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INEQUALITIES WITH INFINITE CONVEX COMBINATIONS

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Abstract. The aim of the paper is to extend the discrete form of Jensen's inequality to infinite convex combinations and sequences of convex combinations. As the complement, the most interesting results are applied to the well known discrete and integral inequalities.

1. Introduction

The primary concept of convexity is based on a bounded closed interval of real numbers, its convex combinations, and corresponding convex function. Let $[a, b]$ be a closed interval with $a < b$. A combination $\alpha x + \beta y$ of points $x, y \in [a, b]$ and coefficients $\alpha, \beta \in [0, 1]$ is said to be convex if $\alpha + \beta = 1$. A function $f : [a, b] \to \mathbb{R}$ is said to be convex if the inequality

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y) \tag{1.1}$$

holds for every convex combination $\alpha x + \beta y$ of points $x, y \in [a, b]$. The convex function f is bounded by two lines. A support line h_1 at an interior point $x_0 \in (a, b)$ with a slope coefficient $k \in [f'(x_0-), f'(x_0+)]$ expressed by

$$h_1(x) = k(x - x_0) + f(x_0)$$

is the lower bound of f. The secant line h_2 at the endpoints a and b expressed by

$$h_2(x) = \frac{b - x}{b - a} f(a) + \frac{x - a}{b - a} f(b)$$

is the upper bound of f. So, the double inequality

$$h_1(x) \leq f(x) \leq h_2(x) \tag{1.2}$$

holds for every point $x \in [a, b]$. This initial inequality can be upgraded as follows.

Given a positive integer n, let $\sum_{i=1}^{n} \lambda_i x_i$ be an n-membered convex combination ($\lambda_i \in [0, 1]$ and $\sum_{i=1}^{n} \lambda_i = 1$) of points $x_i \in [a, b]$. If $\alpha a + \beta b$ is the convex combination of endpoints

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a and b that satisfies $\alpha a+\beta b=\sum_{i=1}^{n}\lambda_i x_i$, then each convex function $f:[a,b]\to\mathbb{R}$ satisfies the double inequality

$$f(\alpha a+\beta b)\le\sum_{i=1}^{n}\lambda_i f(x_i)\le\alpha f(a)+\beta f(b). \tag{1.3}$$

This fundamental convex function inequality expresses the nature of growth of a convex function. The convex function values, taken in the forms of convex combinations, grow from the center to the ends. Its version, with $\sum_{i=1}^{n}\lambda_i x_i$ instead of $\alpha a+\beta b$ on the left side, represents the extended discrete form of Jensen's inequality. We want to replace n with infinity.

For more details on convex sets and functions, see books [9] and [10].

2. Main results

Let $\mathbb{X}$ be a vector space over the field $\mathbb{R}$. Without employing any convergence in the space $\mathbb{X}$, we formally define the notion of an infinite convex combination which contains the infinite number of space points. We only assume that the coefficients sum converges to number 1 in the field $\mathbb{R}$.

Definition 2.1. An infinite linear combination $\sum_{i=1}^{\infty}\lambda_i x_i$ of points $x_i\in\mathbb{X}$ is said to be convex if coefficients $\lambda_i\in[0,1]$ and $\sum_{i=1}^{\infty}\lambda_i=1$.

Infinite convex combinations are prone to convergence, which is best illustrated in the next lemma and corollary.

Lemma 2.1. *An infinite convex combination $\sum_{i=1}^{\infty}\lambda_i t_i$ of points $t_i\in[0,1]$ converges in $[0,1]$.*

Proof. The nth partial sum $\tau_n=\sum_{i=1}^{n}\lambda_i t_i$ is within the bounds

$$0\le\sum_{i=1}^{n}\lambda_i t_i\le\sum_{i=1}^{n}\lambda_i\le 1.$$

Furthermore, the partial sums τ_n and τ_{n+1} maintain the order

$$\tau_n=\sum_{i=1}^{n}\lambda_i t_i\le\sum_{i=1}^{n+1}\lambda_i t_i=\tau_{n+1}.$$

So, the sequence $(\tau_n)_{n=1}^{\infty}$ is bounded and nondecreasing, which ensures that it converges to some number $t\in[0,1]$. Written down as the series, it stands as

$$\sum_{i=1}^{\infty}\lambda_i t_i=t\in[0,1],$$

which completes the proof. □

The above initial lemma also applies to any real bounded closed interval.

