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MONOTONICITY AND INEQUALITIES RELATED TO THE k-GAMMA AND k-DIGAMMA FUNCTIONS

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ABSTRACT. In this paper, we mainly present some monotonicity and inequalities related to the k-gamma and k-digamma functions.

1. INTRODUCTION

The Euler gamma function is defined all positive real numbers x by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$

The logarithmic derivative of $\Gamma(x)$ is called the psi or digamma function. That is

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty}\frac{x}{n(n+x)},$$

where $\gamma = 0.5772\ldots$ is the Euler-Mascheroni constant. The polygamma functions $\psi^{(m)}(x)$ for $m \in \mathbb{N}$ are defined by

$$\psi^{(m)}(x) = \frac{d^m}{dx^m}\psi(x) = (-1)^m m!\sum_{n=0}^{\infty}\frac{1}{(n+x)^{m+1}}, x > 0.$$

The gamma, digamma and polygamma functions play an important role in the theory of special functions, and are closely related to factorial, fractional differential equations, mathematical physics and crops up in many unexpected place in analysis. The reader may see reference ([3]). Some of the work about origin, history, the complete monotonicity, and inequalities of these special functions can be found in ([11], [18, 19]) and the references therein.

In 2007, Díaz and Pariguan [4] defined the $k-$analogue of the gamma function for $k > 0$ and $x > 0$ as

$$\Gamma_k(x) = \int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}dt = \lim_{n\to\infty}\frac{n!k^n(nk)^{\frac{x}{k}-1}}{x(x+k)\cdots(x+(n-1)k)},$$

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where $\lim_{k\to 1}\Gamma_k(x)=\Gamma(x)$. Similarly, we may define the $k-$analogue of the digamma and polygamma functions as

$$\psi_k(x)=\frac{d}{dx}\ln\Gamma_k(x) \quad \text{and} \quad \psi_k^{(m)}(x)=\frac{d^m}{dx^m}\psi_k(x).$$

It is well known that the $k-$analogues of the digamma and polygamma functions satisfy the following recursive formula and series identities (See [4, 12–14, 20]).

$$\Gamma_k(x+k)=x\Gamma_k(x), \quad x>0, \tag{1.1}$$

$$\begin{aligned}\psi_k(x)&=\tfrac{\ln k-\gamma}{k}-\tfrac{1}{x}+\textstyle\sum_{n=1}^{\infty}\tfrac{x}{nk(nk+x)}\\ &=\tfrac{\ln k-\gamma}{k}+\textstyle\int_0^{\infty}\tfrac{e^{-kt}-e^{-xt}}{1-e^{-kt}}dt,\\ &=\tfrac{\ln k-\gamma}{k}+\textstyle\int_0^{1}\tfrac{t^{k-1}-t^{x-1}}{1-t^k}dt,\end{aligned} \tag{1.2}$$

and

$$\begin{aligned}\psi_k^{(m)}(x)&=(-1)^{m+1}m!\textstyle\sum_{n=0}^{\infty}\tfrac{1}{(nk+x)^{m+1}}\\ &=(-1)^{m+1}\textstyle\int_0^{\infty}\tfrac{1}{1-e^{-kt}}t^m e^{-xt}dt.\end{aligned} \tag{1.3}$$

In particular, more properties of the functions $\psi_k(x)$ and $\psi_k^{(m)}(x)$ can be found in [14,20]. It's worth noting that many mathematicians have studied k-generalizations look like the above form, such as k-hypergeometric function, k-hypergeometric differential equations, Appell k-series, k-functions, Kummar k-confluent hypergeometric function, Riemann-Liouville k- and Hadamard k-fractional derivatives, (k,ρ)-fractional integral operator et. al. The readers may refer to references [6–10, 15–17].

The following recurrence and asymptotic formulas are often encountered in the literature:

$$\psi_k{}^{(m)}(x+k)=\psi_k{}^{(m)}(x)+(-1)^m\frac{m!}{x^{m+1}}(x>0, m=0,1\ldots\ldots). \tag{1.4}$$

In [2], Alzer obtained some inequalities for the gamma and polygamma functions. Later, in [1], Alzer proved the following extension of this result. Let $M_n^{[r]}(x_j,p_j)$ be the weighted power mean of $x_1,\ldots,x_n$ of order r. The inequality

$$\Gamma\left(M_n^{[r]}(x_j,p_j)\right)\le M_n^{[r]}(\Gamma(x_j),p_j)$$

holds for all $x_j>0$ and $p_j>0, j=1,2\ldots,n, n\ge 2$ with $\sum\limits_{j=1}^{n}p_j=1$ if and only if $0.01317\ldots\le r\le 11.29416\ldots$. It is natural to look for an extension of these results to k-gamma and k-digamma functions. This is the main object of this paper.

2. Lemmas

Lemma 2.1. *Let $g_k(x)=kx\psi_k(kx)-x\log(k)$ and let $\gamma=0.57721...$ be Euler-Mascheroni constant.*

(1) $g_k^{''}$ is strictly completely monotonic on $(0,\infty)$.

