-1}g(s)ds\right)f'(t)dt &= \left.\left(\int_x^t (x-s)^{\alpha-1}g(s)ds\right)f(t)\right|_a^x \\ &\quad -\int_a^x f(t)\left(\int (x-s)^{\alpha-1}g(s)ds\right) \\ &= \left(\int_x^a (x-s)^{\alpha-1}g(s)ds\right)f(a) \\ &\quad -\int (x-t)^{\alpha-1}g(t)f(t)dt \\ &= \Gamma(\alpha)\left[f(a)J_{a+}^{\alpha}g(x)-J_{a+}^{\alpha}(fg)(x)\right].\end{aligned} \tag{2.3}$$

Similarly, we have

$$\begin{aligned}\int_a^x\left(\int_a^t (x-s)^{\alpha-1}g(s)ds\right)f'(t)dt &= \left(\int_a^t (x-s)^{\alpha-1}g(s)ds\right)f(x) \\ &\quad -\int_a^x (x-t)^{\alpha-1}g(t)f(t)dt \\ &= \Gamma(\alpha)\left[f(x)J_{a+}^{\alpha}g(x)-J_{a+}^{\alpha}(fg)(x)\right],\end{aligned} \tag{2.4}$$

$$\begin{aligned}\int_x^b\left(\int_x^t (s-x)^{\alpha-1}g(s)ds\right)f'(t)dt &= \left(\int_x^t (s-x)^{\alpha-1}g(s)ds\right)f(b) \\ &\quad -\int_x^b (t-x)^{\alpha-1}g(t)f(t)dt \\ &= \Gamma(\alpha)\left[f(b)J_{b-}^{\alpha}g(x)-J_{b-}^{\alpha}(fg)(x)\right]\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\int_x^b\left(\int_b^t (s-x)^{\alpha-1}g(s)ds\right)f'(t)dt &= \left(\int_b^t (s-x)^{\alpha-1}g(s)ds\right)f(x) \\ &\quad -\int_x^b (t-x)^{\alpha-1}g(t)f(t)dt \\ &= \Gamma(\alpha)\left[f(x)J_{b-}^{\alpha}g(x)-J_{b-}^{\alpha}(fg)(x)\right].\end{aligned} \tag{2.6}$$

By the identities (2.3), (2.4), (2.5) and (2.6) we obtain

$$
\begin{aligned}
&\int_a^b K_\lambda(x,t)f'(t)dt \qquad (2.7)\\
=\;& \lambda\Gamma(\alpha)\left[f(a)J_{a+}^{\alpha}g(x)-J_{a+}^{\alpha}(fg)(x)\right]-(1-\lambda)\Gamma(\alpha)\left[f(x)J_{a+}^{\alpha}g(x)-J_{a+}^{\alpha}(fg)(x)\right]\\
&+\lambda\Gamma(\alpha)\left[f(b)J_{b-}^{\alpha}g(x)-J_{b-}^{\alpha}(fg)(x)\right]+(1-\lambda)\Gamma(\alpha)\left[f(x)J_{b-}^{\alpha}g(x)-J_{b-}^{\alpha}(fg)(x)\right]\\
=\;& \lambda\Gamma(\alpha)\left[f(a)J_{a+}^{\alpha}g(x)-f(b)J_{b-}^{\alpha}g(x)\right]+(1-\lambda)\Gamma(\alpha)f(x)\left[J_{a+}^{\alpha}g(x)-J_{b-}^{\alpha}g(x)\right]\\
&-\Gamma(\alpha)\left[J_{a+}^{\alpha}(fg)(x)+J_{b-}^{\alpha}(fg)(x)\right].
\end{aligned}
$$

If we divide the both sides of (2.7) by $\Gamma(\alpha)$, then we establish the required result (2.2). □

Remark 2.1. If we choose $\alpha = 1$ in Lemma 2.1, then Lemma 2.1 reduces to Lemma 4 in [13] proved by Erden and Sarıkaya.

Corollary 2.1. *If we choose $\lambda = 1$ in Lemma 2.1, then we have the following weighted fractional equality*

$$
\begin{aligned}
&\Gamma(\alpha)\left[f(a)J_{a+}^{\alpha}g(x)+J_{a+}^{\alpha}(fg)(x)\right]-\Gamma(\alpha)\left[f(b)J_{b-}^{\alpha}g(x)-J_{b-}^{\alpha}(fg)(x)\right]\\
=\;&\int_a^x\left(\int_x^t(x-s)^{\alpha-1}g(s)ds\right)f'(t)dt+\int_x^b\left(\int_x^t(s-x)^{\alpha-1}g(s)ds\right)f'(t)dt.
\end{aligned}
$$

Remark 2.2. In Corollary 2.1, if we take $\alpha = 1$, then Corollary 2.1 reduces to Lemma 1.1.