Corollary 2.1. *An infinite convex combination $\sum_{i=1}^{\infty}\lambda_i x_i$ of points $x_i\in[a,b]$ converges in $[a,b]$.*

Proof. Since $a \leq x_i \leq b$, then
$$0 \leq \frac{x_i - a}{b - a} \leq 1.$$
By applying Lemma 2.1 to the numbers $t_i = (x_i - a)/(b - a)$, it follows that
$$\sum_{i=1}^{\infty} \lambda_i \frac{x_i - a}{b - a} = t \in [0, 1],$$
and consequently
$$\sum_{i=1}^{\infty} \lambda_i x_i = (1 - t)a + tb \in [a, b],$$
which ends the proof. □

In terms of functions, in the proof of the above corollary, we utilized the affine function $x \mapsto (x - a)/(b - a)$ mapping the interval $[a, b]$ onto the unit interval $[0, 1]$.

Corollary 2.2. *Let X be a nonempty set, let $g : X \to \mathbb{R}$ be a function with the image in $[a, b]$, and let $\sum_{i=1}^{\infty} \lambda_i g(x_i)$ be an infinite convex combination of points $g(x_i)$ with $x_i \in X$.*

Then the above combination converges in $[a, b]$.

Proof. Applying Lemma 2.1 to the numbers $t_i = (g(x_i) - a)/(b - a)$, we obtain
$$\sum_{i=1}^{\infty} \lambda_i g(x_i) = (1 - t)a + tb \in [a, b],$$
which proves the assertion of the corollary. □

The above corollaries enable an extension of the inequality in formula (1.3) to infinite convex combinations.

Theorem 2.1. *Let $\sum_{i=1}^{\infty} \lambda_i x_i$ be an infinite convex combination of points $x_i \in [a, b]$, and let $\alpha a + \beta b$ be the convex combination that satisfies $\alpha a + \beta b = \sum_{i=1}^{\infty} \lambda_i x_i$.*

Then each convex function $f : [a, b] \to \mathbb{R}$ satisfies the double inequality
$$f(\alpha a + \beta b) \leq \sum_{i=1}^{\infty} \lambda_i f(x_i) \leq \alpha f(a) + \beta f(b). \tag{2.1}$$

Proof. Respecting the fact that the convex function f may be discontinuous at the endpoints a and b, we will discus two cases.

The first case. Suppose that $\sum_{i=1}^{\infty} \lambda_i x_i = a$. Then $\sum_{i=1}^{\infty} \lambda_i (x_i - a) = 0$ with $\lambda_i \geq 0$ and $x_i - a \geq 0$. Thus $\lambda_i \neq 0$ implies $x_i = a$, and therefore
$$\sum_{i=1}^{\infty} \lambda_i f(x_i) = \sum_{i=1}^{\infty} \lambda_i f(a) = f(a).$$
Thus the trivial double inequality $f(a) \leq f(a) \leq f(a)$ represents formula (2.1). A similar applies if $\sum_{i=1}^{\infty} \lambda_i x_i = b$.

The second case. Suppose that $\sum_{i=1}^{\infty} \lambda_i x_i \in (a,b)$. Let $n \geq 2$ be an integer, and let $\varepsilon_n = 1 - \sum_{i=1}^{n-1} \lambda_i$ be the additional coefficient. Applying the double inequality in formula (1.3) to the convex combinations equality

$$\alpha_n a + \beta_n b = \sum_{i=1}^{n-1} \lambda_i x_i + \varepsilon_n x_n, \tag{2.2}$$

we obtain

$$f(\alpha_n a + \beta_n b) \leq \sum_{i=1}^{n-1} \lambda_i f(x_i) + \varepsilon_n f(x_n) \leq \alpha_n f(a) + \beta_n f(b). \tag{2.3}$$

Now, we have to consider the reflection moment as n approaches infinity. Applying the reflection moment to formula (2.2), we get $\lim_{n\to\infty} \varepsilon_n x_n = 0$ and

$$\lim_{n\to\infty} (\alpha_n a + \beta_n b) = \sum_{i=1}^{\infty} \lambda_i x_i = \alpha a + \beta b.$$

Using this limit, and employing the continuity of the restriction $f/(a,b)$, we obtain

$$\lim_{n\to\infty} f(\alpha_n a + \beta_n b) = f\big(\lim_{n\to\infty} (\alpha_n a + \beta_n b)\big) = f(\alpha a + \beta b).$$

Since $\lim_{n\to\infty} \varepsilon_n f(x_n) = 0$, $\lim_{n\to\infty} \alpha_n = \alpha$ and $\lim_{n\to\infty} \beta_n = \beta$, the application of the reflection moment to formula (2.3) results so that it approaches formula (2.1). $\square$

The inequality in formula (2.1) provides the extended discrete form of Jensen's inequality (see [3]) for infinite convex combinations.