(2) $g_k^{'}$ is strictly increasing on $(0,\infty)$ with $g_k^{'}(0^+)=-\gamma$ and $\lim\limits_{x\to\infty}g_k^{'}(x)=\infty$. The only positive zero of g_k is given by $r_0=0.21609....$

(3)g_k is strictly decreasing on $(0,r_0]$ and strictly increasing on $[r_0,\infty)$ with $g_k(0^+)=-1$ and $\lim\limits_{x\to\infty}g_k(x)=\infty$.The only positive zero of $g_k^{'}$ is given by $r_1=1.46163....$

Proof. (1) Applying the series representation

$$\psi_k^{(m)}(x) = (-1)^{m+1} m! \sum_{v=0}^{\infty} \frac{1}{(kv+x)^{m+1}}, x > 0, k > 0, m = 1, 2 \ldots,$$

we obtain for $x > 0$, $k > 0$ and $n \geq 2$:

$$(-1)^n (g_k^{''}(x))^{(n)} = nk^n (-1)^n \psi_k^{(n-1)}(kx) - k^{n+1}(-1)^{n+1} x \psi_k^{(n)}(kx)$$
$$= k^n n! \sum_{v=0}^{\infty} \frac{kv}{(kv+kx)^{n+1}} > 0.$$

(2) We have $g_k^{''}(x) > 0$ for $x > 0$ and $k > 0$, so that $g_k^{'}$ is strictly increasing on $(0, \infty)$. Applying the recurrence formulas

$$\psi_k(kx) = \psi_k(kx + k) - \frac{1}{kx} \tag{2.1}$$

and

$$\psi_k^{'}(kx) = \psi_k^{'}(kx + k) + \frac{1}{(kx)^2},$$

we get $g_k^{'}(x) = k\psi_k(kx + k) + k^2 x \psi_k^{'}(kx + k) - \log(k)$, which leads to $g_k^{'}(0^+) = k\psi_k(k) - \log(k) = -\gamma$. Moreover, since $\psi_k^{'}$ is positive on $(0, \infty)$ and

$$k\psi_k(kx) = \log(k) + O(\frac{1}{x})(x \to \infty), \tag{2.2}$$

we have $\lim_{x\to\infty} g_k^{'}(x) = \lim_{x\to\infty} (k\psi_k(kx + k) + x\psi_k^{'}(kx + k) - \log(k)) = \infty$.

(3) The function $g_k^{'}$ is negative on $(0, r_0)$ and positive on (r_0, ∞), where $r_0 = 0.21609...$ is the only positive solution of $k\psi_k(kx + k) + x\psi_k^{'}(kx + k) - \log(k) = 0$. From (2.1) and (2.2) we obtain $g_k(0^+) = -1$ and $\lim_{x\to\infty} g_k(x) = \infty$. □

Lemma 2.2. *The function*

$$v_k(x) = k\psi_k(kx) - \log(k) - x(k\psi_k(kx) - \log(k))^2$$

is negative on $(0, r_1) \cup (r_2, \infty)$ *and positive on* (r_1, r_2)*. Here,* $r_1 = 1.46163...$ *is the only positive solution of* $k\psi_k(kx) - \log(k) = 0$*, and* $r_2 = 2.08907...$ *is the only positive solution of* $x(k\psi_k(kx) - \log(k)) = 1$*.*

Proof. Let $h_k(x) = 1 - x(k\psi_k(kx) - \log(k))$. From Lemma 2.1, (3) we conclude that h_k is strictly increasing on $(0, r_0)$ and strictly decreasing on (r_0, ∞) with $h_k(0^+) = 2$ and $\lim_{x\to\infty} h_k(x) = -\infty$. This implies that h_k is positive on $(0, r_2)$ and negative on (r_2, ∞). Thus, we get: if $0 < x < r_1$ or $x > r_2$, then $v_k(x) = k\psi_k(kx) - \log(k) - x(k\psi_k(kx) - \log(k))^2 < 0$; and, if $r_1 < x < r_2$, then $v_k(x) = k\psi_k(kx) - \log(k) - x(k\psi_k(kx) - \log(k))^2 > 0$. □

Lemma 2.3. *Let* $w_k(x) = k^2 x^2 \psi_k^{'}(kx)$*. The function* $w_k^{''}$ *is strictly completely monotonic on* $(0, \infty)$*.*

Proof. Let $x > 0$, $k > 0$ and $n \geq 2$. A simple calculation gives

$$w_k^{(n)}(x) = n(n-1)k^n \psi_k^{(n-1)}(kx) + 2nk^{n+1} x \psi_k^{(n)}(kx) + k^{n+2} x^2 \psi_k^{(n+1)}(kx).$$

Using the integral representation

$$\psi_k^{(m)}(x) = (-1)^{m+1} \int_0^{\infty} e^{-xt} \frac{t^m}{1-e^{-kt}} dtx > 0, k > 0, m = 1, 2 \ldots,$$

and the convolution theorem for Laplace transforms we obtain

$$\frac{1}{k^{n+2}x^2}(-1)^n w_k^{(n)}(x) = \int_0^\infty e^{-ktx}\delta_{n,k}(t)dt, \tag{2.3}$$

where

$$\delta_{n,k}(t) = n(n-1)t\int_0^t \frac{s^{n-1}}{1-e^{-ks}}ds - n(n+1)\int_0^t \frac{s^n}{1-e^{-ks}}ds + \frac{t^{n+1}}{1-e^{-kt}}.$$