Corollary 2.2. *If we choose $\lambda = 0$ in Lemma 2.1, then we have the following weighted fractional equality*

$$
\begin{aligned}
&\left[J_{a+}^{\alpha}g\left(x\right)+J_{b-}^{\alpha}g\left(x\right)\right]f(x)-\left[J_{a+}^{\alpha}\left(fg\right)\left(x\right)+J_{b-}^{\alpha}\left(fg\right)\left(x\right)\right]\\
=\;&\int_a^x\left(\int_a^t(x-s)^{\alpha-1}g(s)ds\right)f'(t)dt+\int_x^b\left(\int_b^t(s-x)^{\alpha-1}g(s)ds\right)f'(t)dt.
\end{aligned}
$$

Corollary 2.3. *In Lemma 2.1, let $g(t) = 1$ for all $t \in [a,b]$. Then we have the following identity*

$$\lambda\left[(x-a)^{\alpha} f(a) + (b-x)^{\alpha} f(b)\right]$$

$$+(1-\lambda)\left[(x-a)^{\alpha} + (b-x)^{\alpha}\right] f(x) - \Gamma(\alpha+1)\left[J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(x)\right]$$

$$= \alpha \int_a^b P_\lambda(x,t) f'(t)dt$$

where

$$P_\lambda(x,t) = \begin{cases} \lambda \int_x^t (x-s)^{\alpha-1} ds + (1-\lambda) \int_a^t (x-s)^{\alpha-1} ds, & a \le t < x \\ \lambda \int_x^t (s-x)^{\alpha-1} ds + (1-\lambda) \int_b^t (s-x)^{\alpha-1} ds, & x \le t \le b. \end{cases}$$

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b), $g : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ with $a < b$ and $f' \in L[a,b]$. If $|f'|$ is convex on $[a,b]$, then for all $x \in [a,b]$ and $\alpha > 0$ we have the following inequality for fractional integrals*

$$\begin{aligned}
&|\Lambda_\lambda(f,g)| \qquad (2.8)\\
\le\ & \frac{1}{\Gamma(\alpha+3)(b-a)} \Big\{ (x-a)^{\alpha+1} \|g\|_{[a,x],\infty} \\
& [(2\lambda-1)\left[(\alpha+2)(b-x) + (\alpha+1)(x-a)\right] \\
& +(1-\lambda)(\alpha+1)(\alpha+2)\left(b - \frac{a+x}{2}\right)\Big] |f'(a)| \\
& +(b-x)^{\alpha+2} \|g\|_{[x,b],\infty} \left[\lambda + (1-\lambda)\left(\frac{(\alpha+1)(\alpha+2)}{2} - 1\right)\right] |f'(a)| \\
& +(x-a)^{\alpha+2} \|g\|_{[a,x],\infty} \left[\lambda + (1-\lambda)\left(\frac{(\alpha+1)(\alpha+2)}{2} - 1\right)\right] |f'(b)| \\
& +(b-x)^{\alpha+1} \|g\|_{[x,b],\infty} [(2\lambda-1)\left[(\alpha+1)(b-x) + (\alpha+2)(x-a)\right] \\
& +(1-\lambda)(\alpha+1)(\alpha+2)\left(\frac{x+b}{2} - a\right)\Big] |f'(b)| \Big\}
\end{aligned}$$

$$\leq \frac{\|g\|_{\infty}}{\Gamma(\alpha+3)(b-a)}\Big\{(x-a)^{\alpha+1}\left[(2\lambda-1)\left[(\alpha+2)(b-x)+(\alpha+1)(x-a)\right]\right.$$
$$\left.+(1-\lambda)(\alpha+1)(\alpha+2)\left(b-\frac{a+x}{2}\right)\right]|f'(a)|$$
$$+(b-x)^{\alpha+2}\left[\lambda+(1-\lambda)\left(\frac{(\alpha+1)(\alpha+2)}{2}-1\right)\right]|f'(a)|$$
$$+(x-a)^{\alpha+2}\left[\lambda+(1-\lambda)\left(\frac{(\alpha+1)(\alpha+2)}{2}-1\right)\right]|f'(b)|$$
$$+(b-x)^{\alpha+1}\left[(2\lambda-1)\left[(\alpha+1)(b-x)+(\alpha+2)(x-a)\right]\right.$$
$$\left.+(1-\lambda)(\alpha+1)(\alpha+2)\left(\frac{x+b}{2}-a\right)\right]|f'(b)|\Big\}.$$