Corollary 2.3. *Let $\sum_{i=1}^{\infty} \lambda_i x_i$ be an infinite convex combination of points $x_i \in [a,b]$, and let $c = \sum_{i=1}^{\infty} \lambda_i x_i$ be its sum.*

Then each convex function $f : [a,b] \to \mathbb{R}$ satisfies the double inequality

$$f\left(\sum_{i=1}^{\infty} \lambda_i x_i\right) \leq \sum_{i=1}^{\infty} \lambda_i f(x_i) \leq \frac{b-c}{b-a} f(a) + \frac{c-a}{b-a} f(b). \tag{2.4}$$

Proof. By combining the equalities $\alpha a + \beta b = c$ and $\beta = 1 - \alpha$, it follows that $\alpha = (b-c)/(b-a)$ and $\beta = (c-a)/(b-a)$. $\square$

The double inequality in formula (2.4) presents the fundamental convex function inequality on the bounded closed interval, including its infinite convex combinations.

The double inequality in formula (1.3) can also be extended to sequences of convex combinations. Because of the possible convergence to the interval endpoints, we have to use a continuous convex function f.

Theorem 2.2. *Let $(c_n)_{n=1}^{\infty}$ be a convergent sequence of convex combinations $c_n = \sum_{i=1}^{m_n} \lambda_{ni} x_{ni}$ of points $x_{ni} \in [a,b]$, and let $\alpha a + \beta b$ be the convex combination that satisfies $\alpha a + \beta b = \lim_{n\to\infty} c_n$.*

Then each continuous convex function $f : [a,b] \to \mathbb{R}$ satisfies the double inequality

$$f(\alpha a + \beta b) \leq \lim_{n\to\infty} \sum_{i=1}^{m_n} \lambda_{ni} f(x_{ni}) \leq \alpha f(a) + \beta f(b). \tag{2.5}$$

Proof. Applying the proof of the second case of Theorem 2.1 to the convex combinations equality

$$\alpha_n a + \beta_n b = \sum_{i=1}^{m_n} \lambda_{ni} x_{ni},$$

and utilizing the continuity of f, we can attain the inequality in formula (2.5). $\square$

In the statement of the above theorem, the continuity of f can not be omitted. It refers to the left-hand side (containing the left and middle members) of the inequality in formula (2.5). Using points $x_{n1} = a + 1/n$ and $x_{n2} = b - 1/n$, let us take the sequence of binomial convex combinations

$$c_n = \frac{n-1}{n} x_{n1} + \frac{1}{n} x_{n2}$$

for all sufficiently large integers n, providing that points $x_{n1}, x_{n2} \in [a, b]$. It is obvious that $\lim_{n\to\infty} c_n = a$. Let f be a convex function that has a discontinuity at the endpoint a. Then, as regards formula (2.5), the left member is

$$f\big(\lim_{n\to\infty} c_n\big) = f(a),$$

and the middle member is

$$\lim_{n\to\infty} \left(\frac{n-1}{n} f(x_{n1}) + \frac{1}{n} f(x_{n2}) \right) = f(a+).$$

Since the function f satisfies $f(a) > f(a+)$, the left member is greater than the middle member in the observed case.

We point out the next applicable variant of Theorem 2.2 as the extended discrete form of Jensen's inequality for convergent sequences of convex combinations. It can be useful in the creation of integral inequalities.

Corollary 2.4. *Let $(c_n)_{n=1}^{\infty}$ be a convergent sequence of convex combinations $c_n = \sum_{i=1}^{m_n} \lambda_{ni} x_{ni}$ of points $x_{ni} \in [a, b]$, and let $c = \lim_{n\to\infty} c_n$ be its limit.*

Then each continuous convex function $f : [a, b] \to \mathbb{R}$ satisfies the double inequality

$$f\left(\lim_{n\to\infty} \sum_{i=1}^{m_n} \lambda_{ni} x_{ni} \right) \le \lim_{n\to\infty} \sum_{i=1}^{m_n} \lambda_{ni} f(x_{ni}) \le \frac{b-c}{b-a} f(a) + \frac{c-a}{b-a} f(b). \tag{2.6}$$

Different forms of Jensen's inequality were considered in [6] and [8], and the extension of Jensen's inequality to affine combinations was demonstrated in [5].