Differentiation yields

$$\delta^{'}_{n,k}(t) = n(n-1)t\int_0^t \frac{s^{n-1}}{1-e^{-ks}}ds - \frac{(n-1)t^n(1-e^{-kt}) + t^{n+1}ke^{-kt}}{(1-e^{-kt})^2}$$

and

$$\frac{1}{k}e^{2kt}t^{-n}(1-e^{-kt})^3\delta^{''}_{n,k}(t) = 2 + kt + (kt-2)e^{kt} = \sum_{v=3}^{\infty}(v-2)\frac{(kt)^v}{v!}.$$

We have $\delta_{n,k}(0^+) = \delta^{'}_{n.k}(0^+) = 0$ and $\delta^{''}_{n.k}(t) > 0$ for $t > 0$. This implies that $\delta_{n,k}$ is positive on $(0,\infty)$, so that (2.3) leads to $(-1)^n w_k^{(n)}(x) > 0$ for $x > 0$ and $k > 0$. □

Lemma 2.4. *Let*

$$f_k(x) = \frac{k\psi_k(kx) - \log(x) + xk^2\psi_k^{'}(kx)}{k\psi_k(kx) - \log(x) - x(k\psi_k(kx) - \log(x))^2},$$

and let r_0, r_1, r_2 be defined as in Lemma 2.1 and Lemma 2.2, respectively. Then we have

$$\sup_{0<x<r_0} f_k(x) = 0.01317\ldots \quad \text{and} \quad \inf_{r_1<x<r_2} f_k(x) = 11.29416\ldots.$$

There exists precisely one number $x_0 \in (0, r_0)$ with $f_k(x_0) = \sup_{0<x<r_0} f_k(x)$, and there exists precisely one number $y_0 \in (r_1, r_2)$ with $f_k(y_0) = \inf_{r_1<x<r_2} f_k(x)$.

Proof. Let $x > 0$, $k > 0$ and $c \in R$, and $Q_{c,k}(x) = (1-c)g_k(x) + c(g_k(x))^2 + w_k(x)$ where g_k and w_k are defined in Lemma 2.1 and Lemma 2.3, respectively. We distinguish two cases.

Case 1. $c = 0.013179$ and $x \in (0, r_0)$. Differentiation gives

$$Q^{''}_{c,k}(x) = (1-c)g^{''}_k(x) + 2c[(g^{'}_k(x))^2 + g_k(x)g^{''}_k(x)] + w^{''}_k(x) \tag{2.4}$$

From Lemma 2.1, (1), (3) and Lemma 2.3 we obtain

$$Q^{''}_{c,k}(x) > [1-c+2cg_k(x)]g^{''}_k(x) > [1-c+2cg_k(r_0)]g^{''}_k(x) > 0.95g^{''}_k(x) > 0.$$

Hence, $Q_{c,k}$ is strictly convex on $(0, r_0)$. Let $x_1 = 0.100205$ and $x_2 = 0.100209$. We have $\mathrm{Q}^{'}_{\mathrm{c,k}}(x_1) < 0 < \mathrm{Q}^{'}_{\mathrm{c,k}}(x_2)$. Thus, there exists a number $z_0 \in (x_1, x_2)$ such that

$$Q_{c,k}(x) \geq Q_{c,k}(z_0) \quad (0 < x < r_0, k > 0)\ .$$

Since

$$Q_{c,k}(z_0) > Q_{c,k}(x_1) + (x_2 - x_1)\mathrm{Q}^{'}_{\mathrm{c,k}}(x_1) = 0.000014\ldots,$$

we conclude that $Q_{c,k}$ is positive on $(0, r_0)$. This leads to

$$f_k(x) < c = 0.013179 \quad (0 < x < r_0, k > 0)\ .$$

Since $f_k(0.100208) = 0.013172...$, we obtain

$$\sup_{0<x<r_0} f_k(x) = 0.01317....$$

Let $c_0 = \sup_{0<x<r_0} f_k(x)$. We have $f_k(0^+) = f_k(r_0) = 0$, which implies that there exists a number $x_0 \in (0, r_0)$ with $f_k(x_0) = c_0$. We assume that there exists a number $x_0^* \in (0, r_0)$ such that $x_0^* \neq x_0$ and $f_k(x_0^*) = c_0$. Then we get

$$Q_{c_0,k}(x) \geq 0 = Q_{c_0,k}(x_0) = Q_{c_0,k}(x_0^*) \tag{2.5}$$

Since $c_0 \leq 0.01318 = \mathrm{c}^{'}$ we obtain from (2.4):

$$Q^{''}_{c_0,k}(x) > [1 - c_0 + 2c_0 g_k(r_0)]g^{''}_k(x) \geq [1 - c^{'} + 2c^{'} g_k(r_0)]g^{''}_k(x) > 0.95g^{''}_k(x) > 0.$$

Thus, $Q_{c_0,k}$ is non-negative and strictly convex on $(0, r_0)$, so that (2.5) gives $x_0 = x_0^*$.

Case 2. $c = 11.29416$ and $x \in (r_1, r_2)$. From Lemma 2.1, (1), (3) and Lemma 2.3 we get

$$Q^{''}_{c,k}(x) > (1 - c)g^{''}_k(x) + 2c(g^{'}_k(x))^2 = l_{c,k}(x),$$

say.