Proof. By taking modulus in Lemma 2.1, we have

$$|\Lambda_\lambda(f,g)| \tag{2.9}$$
$$= \frac{1}{\Gamma(\alpha)}\left|\int_a^b K_\lambda(x,t)f'(t)dt\right|$$
$$\leq \frac{1}{\Gamma(\alpha)}\left|\lambda\int_a^t\left(\int_x^t(x-s)^{\alpha-1}g(s)ds\right)f'(t)dt+(1-\lambda)\int_a^x\left(\int_a^t(x-s)^{\alpha-1}g(s)ds\right)f'(t)dt\right|$$
$$+\frac{1}{\Gamma(\alpha)}\left|\lambda\int_x^b\left(\int_x^t(s-x)^{\alpha-1}g(s)ds\right)f'(t)dt+(1-\lambda)\int_x^b\left(\int_b^t(s-x)^{\alpha-1}g(s)ds\right)f'(t)dt\right|$$
$$\leq \frac{\lambda}{\Gamma(\alpha)}\int_a^t\left|\int_x^t(x-s)^{\alpha-1}g(s)ds\right||f'(t)|\,dt+\frac{(1-\lambda)}{\Gamma(\alpha)}\int_a^x\left|\int_a^t(x-s)^{\alpha-1}g(s)ds\right||f'(t)|\,dt$$
$$+\frac{1}{\Gamma(\alpha)}\lambda\int_x^b\left|\int_x^t(s-x)^{\alpha-1}g(s)ds\right||f'(t)|\,dt+\frac{1}{\Gamma(\alpha)}(1-\lambda)\int_x^b\left|\int_b^t(s-x)^{\alpha-1}g(s)ds\right||f'(t)|\,dt.$$

Using the facts that g is continuous on $[a,b]$ and $|f'|$ is convex on $[a,b]$, we obtain

$$\int_a^t\left|\int_x^t(x-s)^{\alpha-1}g(s)ds\right||f'(t)|\,dt \tag{2.10}$$
$$\leq \|g\|_{[a,x],\infty}\int_a^t\left|\int_x^t(x-s)^{\alpha-1}ds\right||f'(t)|\,dt$$

$$
\begin{aligned}
&= \frac{\|g\|_{[a,x],\infty}}{\alpha} \int_a^x (x-t)^{\alpha} |f'(t)| \, dt \\
&\leq \frac{\|g\|_{[a,x],\infty}}{\alpha(b-a)} \int_a^x (x-t)^{\alpha} \left[(b-t) |f'(a)| + (t-a) |f'(b)| \right] dt \\
&= \frac{\|g\|_{[a,x],\infty}}{\alpha(b-a)} \left[|f'(a)| \int_a^x (x-t)^{\alpha}(b-t) dt + |f'(b)| \int_a^x (x-t)^{\alpha}(t-a) dt \right] \\
&= \frac{\|g\|_{[a,x],\infty}}{\alpha(b-a)} \left[|f'(a)| \frac{(\alpha+2)(b-x)+(\alpha+1)(x-a)}{(\alpha+1)(\alpha+2)} (x-a)^{\alpha+1} \right. \\
&\quad \left. + |f'(b)| \frac{(x-a)^{\alpha+2}}{(\alpha+1)(\alpha+2)} \right] \\
&= \frac{\|g\|_{[a,x],\infty}}{\alpha(\alpha+1)(\alpha+2)(b-a)} \\
&\quad \times \left[(x-a)^{\alpha+1} \left[(\alpha+2)(b-x) + (\alpha+1)(x-a) \right] |f'(a)| + (x-a) |f'(b)| \right].
\end{aligned}
$$

Similarly, we establish

$$
\begin{aligned}
&\int_a^x \left| \left(\int_a^t (x-s)^{\alpha-1} g(s) ds \right) \right| |f'(t)| \, dt \qquad (2.11) \\
&\leq \|g\|_{[a,x],\infty} \int_a^x \left| \left(\int_a^t (x-s)^{\alpha-1} ds \right) \right| |f'(t)| \, dt \\
&\leq \frac{\|g\|_{[a,x],\infty}}{\alpha(b-a)} \left\{ \left[(x-a)^{\alpha+1} \left(b - \frac{a+x}{2} \right) - \frac{(\alpha+2)(b-x)+(\alpha+1)(x-a)}{(\alpha+1)(\alpha+2)} (x-a)^{\alpha+1} \right] |f'(a)| \right. \\
&\quad \left. + \left[(x-a)^{\alpha+2} \left(\frac{1}{2} - \frac{1}{(\alpha+1)(\alpha+2)} \right) |f'(b)| \right] \right\},
\end{aligned}
$$

$$
\begin{aligned}
&\int_x^b \left| \int_x^t (s-x)^{\alpha-1} g(s) ds \right| |f'(t)| \, dt \qquad (2.12) \\
&\leq \|g\|_{[x,b],\infty} \int_x^b \left| \int_x^t (s-x)^{\alpha-1} ds \right| |f'(t)| \, dt \\
&\leq \frac{\|g\|_{[x,b],\infty}}{\alpha(\alpha+1)(\alpha+2)(b-a)} \left[|f'(a)| (b-x) + |f'(b)| \left[(b-x)(\alpha+1) + (x-a)(\alpha+2) \right] \right],
\end{aligned}
$$

and

$$\int_x^b \left| \int_b^t (s-x)^{\alpha-1} g(s) ds \right| |f'(t)| \, dt \tag{2.13}$$

$$\leq \|g\|_{[x,b],\infty} \int_x^b \left| \int_b^t (s-x)^{\alpha-1} ds \right| |f'(t)| \, dt$$

$$\leq \frac{\|g\|_{[x,b],\infty}}{\alpha(b-a)} \left\{ \left[(b-x)^{\alpha+2} \left(\frac{1}{2} - \frac{1}{(\alpha+1)(\alpha+2)} \right) |f'(a)| \right] \right.$$

$$\left. + (b-x)^{\alpha+1} \left[\left(\frac{b+x}{2} - a \right) - \frac{(\alpha+1)(b-x)+(\alpha+2)(x-a)}{(\alpha+1)(\alpha+2)} \right] |f'(b)| \right\}.$$

If we substitute the inequalities (2.10)-(2.13) in (2.9), then we obtain the first inequality in (2.8).