3. Products of convex combinations

In this section, we present the extended discrete form of Jensen's inequality for the product of convex combinations. To ensure a cause-effect relationship, we deal with finite and infinite convex combinations.

The section is based on the fact that the product of convex combinations produces a convex combination. If $\sum_{i=1}^{n} \lambda_i x_i$ is an n-membered convex combination of points x_i, and if $\sum_{j=1}^{m}\kappa_j y_j$ is an m-membered convex combination of points y_j, then their product

$$\Big(\sum_{i=1}^{n}\lambda_i x_i\Big)\Big(\sum_{j=1}^{m}\kappa_j y_j\Big)=\sum_{i=1}^{n}\sum_{j=1}^{m}\lambda_i\kappa_j x_i y_j \tag{3.1}$$

is the nm-membered convex combination of points $x_i y_j$ because the coefficients $\lambda_i\kappa_j$ are nonnegative and their sum is equal to

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\lambda_i\kappa_j=\Big(\sum_{i=1}^{n}\lambda_i\Big)\Big(\sum_{j=1}^{m}\kappa_j\Big)=1\cdot 1=1.$$

Relying on the product representation in formula (3.1) and the inequality in formula (1.3), we get the following.

Lemma 3.1. *Let $\sum_{i=1}^{n}\lambda_i x_i$ and $\sum_{j=1}^{m}\kappa_j y_j$ be convex combinations of real points x_i and y_j, let $[a,b]$ be a closed interval containing the products $x_i y_j$, and let $\alpha a+\beta b$ be the convex combination that satisfies $\alpha a+\beta b=\big(\sum_{i=1}^{n}\lambda_i x_i\big)\big(\sum_{j=1}^{m}\kappa_j y_j\big)$.*

Then each convex function $f:[a,b]\to\mathbb{R}$ satisfies the double inequality

$$f(\alpha a+\beta b)\le\sum_{i=1}^{n}\sum_{j=1}^{m}\lambda_i\kappa_j f(x_i y_j)\le\alpha f(a)+\beta f(b). \tag{3.2}$$

The above lemma can be expanded to infinite convex combinations. The product of infinite convex combinations $\sum_{i=1}^{\infty}\lambda_i x_i$ and $\sum_{j=1}^{\infty}\kappa_j y_j$ of points x_i and y_j as the Cauchy convolution

$$\Big(\sum_{i=1}^{\infty}\lambda_i x_i\Big)\Big(\sum_{j=1}^{\infty}\kappa_j y_j\Big)=\sum_{j=1}^{\infty}\sum_{i=1}^{j}\lambda_i\kappa_{j-i+1}x_i y_{j-i+1} \tag{3.3}$$

is the infinite convex combination of points $x_i y_{j-i+1}$ because the coefficients $\lambda_i\kappa_{j-i+1}$ are nonnegative and their sum is equal to

$$\sum_{j=1}^{\infty}\sum_{i=1}^{j}\lambda_i\kappa_{j-i+1}=\Big(\sum_{i=1}^{\infty}\lambda_i\Big)\Big(\sum_{j=1}^{\infty}\kappa_j\Big)=1\cdot 1=1.$$

Using the Cauchy product in formula (3.3), we are expanding Theorem 2.1.

Theorem 3.1. *Let $\sum_{i=1}^{\infty}\lambda_i x_i$ and $\sum_{j=1}^{\infty}\kappa_j y_j$ be infinite convex combinations of real points x_i and y_j such that the sequences $(x_i)_{i=1}^{\infty}$ and $(y_j)_{j=1}^{\infty}$ are bounded, let $[a,b]$ be a closed interval containing the sequence $((x_i y_{j-i+1})_{i=1}^{j})_{j=1}^{\infty}$, and let $\alpha a+\beta b$ be the convex combination that satisfies $\alpha a+\beta b=\big(\sum_{i=1}^{\infty}\lambda_i x_i\big)\big(\sum_{j=1}^{\infty}\kappa_j y_j\big)$.*

Then each convex function $f:[a,b]\to\mathbb{R}$ satisfies the double inequality

$$f(\alpha a+\beta b)\le\sum_{j=1}^{\infty}\sum_{i=1}^{j}\lambda_i\kappa_{j-i+1}f(x_i y_{j-i+1})\le\alpha f(a)+\beta f(b). \tag{3.4}$$