Lemma 2.1 implies that $l_{c,k}$ is strictly increasing on $[r_0, \infty)$. Hence, we get

$$Q^{''}_{c,k}(x) > l_{c,k}(x) > l_{c,k}(1.4616) = 38.58....$$

We set $x_1^* = 1.74747$ and $x_2^* = 1747471$. Then we have $Q^{'}_{c,k}(x_1^*) < 0 < Q^{'}_{c,k}(x_2^*)$. This implies that there exists a number $\mathrm{z}_0^* \in (x_1^*, x_2^*)$ such that $Q_{c,k}$ attains its absolute minimum at z_0^*. The strict convexity of $Q_{c,k}$ yields

$$Q_{c,k}(z_0^*) > Q_{c,k}(x_1^*) + (x_2^* - x_1^*)Q^{'}_{c,k}(x_1^*) = 0.0000016....$$

Hence, $Q_{c,k}$ is positive on (r_1, r_2). Applying Lemma 2.2 we get

$$f_k(x) > c = 11.29416 \quad (r_1 < x < r_2, k > 0)$$

Since $f_k(1.747471) = 11.294166...$, we obtain

$$\inf_{r_1<x<r_2} f_k(\mathrm{x}) = 11.29416....$$

Let $c_1 = \inf_{r_1<x<r_2} f_k(\mathrm{x})$. We have $f_k > 0$ on (r_1, r_2) and $f_k(r_1^+) = f_k(r_2^-) = \infty$. This implies that there exists a number $y_0 \in (r_1, r_2)$ with $f_k(y_0) = c_1$. We suppose that there exists a number $y_0^* \in (r_1, r_2)$ such that $y_0^* \neq y_0$ and $f_k(y_0^*) = c_1$. Then we obtain

$$Q_{c_1,k}(x) \geq 0 = Q_{c_1,k}(y_0^*) = Q_{c_1,k}(y_0) \quad (r_1 < x < r_2, k > 0) \tag{2.6}$$

We have

$$Q^{''}_{c_1,k}(x) > (1 - c_1)g^{''}_k(x) + 2c_1(g^{'}_k(x))^2 = l_{c_1,k}(x) > l_{c_1,k}(r^*),$$

where $r^* = 1.4616$. Since $(\partial/\partial t)l_{t,k}(r^*) = -g^{''}_k(r^*) + 2(g^{'}_k(r^*))^2 = 3.35...$, we conclude from $c_1 \geq c$ that $l_{c_1,k}(r^*) \geq l_{c,k}(r^*) = 38.5...$. Therefore, $Q_{c_1,k}$ is strictly convex and non-negative on (r_1, r_2). From (2.6) we get $y_0^* = y_0$. □

Lemma 2.5. *([5]) Let $u \in C^1(0, \infty)$ with $u(1) = 0$ and $v \in C^1(0, \infty)$ such that $v < 0$ on $(0, 1)$, $v > 0$ on $(1, \infty)$ and $v^{'} > 0$ on $(0, \infty)$. If $u^{'}/v^{'}$ is strictly increasing on $(0, \infty)$, then u/v is also strictly increasing on $(0, \infty)$.*

3. Main results

In the section, we denote by $r_0 = 0.21609...$ the only positive solution of $k\psi_k(kx) - \log(k) + xk^2\psi_k^{'}(kx) = 0$; $r_1 = 1.46163...$ is the only positive zero of ψ_k, and $r_2 = 2.08907...$ is the only positive solution of $kx\psi_k(kx) - x\log(k) = 1$. Our results read as follows.

Theorem 3.1. *Let* r *be a real number,* $k > 0$ *and let* $n \geq 2$ *be an integer. The inequality*

$$\frac{1}{k^{Mn^{[r]}(x_j,p_j)-1}}\Gamma_k(kMn^{[r]}(x_j,p_j)) \leq Mn^{[r]}(\frac{1}{k^{x_j-1}}\Gamma_k(kx_j),p_j) \tag{3.1}$$

holds for all positive real numbers x_j *and* $p_j (j = 1, ..., n)$ *with* $\sum_{j=1}^{n} p_j = 1$ *if and only if*

$$\alpha \leq r \leq \beta, \tag{3.2}$$

where

$$M_n^{[r]}(x_j,p_j) = (\sum_{j=1}^{n} p_j x_j^r)^{1/r} \quad (r \neq 0),$$

and

$$\alpha = \sup_{0<x<r_0} \frac{k\psi_k(kx) - \log(k) + xk^2\psi_k^{'}(kx)}{k\psi_k(kx) - \log(k) + x(k\psi_k(kx) - \log(k))^2}, \tag{3.3}$$

$$\beta = \inf_{r_1<x<r_2} \frac{k\psi_k(kx) - \log(k) + xk^2\psi_k^{'}(kx)}{k\psi_k(kx) - \log(k) + x(k\psi_k(kx) - \log(k))^2}. \tag{3.4}$$

Let $\alpha \leq r \leq \beta$*; then the sign of equality is valid in* (3.1) *if and only if* $x_1 = ... = x_n$.