The proof of second inequality in (2.8) is obvious from the facts that

$$\|g\|_{[a,x],\infty} \leq \|g\|_{[a,b],\infty} = \|g\|_\infty \text{ and } \|g\|_{[x,b],\infty} \leq \|g\|_{[a,b],\infty} = \|g\|_\infty \tag{2.14}$$

for all $x \in [a,b]$. □

Remark 2.3. If we choose $\alpha = 1$ in Theorem 2.1, then Theorem 2.1 reduces to Theorem 5 in [13] proved by Erden and Sarikaya.

Corollary 2.4. *If we choose $\lambda = 1$ in Theorem 2.1, then we have the following weighted fractional trapezoid inequality*

$$\left| \left[f(a) J_{a+}^\alpha g(x) + f(b) J_{b-}^\alpha g(x) \right] - \left[J_{a+}^\alpha (fg)(x) + J_{b-}^\alpha (fg)(x) \right] \right|$$

$$\leq \frac{1}{\Gamma(\alpha+3)(b-a)}$$

$$\times \left\{ \left[(x-a)^{\alpha+1} \left[(\alpha+2)(b-x) + (\alpha+1)(x-a) \right] \|g\|_{[a,x],\infty} + (b-x)^{\alpha+2} \|g\|_{[x,b],\infty} \right] |f'(a)| \right.$$

$$\left. + (x-a)^{\alpha+2} \|g\|_{[a,x],\infty} + b-x)^{\alpha+1} \left[(\alpha+1)(b-x) + (\alpha+2)(x-a) \|g\|_{[x,b],\infty} \right] |f'(b)| \right.$$

$$\leq \frac{\|g\|_\infty}{\Gamma(\alpha+3)(b-a)} \left\{ \left[(x-a)^{\alpha+1} \left[(\alpha+2)(b-x) + (\alpha+1)(x-a) \right] + (b-x)^{\alpha+2} \right] |f'(a)| \right.$$

$$\left. + (x-a)^{\alpha+2} + b-x)^{\alpha+1} \left[(\alpha+1)(b-x) + (\alpha+2)(x-a) \right] |f'(b)| \right\}.$$

Corollary 2.5. *If we choose $\lambda = 0$ in Theorem 2.1, then we have the following weighted fractional Ostrowski type inequality*

$$\begin{aligned}
&\left|\left[J_{a+}^{\alpha}g(x)+J_{b-}^{\alpha}g(x)\right]f(x)-\left[J_{a+}^{\alpha}(fg)(x)+J_{b-}^{\alpha}(fg)(x)\right]\right| \\
\leq\ & \frac{1}{\Gamma(\alpha+3)(b-a)}\left\{\left[(b-x)^{\alpha+2}\|g\|_{[x,b],\infty}\left(\frac{(\alpha+1)(\alpha+2)}{2}-1\right)\right.\right. \\
&+(x-a)^{\alpha+1}\|g\|_{[a,x],\infty} \\
&\left.\times\left[(\alpha+1)(\alpha+2)\left(b-\frac{a+x}{2}\right)-\left[(\alpha+2)(b-x)+(\alpha+1)(x-a)\right]\right]\right]\left|f'(a)\right| \\
&+\left[(x-a)^{\alpha+2}\|g\|_{[a,x],\infty}\left(\frac{(\alpha+1)(\alpha+2)}{2}-1\right)\right. \\
&+(b-x)^{\alpha+1}\|g\|_{[x,b],\infty} \\
&\left.\times\left[(\alpha+1)(\alpha+2)\left(\frac{x+b}{2}-a\right)-\left[(\alpha+1)(b-x)+(\alpha+2)(x-a)\right]\right]\right]\left|f'(b)\right| \\
\leq\ & \frac{\|g\|_{\infty}}{\Gamma(\alpha+3)(b-a)}\left\{\left[(b-x)^{\alpha+2}\left(\frac{(\alpha+1)(\alpha+2)}{2}-1\right)\right.\right. \\
&\left.+(x-a)^{\alpha+1}\left[(\alpha+1)(\alpha+2)\left(b-\frac{a+x}{2}\right)-\left[(\alpha+2)(b-x)+(\alpha+1)(x-a)\right]\right]\right]\left|f'(a)\right| \\
&+\left[(x-a)^{\alpha+2}\left(\frac{(\alpha+1)(\alpha+2)}{2}-1\right)\right. \\
&\left.\left.+(b-x)^{\alpha+1}\left[(\alpha+1)(\alpha+2)\left(\frac{x+b}{2}-a\right)-\left[(\alpha+1)(b-x)+(\alpha+2)(x-a)\right]\right]\right]\left|f'(b)\right|\right\}.
\end{aligned}$$