Proof. Suppose that $\sum_{i=1}^{\infty}\lambda_i x_i$ converges to x and $\sum_{j=1}^{\infty}\kappa_j y_j$ converges to y. Then the product $\sum_{j=1}^{\infty}\sum_{i=1}^{j}\lambda_i\kappa_{j-i+1}x_i y_{j-i+1}$ converges to $xy=\alpha a+\beta b$ because both factors converge absolutely. This product fits into Theorem 2.1. $\square$

4. Applications to discrete and integral inequalities

Due to Corollary 2.1, we can consider the quasi-arithmetic means related to infinite convex combinations. Let $(x_i)_{i=1}^{\infty}$ be a sequence of points $x_i \in [a,b]$, let $(\lambda_i)_{i=1}^{\infty}$ be a sequence of nonnegative coefficients such that $\sum_{i=1}^{\infty} \lambda_i = 1$, and let $\varphi : [a,b] \to \mathbb{R}$ be a strictly monotone continuous function. The discrete quasi-arithmetic mean of points x_i with respect to coefficients λ_i and the function φ is the number

$$M_\varphi(x_i;\lambda_i) = \varphi^{-1}\Bigg(\sum_{i=1}^{\infty} \lambda_i \varphi(x_i)\Bigg). \tag{4.1}$$

The number $M_\varphi(x_i;\lambda_i)$ belongs to the interval $[a,b]$ because the infinite convex combination $\sum_{i=1}^{\infty} \lambda_i \varphi(x_i)$ converges in the image of φ by Corollary 2.2 (the image of φ is the bounded closed interval).

The framework of quasi-arithmetic means includes a pair of strictly monotone continuous functions $\varphi, \psi : [a,b] \to \mathbb{R}$ and the following definition. The function ψ is said to be *φ-convex* (*φ-concave*) if the composition $\psi \circ \varphi^{-1}$ is convex (concave). In the case of infinite convex combinations, the basic result on quasi-arithmetic means applies as follows.

Lemma 4.1. *Let $\varphi, \psi : [a,b] \to \mathbb{R}$ be strictly monotone continuous functions, and let $\sum_{i=1}^{\infty} \lambda_i x_i$ be an infinite convex combination of points $x_i \in [a,b]$.*

If either ψ is increasing and φ-convex or ψ is decreasing and φ-concave, then

$$M_\varphi(x_i;\lambda_i) \le M_\psi(x_i;\lambda_i). \tag{4.2}$$

If either ψ is decreasing and φ-convex or ψ is increasing and φ-concave, then the reverse inequality is valid in formula (4.2).

Proof. We prove the case that ψ is increasing and φ-convex. Using the left-hand side of the inequality in formula (2.4) with the convex combination $\sum_{i=1}^{\infty} \lambda_i \varphi(x_i)$ and the convex function $\psi \circ \varphi^{-1}$, we get

$$(\psi \circ \varphi^{-1})\Bigg(\sum_{i=1}^{\infty} \lambda_i \varphi(x_i)\Bigg) \le \sum_{i=1}^{\infty} \lambda_i \psi(x_i).$$

Acting with the increasing function ψ^{-1} to the above inequality, we obtain the inequality in formula (4.2). ☐

The power means are represented by power functions $\varphi(x) = x^r$ with $x > 0$ and $r \neq 0$ ($r = 0$ as the limit case) in formula (4.1). Let $\sum_{i=1}^{\infty} \lambda_i x_i$ be an infinite convex combination of positive real numbers x_i such that the sequence $(x_i)_{i=1}^{\infty}$ is bounded. If $r \neq 0$, then the power mean of order r is

$$M_r(x_i;\lambda_i) = \left(\sum_{i=1}^{\infty} \lambda_i x_i^r\right)^{1/r}. \tag{4.3}$$

Letting r tend to 0, we have the limit case as

$$M_0(x_i;\lambda_i) = \lim_{r\to 0} M_r(x_i;\lambda_i) = \exp\left(\sum_{i=1}^{\infty} \lambda_i \ln x_i\right) = \prod_{i=1}^{\infty} x_i^{\lambda_i}. \tag{4.4}$$

It can be proved that $M_r(x_i;\lambda_i) \le M_s(x_i;\lambda_i)$ if $r \le s$. Applying this inequality to the harmonic mean M_{-1}, geometric mean M_0 and arithmetic mean M_1, we gain the extension of the most famous mean inequality to infinite convex combinations.