Proof. First, we assume that $r \in [\alpha, \beta]$, where α and β are given in (3.3) and (3.4), respectively. In order to prove (3.1) we may suppose that

$$0 < x_n \leq x_{n-1} \leq ... \leq x_2 \leq x_1, \quad x_n < x_1. \tag{3.5}$$

We define

$$F_k(x_1,...,x_n) = \sum_{j=1}^{n} p_j(\frac{1}{k^{x_j-1}}\Gamma_k(kx_j))^r - [\frac{1}{k^{Mn^{[r]}(x_j,p_j)-1}}\Gamma_k(k\sum_{j=1}^{n} p_j x_j^r)^{1/r})]^r$$

and

$$F_{k,q}(x) = F_k(x,...,x,x_{q+1},...,x_n) \quad (q \in \{1,...,n-1\})$$

In what follows, we establish that $F_{k,q}$ is strictly increasing on $[x_{q+1}, \infty)$. Since $F_{k,q-1}(x_q) = F_{k,q}(x_q)$ for $q = 2, ..., n-1$, we obtain from (3.5) that

$$\begin{aligned} F_k(x_1,...,x_n) = F_{k,1}(1) \geq F_{k,1}(2) = F_{k,2}(2) \geq F_{k,2}(3) \\ \geq ... \geq F_{k,n-1}(x_{n-1}) \geq F_{k,n-1}(x_n) = F_k(x_n,...,x_n) = 0 \end{aligned} \tag{3.6}$$

Moreover, since $F_{k,q}$ is strictly monotonic, we conclude from $x_1 > x_n$ that at least one of the inequalities in (3.6) is strict. Hence, we get (3.1) with "<" instead of "≤".

It remains to prove that $F_{k,q}^{'}(x) > 0$ for $x > x_{q+1}$. Let

$$G_k(x) = x^{1-r}(\frac{\Gamma_k(kx)}{k^{x-1}})^{r-1}\frac{k\Gamma_k^{'}(kx) - \ln k\Gamma_k(kx)}{k^{x-1}}$$

and

$$y=(x^r\sum_{j=1}^{p}p_j+\sum_{j=p+1}^{n}p_jx_j^r)^{\frac{1}{r}}.$$

Then we obtain

$$\frac{1}{r}(\sum_{j=1}^{p}p_j)^{-1}x^{1-r}F_{k,q}^{'}(x)=G_k(x)-G_k(y). \tag{3.7}$$

Differentiation gives

$$x^r(\frac{\Gamma_k(kx)}{k^{x-1}})^{-r}G_k^{'}(x)=u_k(x)-rv_k(x) \tag{3.8}$$

where

$$u_k(x)=k\psi_k(kx)-\log(k)+xk^2\psi_k^{'}(kx)$$

and

$$v_k(x)=k\psi_k(kx)-\log(k)-x(k\psi_k(kx)-\log(k))^2.$$

Let f_k, x_0 and y_0 be as in Lemma 2.4. We consider three cases.

Case 1. $0<x<x_0$ or $x_0<x<r_0$. Applying Lemma 2.2 and Lemma 2.4 we get

$$v_k(x)<0 \quad and \quad r\geq\sup_{0<t<r_0}f_k(t)>f_k(x)=\tfrac{u_k(x)}{v_k(x)} \tag{3.9}$$

so that (3.9) implies $u_k(x)-rv_k(x)>0$.

Case 2. $r_0<x<r_1$ or $x>r_2$. From Lemma 2.1, (2) and Lemma 2.2 we conclude that

$$v_k(x)<u_k(x)=\frac{d}{dx}(x(k\psi_k(kx)-\log(k))).$$

Hence, we have $u_k(x)-rv_k(x)>0$.

Case 3. $r_1<x<y_0$ or $y_0<x<r_2$. Lemma 2.2 and Lemma 2.4 give

$$v_k(x)>0 \quad and \quad r\leq\inf_{r_1<t<r_2}f_k(t)<f_k(x)=\tfrac{u_k(x)}{v_k(x)},$$

so that we obtain again $u_k(x)-rv_k(x)>0$.

Thus, we conclude from (3.8) that $G_k^{'}(x)>0$ for $x\in(0,\infty)-\{x_0,r_0,r_1,y_0,r_2\}$, which implies that G_k is strictly increasing on $(0,\infty)$. Since $0<x_n\leq x_{n-1}\leq...\leq x_{q+1}<x$, we have

$$y=(x^r\sum_{j=1}^{q}p_j+\sum_{j=q+1}^{n}p_jx_j^r)^{\frac{1}{r}}<x \quad and \quad G_k(y)<G_k(x),$$

so that (3.7) leads to $F_{k,q}^{'}(x)>0$.

Now, we assume that (3.1) is valid for all $x_j>0$ and weights $p_j(j=1,...,n)$. We conclude that $r\neq 0$,(see [1, p.2]). We set $x_1=x$ and $x_2=...=x_n=y$. Then we obtain

$$\begin{aligned}&k^{1-[p_1x^r+(1-p_1)y^r)]^{1/r}}\Gamma_k(k(p_1x^r+(1-p_1)y^r)^{1/r})\\&\leq[p_1(k^{1-x}\Gamma_k(kx))^r+(1-p_1)(k^{1-y}\Gamma_k(ky))^r]^{1/r}\end{aligned} \tag{3.10}$$

Let $r<0$. If x tends to 0, then (3.10) yields $\infty\leq(1-p_1)^rk^{1-y}\Gamma_k(ky)$. Thus, we have $r>0$. We define for $k,x,y>0$:

$$\begin{aligned}H_k(x,y)&=p_1(k^{1-x}\Gamma_k(kx))^r+(1-p_1)(k^{1-y}\Gamma_k(ky))^r\\&-[k^{1-[p_1x^r+(1-p_1)y^r)]^{1/r}}\Gamma_k(k(p_1x^r+(1-p_1)y^r)^{1/r}]^r.\end{aligned}$$

From (3.10) we get $H_k(x,y) \geq 0 = H_k(y,y)$. Since $\frac{\partial}{\partial x}H_k(x,y)\Big|_{x=y} = 0$, we obtain

$$0 \leq \frac{\partial^2 H_k(x,y)}{\partial x^2}\Bigg|_{x=y} = (1-p_1)p_1 r y^{-1}(k^{1-y}\Gamma_k(ky))^r[u_k(y) - rv_k(y)] \tag{3.11}$$

We consider two cases.