Theorem 2.2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ and $f'\in L[a,b]$. If $g:[a,b]\to\mathbb{R}$ is continuous and if $|f'|^{q}$, $q>1$, is convex on $[a,b]$, then for all $x\in[a,b]$ and $\alpha>0$ we have the following inequality for fractional integrals,*

$$\begin{aligned}
&\left|\Lambda_{\lambda}(f,g)\right| \qquad (2.15)\\
\leq\ & \frac{\|g\|_{[a,x],\infty}}{\Gamma(\alpha+1)}\frac{(x-a)^{\alpha+1}}{(b-a)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}\left[\lambda\left[\left(b-\frac{a+x}{2}\right)\left|f'(a)\right|^{q}+\frac{(x-a)}{2}\left|f'(b)\right|^{q}\right]^{\frac{1}{q}}\right. \\
&\left.+(1-\lambda)(\alpha p)^{\frac{1}{p}}\left[\left(b-\frac{a+x}{2}\right)\left|f'(a)\right|^{q}+\frac{(x-a)}{2}\left|f'(b)\right|^{q}\right]^{\frac{1}{q}}\right]
\end{aligned}$$

$$+\frac{\| g \|_{[x,b],\infty}}{\Gamma(\alpha+1)} \frac{(b-x)^{\alpha+1}}{(b-a)^{\frac{1}{q}} (\alpha p+1)^{\frac{1}{p}}} \left[\lambda \left[\frac{(b-x)}{2} |f'(a)|^q + \left(\frac{x+b}{2} - a \right) |f'(b)|^q \right]^{\frac{1}{q}} \right.$$

$$\left. + (1-\lambda) (\alpha p)^{\frac{1}{p}} \left[\left(\frac{b-x}{2} \right) \left| f'(a) \right|^q + \left(\frac{x+b}{2} - a \right) \left| f'(b) \right|^q \right]^{\frac{1}{q}} \right]$$

$$\leq \frac{\|g\|_\infty}{\Gamma(\alpha+1)(b-a)^{\frac{1}{q}} (\alpha p+1)^{\frac{1}{p}}}$$

$$\left\{ (x-a)^{\alpha+1} \left[\lambda \left[\left(b - \frac{a+x}{2} \right) \left| f'(a) \right|^q + \frac{(x-a)}{2} \left| f'(b) \right|^q \right]^{\frac{1}{q}} \right. \right.$$

$$\left. + (1-\lambda) (\alpha p)^{\frac{1}{p}} \left[\left(b - \frac{a+x}{2} \right) \left| f'(a) \right|^q + \frac{(x-a)}{2} \left| f'(b) \right|^q \right]^{\frac{1}{q}} \right]$$

$$+(b-x)^{\alpha+1} \left[\lambda \left[\frac{(b-x)}{2} |f'(a)|^q + \left(\frac{x+b}{2} - a \right) |f'(b)|^q \right]^{\frac{1}{q}} \right.$$

$$\left. \left. + (1-\lambda) (\alpha p)^{\frac{1}{p}} \left[\left(\frac{b-x}{2} \right) \left| f'(a) \right|^q + \left(\frac{x+b}{2} - a \right) \left| f'(b) \right|^q \right]^{\frac{1}{q}} \right] \right\}.$$

Proof. By taking modulus in Lemma 2.1, we have

$$|\Lambda_\lambda(f,g)| \tag{2.16}$$

$$\leq \frac{\lambda}{\Gamma(\alpha)} \int_a^t \left| \int_x^t (x-s)^{\alpha-1} g(s) ds \right| |f'(t)|\, dt + \frac{(1-\lambda)}{\Gamma(\alpha)} \int_a^x \left| \int_a^t (x-s)^{\alpha-1} g(s) ds \right| |f'(t)|\, dt$$

$$+\frac{\lambda}{\Gamma(\alpha)} \int_x^b \left| \int_x^t (s-x)^{\alpha-1} g(s) ds \right| |f'(t)|\, dt + \frac{(1-\lambda)}{\Gamma(\alpha)} \int_x^b \left| \int_b^t (s-x)^{\alpha-1} g(s) ds \right| |f'(t)|\, dt$$

$$\leq \frac{\| g \|_{[a,x],\infty}}{\Gamma(\alpha+1)} \left[\lambda \int_a^t (x-t)^\alpha |f'(t)|\, dt + (1-\lambda) \int_a^x [(x-a)^\alpha - (x-t)^\alpha] |f'(t)|\, dt \right]$$

$$+\frac{\| g \|_{[x,b],\infty}}{\Gamma(\alpha+1)} \left[\lambda \int_x^b (t-x)^\alpha |f'(t)|\, dt + (1-\lambda) \int_x^b [(b-x)^\alpha - (t-x)^\alpha] |f'(t)|\, dt \right].$$