Corollary 4.1. *Let $\sum_{i=1}^{\infty}\lambda_i x_i$ be an infinite convex combination of positive real numbers x_i such that the sequence $(x_i)_{i=1}^{\infty}$ is bounded.*

Then the infinite form of the harmonic-geometric-arithmetic mean inequality stands as

$$\left(\sum_{i=1}^{\infty}\lambda_i x_i^{-1}\right)^{-1} \le \prod_{i=1}^{\infty} x_i^{\lambda_i} \le \sum_{i=1}^{\infty}\lambda_i x_i. \tag{4.5}$$

Double inequality in formula (2.6) can be upgraded to the extended integral form of Jensen's inequality (see [4]) by using an integrable (including boundedness) function g. For this purpose, we employ the sequence of convex combinations of points $g(x_{ni})$ that converges to the integral arithmetic mean of g.

Corollary 4.2. *Let $g:[u,v]\to\mathbb{R}$ be an integrable function with the image in $[a,b]$, and let $c = 1/(v-u)\int_u^v g(x)\,dx$ be the integral arithmetic mean of g.*

Then each convex function $f:[a,b]\to\mathbb{R}$ satisfies the double inequality

$$f\left(\frac{1}{v-u}\int_u^v g(x)\,dx\right) \le \frac{1}{v-u}\int_u^v f(g(x))\,dx \le \frac{b-c}{b-a}f(a) + \frac{c-a}{b-a}f(b). \tag{4.6}$$

Proof. We will observe two cases, depending on the position of c.

The first case. Suppose that $c = a$. Then the function g is almost everywhere equal to a, and the trivial double inequality $f(a)\le f(a)\le f(a)$ represents formula (4.6). A similar applies if $c = b$.

The second case. Suppose that $c\in(a,b)$. Let $n\ge 2$ be an integer, let $\triangle_{ni}$ for $i=1,\dots,n$ be members of the partition of the interval $[u,v]$ keeping the same length $|\triangle_{ni}| = (v-u)/n$, let $\lambda_{ni} = 1/n = |\triangle_{ni}|/(v-u)$ be the coefficients, let $x_{ni}\in\triangle_{ni}$ be points, and let

$$c_n = \sum_{i=1}^{n}\lambda_{ni}g(x_{ni}) = \frac{1}{v-u}\sum_{i=1}^{n}|\triangle_{ni}|g(x_{ni})$$

be the convex combination of points $g(x_{ni})$ as the nth integral sum of g over $v-u$. Since g is integrable, it follows that

$$\lim_{n\to\infty} c_n = \frac{1}{v-u}\int_u^v g(x)\,dx = c.$$

Let $[a_1,b_1]\subset(a,b)$ be a closed neighbourhood of the point c. If we put $m_n = n$ in formula (2.6), and if we apply that formula to the convergent sequence $(c_n)_{n=1}^{\infty}$ and continuous convex restriction $f/[a_1,b_1]$, we get

$$f\left(\lim_{n\to\infty}\sum_{i=1}^{n}\lambda_{ni}g(x_{ni})\right) \le \lim_{n\to\infty}\sum_{i=1}^{n}\lambda_{ni}f(g(x_{ni})) \le \frac{b_1-c}{b_1-a_1}f(a_1) + \frac{c-a_1}{b_1-a_1}f(b_1).$$

Putting the integrals into the above inequality, and using the secant lines inequality $h_{[a_1,b_1]}(c)\le h_{[a,b]}(c)$, we reach formula (4.6). $\square$

It is interesting that the convex function f in formula (4.6) does not have to be continuous, while in formula (2.6) it must be continuous. The reason is that the value $1/(v-u)\int_u^v g(x)\,dx$ in formula (4.6) is an integral number, while the value $\lim_{n\to\infty}\sum_{i=1}^{m_n}\lambda_{ni}x_{ni}$ in formula (2.6) is a limit number.

The inequality in formula (4.6) is reduced to the Hermite-Hadamard inequality (see [2] and [1]) if the identity function $g:[a,b]\to[a,b]$ as $g(x)=x$ is used. The reason is that the integral arithmetic mean of g coincides with the interval midpoint, $c=(a+b)/2$.

Corollary 4.3. *Each convex function $f:[a,b]\to\mathbb{R}$ satisfies the double inequality*

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{f(a)+f(b)}{2}. \tag{4.7}$$

Generalizations and refinements of the Hermite-Hadamard inequality on the bounded closed interval can be found in [7].

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