Case 1. $y = x_0$. Then (3.11) implies

$$0 \leq u_k(x_0) - rv_k(x_0) = v_k(x_0)(\sup_{0<x<r_0} f_k(x) - r)$$

Lemma 2.2 yields $v_k(x_0) < 0$, so that we get $\sup\limits_{0<x<r_0} f_k(x) \leq r$.

Case 2. $y = y_0$. Then we conclude from (3.11) that

$$0 \leq u_k(y_0) - rv_k(y_0) = v_k(y_0)(\inf_{r_1<x<r_2} f_k(x) - r)$$

Since $v_k(y_0) > 0$, we obtain $r \leq \inf\limits_{r_1<x<r_2} f_k(x)$. This completes the proof of the Theorem. □

Theorem 3.2. *The function* $f_k(x) = \log(\Gamma_k(kx+k)/k^x)/(x\log(x))$ *is strictly increasing on* $(0,\infty)$.

Proof. We define for $x > 0$ and $k > 0$:

$$u_k(x) = \tfrac{1}{x}\log(\Gamma_k(kx+k)/k^x) \quad \textit{and} \quad v(x) = \log(x).$$

Moreover, let

$$w_k(x) = x^2\left(\frac{u_k'(x)}{v'(x)}\right)' = k^2x^2\psi_k'(kx+k) - kx\psi_k(kx+k) + \log(\Gamma_k(kx+k))$$

Using the integral representations

$$\psi_k'(z) = \textstyle\int_0^\infty e^{-zt}\frac{t}{1-e^{-kt}}dt, \quad \psi_k''(z) = -\int_0^\infty e^{-zt}\frac{t^2}{1-e^{-kt}}dt$$

and

$$\frac{1}{z} = \int_0^\infty e^{-zt}dt$$

$z > 0, k > 0$, and the convolution theorem for Laplace transforms, we obtain for $x > 0$ and $k > 0$:

$$\frac{1}{k^3x^2}w_k'(x) = \frac{1}{kx}\psi_k'(kx+k) + \psi_k''(kx+k) = \int_0^\infty e^{-kxt}h_k(t)dt$$

where

$$h_k(t) = \int_0^\infty \frac{s}{e^{ks}-1} - \frac{t}{e^{kt}-1}ds$$

Since $x \mapsto x/(e^{kx}-1)$ is strictly decreasing on $(0,\infty)$, we get $h_k(t) > 0 (t > 0, k > 0)$, and, hence, $w_k'(x) > 0$ and $w_k(x) > w_k(0) = 0 (x > 0, k > 0)$. This implies that $\frac{u_k'}{v'}$ is strictly increasing on $(0,\infty)$. From the Lemma 2.5 we conclude that $f_k = \frac{u_k}{v}$ is also strictly increasing on $(0,\infty)$. □

Theorem 3.3. *Let $n \geq 0$ be an integer and let $s \in (0,1), k > 0$ be a real number. Then we have for all real numbers $x > 0$:*

$$\frac{n!(1-s)}{[x+\alpha_{n,k}(s)]^{n+1}} < (-1)^n[k^{n+1}\psi_k^{(n)}(kx+k) - k^{n+1}\psi_k^{(n)}(kx+ks)] < \frac{n!(1-s)}{[x+\beta_{n,k}(s)]^{n+1}} \tag{3.12}$$

with the best possible constants

$$\alpha_{n,k}(s) = \left(\frac{n!(1-s)}{(-1)^n[k^{n+1}\psi_k^{(n)}(k) - k^{n+1}\psi_k^{(n)}(ks)]}\right)^{1/(n+1)} \quad \textit{and} \quad \beta_{n,k}(s) = \frac{s}{2}. \tag{3.13}$$

Proof. Let $s \in (0,1)$ be a (fixed) real number. We denote by $f_{n,k}$ the function

$$f_{n,k}(x) = [\frac{\Delta_{n,k}(x)}{n!(1-s)}]^{-1/(n+1)} - x$$

where

$$\Delta_{n,k}(x) = (-1)^n[k^{n+1}\psi_k^{(n)}(kx+k) - k^{n+1}\psi_k^{(n)}(kx+ks)]$$

We shall prove that

$$\lim_{x\to\infty} f_{n,k}(x) = \frac{s}{2} \tag{3.14}$$

and that $f_{n,k}$ is strictly decreasing on $(0,\infty)$. This implies

$$\frac{s}{2} < f_{n,k}(x) < f_{n,k}(0)$$

($x > 0$), which is equivalent to double-inequality (3.12) with $\alpha_{n,k}(s)$ and $\beta_{n,k}(s)$ given in (3.13) Moreover, we conclude that these constants are best possible.