By using the well-known Hölder inequality in (2.16), we have

$$|\Lambda_\lambda(f,g)| \tag{2.17}$$

$$\begin{aligned}
&\leq \frac{\|g\|_{[a,x],\infty}}{\Gamma(\alpha+1)}\left[\lambda\left(\int_a^x (x-t)^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_a^x \left|f^{'}(t)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+(1-\lambda)\left(\int_a^x \left[(x-a)^{\alpha}-(x-t)^{\alpha}\right]^p dt\right)^{\frac{1}{p}}\left(\int_a^x \left|f^{'}(t)\right|^q dt\right)^{\frac{1}{q}}\right] \\
&+\frac{\parallel g\parallel_{[x,b],\infty}}{\Gamma(\alpha+1)}\left[\lambda\left(\int_x^b (t-x)^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_x^b \left|f^{'}(t)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+(1-\lambda)\left(\int_x^b \left[(b-x)^{\alpha}-(t-x)^{\alpha}\right]^p dt\right)^{\frac{1}{p}}\left(\int_x^b \left|f^{'}(t)\right|^q dt\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Since $\left|f^{'}\right|^q$ is convex, we have

$$\begin{aligned}
&\left(\int_a^x (x-t)^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_a^x \left|f^{'}(t)\right|^q dt\right)^{\frac{1}{q}} \qquad (2.18)\\
&= \left(\frac{(x-a)^{\alpha p+1}}{\alpha p+1}\right)^{\frac{1}{p}}\left(\int_a^x \left|f^{'}(t)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \left(\frac{(x-a)^{\alpha p+1}}{\alpha p+1}\right)^{\frac{1}{p}}\left(\int_a^x \left[\frac{b-t}{b-a}\left|f^{'}(a)\right|^q+\frac{t-a}{b-a}\left|f^{'}(b)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&= \frac{(x-a)^{\alpha+1}}{(\alpha p+1)^{\frac{1}{p}}(b-a)^{\frac{1}{q}}}\left[\left(b-\frac{a+x}{2}\right)\left|f^{'}(a)\right|^q+\frac{(x-a)}{2}\left|f^{'}(b)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\left(\int_a^x \left[(x-a)^{\alpha}-(x-t)^{\alpha}\right]^p dt\right)^{\frac{1}{p}}\left(\int_a^x \left|f^{'}(t)\right|^q dt\right)^{\frac{1}{q}} \qquad (2.19)\\
&\leq \left(\int_a^x \left[(x-a)^{\alpha p}-(x-t)^{\alpha p}\right]dt\right)^{\frac{1}{p}}\left(\int_a^x \left|f^{'}(t)\right|^q dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$\leq \left[(x-a)^{\alpha p+1}\left(1-\frac{1}{\alpha p+1}\right)\right]^{\frac{1}{p}} \frac{1}{(b-a)^{\frac{1}{q}}} \left(\int_a^x \left[(b-t)\left|f^{'}(a)\right|^q + (t-a)\left|f^{'}(b)\right|^q\right] dt\right)^{\frac{1}{q}}$$

$$= \left(1-\frac{1}{\alpha p+1}\right)^{\frac{1}{p}} \frac{(x-a)^{\alpha+1}}{(b-a)^{\frac{1}{q}}} \left[\left(b-\frac{a+x}{2}\right)\left|f^{'}(a)\right|^q + \frac{(x-a)}{2}\left|f^{'}(b)\right|^q\right]^{\frac{1}{q}},$$

$$\left(\int_x^b (t-x)^{\alpha p} dt\right)^{\frac{1}{p}} \left(\int_x^b \left|f^{'}(t)\right|^q dt\right)^{\frac{1}{q}} \tag{2.20}$$

$$\leq \left(\frac{(b-x)^{\alpha p+1}}{\alpha p+1}\right)^{\frac{1}{p}} \frac{1}{(b-a)^{\frac{1}{q}}} \left(\int_x^b \left[(b-t)\left|f^{'}(a)\right|^q + (t-a)\left|f^{'}(b)\right|^q\right] dt\right)^{\frac{1}{q}}$$

$$= \frac{(b-x)^{\alpha+1}}{(\alpha p+1)^{\frac{1}{p}}(b-a)^{\frac{1}{q}}} \left[\frac{(b-x)}{2}\left|f'(a)\right|^q + \left(\frac{x+b}{2}-a\right)\left|f'(b)\right|^q\right]^{\frac{1}{q}}$$

and

$$\left(\int_x^b \left[(b-x)^{\alpha} - (t-x)^{\alpha}\right]^p dt\right)^{\frac{1}{p}} \left(\int_x^b \left|f^{'}(t)\right|^q dt\right)^{\frac{1}{q}} \tag{2.21}$$

$$\leq \left(\int_x^b \left[(b-x)^{\alpha p} - (t-x)^{\alpha p}\right] dt\right)^{\frac{1}{p}} \left(\int_x^b \left|f^{'}(t)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \left[(b-x)^{\alpha p+1}\left(1-\frac{1}{\alpha p+1}\right)\right]^{\frac{1}{p}} \frac{1}{(b-a)^{\frac{1}{q}}} \left(\int_a^x \left[(b-t)\left|f^{'}(a)\right|^q + (t-a)\left|f^{'}(b)\right|^q\right] dt\right)^{\frac{1}{q}}$$

$$= \left(1-\frac{1}{\alpha p+1}\right)^{\frac{1}{p}} \frac{(b-x)^{\alpha+1}}{(b-a)^{\frac{1}{q}}} \left[\left(\frac{b-x}{2}\right)\left|f^{'}(a)\right|^q + \left(\frac{x+b}{2}-a\right)\left|f^{'}(b)\right|^q\right]^{\frac{1}{q}}.$$

Here we use the fact that

$$(A-B)^p \leq A^p - B^p$$

for any $A > B \geq 0$ and $p \geq 1$.

If we put (2.18)-(2.21) in (2.17), then we have

$$
\begin{aligned}
&|\Lambda_\lambda(f,g)| \\
\leq\ & \frac{\|g\|_{[a,x],\infty}}{\Gamma(\alpha+1)} \frac{(x-a)^{\alpha+1}}{(b-a)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}} \left[\lambda \left[\left(b - \frac{a+x}{2} \right) \left| f'(a) \right|^q + \frac{(x-a)}{2} \left| f'(b) \right|^q \right]^{\frac{1}{q}} \right. \\
& \left. + (1-\lambda)(\alpha p)^{\frac{1}{p}} \left[\left(b - \frac{a+x}{2} \right) \left| f'(a) \right|^q + \frac{(x-a)}{2} \left| f'(b) \right|^q \right]^{\frac{1}{q}} \right] \\
& + \frac{\| g \|_{[x,b],\infty}}{\Gamma(\alpha+1)} \frac{(b-x)^{\alpha+1}}{(b-a)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}} \left[\lambda \left[\frac{(b-x)}{2} |f'(a)|^q + \left(\frac{x+b}{2} - a \right) |f'(b)|^q \right]^{\frac{1}{q}} \right. \\
& \left. + (1-\lambda)(\alpha p)^{\frac{1}{p}} \left[\left(\frac{b-x}{2} \right) \left| f'(a) \right|^q + \left(\frac{x+b}{2} - a \right) \left| f'(b) \right|^q \right]^{\frac{1}{q}} \right]
\end{aligned}
$$

which completes the proof of first inequality in (2.15).

The proof of second inequality in (2.15) is obvious from the inequalities (2.14). □

Corollary 2.6. *If we choose $\alpha = 1$ in Theorem 2.2, then we have the following inequality*

$$
\begin{aligned}
& \left| \lambda \left[f(a) \int_a^x g(t)dt + f(b) \int_x^b g(t)dt \right] + (1-\lambda) f(x) \int_a^b g(t)dt - \int_a^b f(t)g(t)dt \right| \\
\leq\ & \frac{\|g\|_{[a,x],\infty}}{(b-a)^{\frac{1}{q}}} \frac{(x-a)^2}{(p+1)^{\frac{1}{p}}} \left[\lambda \left[\left(b - \frac{a+x}{2} \right) \left| f'(a) \right|^q + \frac{(x-a)}{2} \left| f'(b) \right|^q \right]^{\frac{1}{q}} \right. \\
& \left. + (1-\lambda)(p)^{\frac{1}{p}} \left[\left(b - \frac{a+x}{2} \right) \left| f'(a) \right|^q + \frac{(x-a)}{2} \left| f'(b) \right|^q \right]^{\frac{1}{q}} \right] \\
& + \frac{\|g\|_{[x,b],\infty}}{(b-a)^{\frac{1}{q}}} \frac{(b-x)^2}{(p+1)^{\frac{1}{p}}} \left[\lambda \left[\frac{(b-x)}{2} |f'(a)|^q + \left(\frac{x+b}{2} - a \right) |f'(b)|^q \right]^{\frac{1}{q}} \right. \\
& \left. + (1-\lambda)(p)^{\frac{1}{p}} \left[\left(\frac{b-x}{2} \right) \left| f'(a) \right|^q + \left(\frac{x+b}{2} - a \right) \left| f'(b) \right|^q \right]^{\frac{1}{q}} \right]. \\
\leq\ & \frac{\|g\|_{\infty}}{(b-a)^{\frac{1}{q}}(p+1)^{\frac{1}{p}}} \left\{ \left[\lambda \left[\left(b - \frac{a+x}{2} \right) \left| f'(a) \right|^q + \frac{(x-a)}{2} \left| f'(b) \right|^q \right]^{\frac{1}{q}} \right. \right. \\
& \left. + (1-\lambda)(p)^{\frac{1}{p}} \left[\left(b - \frac{a+x}{2} \right) \left| f'(a) \right|^q + \frac{(x-a)}{2} \left| f'(b) \right|^q \right]^{\frac{1}{q}} \right] (x-a)^2
\end{aligned}
$$