From the asymptotic formula

$$k\psi_k(kx) = \log(kx) - \frac{1}{2x} - \frac{1}{12x^2} + O(x^{-4})$$

($x \to \infty$), we get

$$\psi_k(kx+k) - \psi_k(kx+ks) = \frac{1-s}{x} - \frac{s(1-s)}{2(x+s)(x+1)} + O(x^{-3}) \tag{3.15}$$

This leads to

$$f_{0,k}(x) = \frac{\frac{1}{2}sx^2(x+s)^{-1}(x+1)^{-1} + O(x^{-1})}{1+O(x^{-1})}$$

which implies (3.14) for $n = 0$. Let $n \geq 1$; from

$$k^{n+1}\psi_k^{(n)}(kx) = (-1)^{n-1}[(n-1)!x^{-n} + \frac{1}{2}n!x^{-n-1} + \frac{1}{12}(n+1)!x^{-n-2} + O(x^{-n-3})]$$

($x \to \infty$), we obtain

$$x^{n+1}\frac{\Delta_{n,k}(x)}{n!(1-s)} = \frac{1+\frac{1}{2}(n-1)(1+s)\frac{1}{x} + O(x^{-2})}{1+n(1+s)\frac{1}{x}+O(x^{-2})} + \frac{\frac{1}{2}(n+1)\frac{1}{x}+O(x^{-2})}{1+(n+1)(1+s)\frac{1}{x}+O(x^{-2})} + O(x^{-2}) \tag{3.16}$$

This implies

$$f_{n,k}(x) = \frac{(1-\frac{s}{2}(n+1)\frac{1}{x} + O(x^{-2}))^{-1/n+1} - 1}{1/x} \tag{3.17}$$

From (3.17) we conclude $\lim_{x\to\infty} f_{n,k}(x) = \frac{s}{2}$, which proves (3.14) for $n \geq 1$. It remains to establish that

$$(\Delta_{n,k}(x))^{\frac{1}{n+1}+1} f'_{n,k}(x) = \frac{1}{n+1}[n!(1-s)]^{\frac{1}{n+1}} \Delta_{n+1,k}(x) - (\Delta_{n,k}(x))^{\frac{1}{n+1}+1} < 0 \tag{3.18}$$

To prove (3.18) for $x > 0, k > 0$ it suffices to show that the function

$$g_{n,k}(x) = -\log(n!(1-s)) + (n+1)\log(n+1) - (n+1)\log(\Delta_{n+1,k}(x)) + (n+2)\log(\Delta_{n,k}(x))$$

is positive on (0,∞). From (3.15) and (3.16) we get for n ≥ 0 :

$$\lim_{x\to\infty} x^{n+1}\Delta_{n,k}(x) = n!(1-s), \tag{3.19}$$

which implies $\lim_{x\to\infty} g_{n,k}(x) = 0$. Therefore, it is enough to establish that $g_{n,k}$ is strictly decreasing on (0,∞). The inequality $g'_{n,k}(x) < 0$ is equivalent to

$$(n+2)(\Delta_{n+1,k}(x))^2 > (n+1)\Delta_{n+2,k}(x)\Delta_{n,k}(x) \tag{3.20}$$

We set

$$u_k(t) = \frac{e^{-kst} - e^{-kt}}{1 - e^{-kt}}$$

(t > 0, k > 0) and make use of the integral representation

$$\psi_k^{(m)}(z) = (-1)^{m+1} \int_0^\infty \frac{1}{1-e^{-kt}} t^m e^{-zt} dt$$

(z > 0, k > 0). Then we get

$$\begin{aligned}(\Delta_{n+1,k}(x))^2 &= (k^{n+2}\textstyle\int_0^\infty e^{-kxt} t^{n+1} u_k(t)dt)^2 \\ &= k^{2n+4}\textstyle\int_0^\infty e^{-kxt}(t^{n+1}u_k(t)) * (t^{n+1}u_k(t))dt\end{aligned}$$

where * denotes Laplace convolution. Moreover, we obtain

$$\begin{aligned}\Delta_{n+2,k}(x)\Delta_{n,k}(x) &= k^{2n+4}\textstyle\int_0^\infty e^{-kxt} t^{n+2} u_k(t)dt \int_0^\infty e^{-kxt} t^n u_k(t)dt \\ &= k^{2n+4}\textstyle\int_0^\infty e^{-kxt}(t^{n+2}u_k(t)) * (t^n u_k(t))dt\end{aligned}$$

Thus, to prove (3.20) it suffices to show that the following inequality holds for (t > 0, k > 0):

$$\begin{aligned}&(n+2)(t^{n+1}u_k(t)) * (t^{n+1}u_k(t)) - (n+1)(t^{n+2}u_k(t)) * (t^n u_k(t)) \\ &= \textstyle\int_0^t u_k(t-x)u_k(x)(t-x)^n x^{n+1}[t(n+2) - (2n+3)x]dx > 0\end{aligned} \tag{3.21}$$

We denote the integral in (3.21) by $I_k(t)$ and we set $P_{a,k}(y) = u_k(a(1-y))u_k(a(1+y))$.