$$+\left[\lambda\left[\frac{(b-x)}{2}\left|f'(a)\right|^{q}+\left(\frac{x+b}{2}-a\right)\left|f'(b)\right|^{q}\right]^{\frac{1}{q}}\right.$$
$$\left.+(1-\lambda)\left(p\right)^{\frac{1}{p}}\left[\left(\frac{b-x}{2}\right)\left|f^{'}(a)\right|^{q}+\left(\frac{x+b}{2}-a\right)\left|f^{'}(b)\right|^{q}\right]^{\frac{1}{q}}\right](b-x)^{2}\Bigg\}.$$

Corollary 2.7. *If we choose $\lambda=1$ in Theorem 2.2, then we have the following weighted fractional trapezoid inequality*

$$\left|\left[f(a)J_{a+}^{\alpha}g\left(x\right)+f(b)J_{b-}^{\alpha}g\left(x\right)\right]-\left[J_{a+}^{\alpha}\left(fg\right)\left(x\right)+J_{b-}^{\alpha}\left(fg\right)\left(x\right)\right]\right|$$
$$\leq\frac{\|g\|_{[a,x],\infty}}{\Gamma(\alpha+1)}\frac{(x-a)^{\alpha+1}}{(b-a)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}\left[\left(b-\frac{a+x}{2}\right)\left|f^{'}(a)\right|^{q}+\frac{(x-a)}{2}\left|f^{'}(b)\right|^{q}\right]^{\frac{1}{q}}$$
$$+\frac{\|g\|_{[x,b],\infty}}{\Gamma(\alpha+1)}\frac{(b-x)^{\alpha+1}}{(b-a)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}\left[\left[\frac{(b-x)}{2}\left|f'(a)\right|^{q}+\left(\frac{x+b}{2}-a\right)\left|f'(b)\right|^{q}\right]^{\frac{1}{q}}\right.$$
$$\leq\frac{\|g\|_{\infty}}{\Gamma(\alpha+1)(b-a)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}$$
$$\Bigg\{(x-a)^{\alpha+1}\left[\left(b-\frac{a+x}{2}\right)\left|f^{'}(a)\right|^{q}+\frac{(x-a)}{2}\left|f^{'}(b)\right|^{q}\right]^{\frac{1}{q}}$$
$$+(b-x)^{\alpha+1}\left[\frac{(b-x)}{2}\left|f'(a)\right|^{q}+\left(\frac{x+b}{2}-a\right)\left|f'(b)\right|^{q}\right]^{\frac{1}{q}}\Bigg\}.$$

Corollary 2.8. *If we choose $\lambda=0$ in Theorem 2.2, then we have the following weighted fractional Ostrowski inequality*

$$\left|\left[J_{a+}^{\alpha}g\left(x\right)+J_{b-}^{\alpha}g\left(x\right)\right]f(x)-\left[J_{a+}^{\alpha}\left(fg\right)\left(x\right)+J_{b-}^{\alpha}\left(fg\right)\left(x\right)\right]\right|$$
$$\leq\frac{\|g\|_{[a,x],\infty}}{\Gamma(\alpha+1)}\frac{(x-a)^{\alpha+1}}{(b-a)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}(\alpha p)^{\frac{1}{p}}\left[\left(b-\frac{a+x}{2}\right)\left|f^{'}(a)\right|^{q}+\frac{(x-a)}{2}\left|f^{'}(b)\right|^{q}\right]^{\frac{1}{q}}$$
$$+\frac{\|g\|_{[x,b],\infty}}{\Gamma(\alpha+1)}\frac{(b-x)^{\alpha+1}}{(b-a)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}(\alpha p)^{\frac{1}{p}}\left[\left(\frac{b-x}{2}\right)\left|f^{'}(a)\right|^{q}+\left(\frac{x+b}{2}-a\right)\left|f^{'}(b)\right|^{q}\right]^{\frac{1}{q}}$$
$$\leq\frac{\|g\|_{\infty}}{\Gamma(\alpha+1)(b-a)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}$$
$$\Bigg\{(1-\lambda)\left(\alpha p\right)^{\frac{1}{p}}\left[\left(b-\frac{a+x}{2}\right)\left|f^{'}(a)\right|^{q}+\frac{(x-a)}{2}\left|f^{'}(b)\right|^{q}\right]^{\frac{1}{q}}$$
$$+(1-\lambda)\left(\alpha p\right)^{\frac{1}{p}}\left[\left(\frac{b-x}{2}\right)\left|f^{'}(a)\right|^{q}+\left(\frac{x+b}{2}-a\right)\left|f^{'}(b)\right|^{q}\right]^{\frac{1}{q}}\Bigg\}.$$

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