Next, we change the variable, $x = \frac{t}{2}(1+y)$, and take into account that $y \mapsto P_{\frac{t}{2},k}(y)(1-y^2)^n y$ is an odd function. Then we get

$$\begin{aligned}I_k(t) &= (\tfrac{t}{2})^{2n+3}\textstyle\int_{-1}^1 P_{\frac{t}{2},k}(y)(1-y^2)^n[1 - 2(n+1)y - (2n+3)y^2]dy \\ &= 2(\tfrac{t}{2})^{2n+3}\textstyle\int_0^1 P_{\frac{t}{2},k}(y)(1-y^2)^n[1-(2n+3)y^2]dy\end{aligned}$$

We shall prove that $y \mapsto P_{a,k}(y)(a > 0, k > 0)$ is strictly decreasing on $(0,1)$. We set $c = (2n+3)^{-\frac{1}{2}}$; then we obtain

$$\begin{aligned} I_k(t)4^{n+1}t^{-(2n+3)} &= \int_0^c P_{\frac{t}{2},k}(y)(1-y^2)^n[1-(\tfrac{y}{c})^2]dy \\ &+ \int_c^1 P_{\frac{t}{2},k}(y)(1-y^2)^n[1-(\tfrac{y}{c})^2]dy \\ &> P_{\frac{t}{2},k}(c)[\int_0^c (1-y^2)^n[1-(\tfrac{y}{c})^2]dy \\ &+ \int_c^1 (1-y^2)^n[1-(\tfrac{y}{c})^2]dy] \\ &= P_{\frac{t}{2},k}(c)[\int_0^1 (1-y^2)^n[1-(2n+3)y^2]dy \\ &= P_{\frac{t}{2},k}(c)[y(1-y^2)^{n+1}]|_0^1 = 0 \end{aligned}$$

It remains to prove that

$$P'_{a,k}(y) < 0 \tag{3.22}$$

$(y \in (0,1))$. We set

$$Q_{a,k}(x) = \log(u_k(ax));$$

then we have

$$P'_{a,k}(y) = P_{a,k}(y)[-Q'_{a,k}(1-y) + Q'_{a,k}(1+y)]$$

Hence, to establish (3.22) it suffices to show that $x \mapsto Q_{a,k}(x)$ is strictly concave on $(0,\infty)$. Elementary calculations reveal that the inequality

$$Q''_{a,k}(x) = (a/u_k(ax))^2[u_k(ax)u''_k(ax) - (u'_k(ax))^2] < 0$$

is equivalent to

$$0 < b^2z^2 - z^{1+b} - 2(b^2-1)z - z^{1-b} + b^2 = R_b(z) \tag{3.23}$$

say, where $z > 1$ and $b \in (0,1)$. From

$$R_b(1) = R'_b(1) = R''_b(1) = 0$$

and

$$R'''_b(z) = b(1-b^2)z^{-b-2}(z^{2b}-1) > 0$$

we conclude the validity of inequality (3.23). This completes the proof of Theorem 3.3. □

Theorem 3.4. *For all integers $n \geq 1$, we have*

$$\frac{1}{2(n+a)} \leq d_{n,k} - C < \frac{1}{2(n+b)}, \tag{3.24}$$

with the best possible constants

$$a = \tfrac{1}{2(1-C)} - 1 = 0.1826... \quad \text{and} \quad b = \tfrac{1}{6}$$

where $d_{n,k} = k\psi_k(kn+k) - \log(kn) + C$

Proof. Since

$$d_{n,k} - C = k\psi_k(kn+k) - \log(kn)$$

double-inequality (3.24) can be written as

$$b < \frac{1}{2}\frac{1}{k\psi_k(kn+k) - \log(kn)} - n \leq a \tag{3.25}$$

In order to prove (3.25) we define for positive real x:

$$f_k(x) = \frac{1}{2}\frac{1}{k\psi_k(kx+k) - \log(kx)} - x$$

Differentiation yields

$$\begin{aligned} f_k^{'}(x)[k\psi_k(kx+k) - \log(kx)]^2 &= \tfrac{1}{2}[\tfrac{1}{x} - k^2\psi_k^{'}(kx+k)] - [k\psi_k(kx+k) - \log(kx)]^2 \\ &= \tfrac{1}{2}[\tfrac{1}{x} + \tfrac{1}{x^2} - k^2\psi_k^{'}(kx)] - [k\psi_k(kx) + \tfrac{1}{x} - \log(kx)]^2 \end{aligned}.$$

Using the inequalities

$$k\psi_k(kx) > \log(kx) - \frac{1}{2x} - \frac{1}{12x^2}$$

and

$$k^2\psi_k^{'}(kx) > \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5}$$

$(x > 0, k > 0)$, we obtain for $x \geq 2.4$:

$$f_k^{'}(x)[k\psi_k(kx+k) - \log(kx)]^2 < \frac{1}{144x^5}(2.4 - x) \leq 0 \tag{3.26}$$

From (3.26) and $f_k(1) = 0.182...$, $f_k(2) = 0.177...$, $f_k(3) = 0.174...$, we conclude that the sequence $f_k(n) = \frac{1}{2(d_{n,k}-C)} - n(n = 1, 2, ...)$ is strictly decreasing. This leads to

$$\lim_{m\to\infty} f_k(m) < f_k(n) \leq f_k(1) = \frac{1}{2(1-C)} - 1$$

$(n = 1, 2, ...)$. It remains to prove that

$$\lim_{m\to\infty} f_k(m) = \frac{1}{6} \tag{3.27}$$

From the representation

$$k\psi_k(kx) = \log(kx) - \frac{1}{2x} - \frac{1}{12x^2} + O(x^{-4})$$

$(x \to \infty)$, we get

$$f_k(x) = (\frac{1}{6} + O(x^{-2}))/(1 + O(x^{-1}))$$

which implies (3.27). This completes the proof of Theorem 3.4. □

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