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TWO SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH POISSON DISTRIBUTION

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Abstract. In the present paper, we determine necessary and sufficient conditions for two subclasses of analytic functions with negative coefficients. Further, we consider an integral operator related to Poisson distribution series.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of the normalized functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, let $\mathcal{T}$ be a subclass of $\mathcal{A}$ consisting of functions of the form,

$$f(z) = z - \sum_{n=2}^{\infty} |a_n|\, z^n, \qquad z \in \mathbb{U}. \tag{1.2}$$

For some $0 \le \alpha < 1$ and $\beta_j \geqslant 0,\ j = 1, 2, \cdots, k$, and functions of the form (1.2), we let $\mathcal{H}(\beta_1, \beta_1, \ldots, \beta_k; \alpha)$ be the subclass of $\mathcal{A}$ satisfying the analytic criteria

$$\Re\left\{\frac{f(z)}{z} + \beta_1 z(\frac{f(z)}{z})' + \beta_2 z^2(\frac{f(z)}{z})'' + \cdots + \beta_k z^k(\frac{f(z)}{z})^{(k)}\right\} > \alpha \qquad (z \in \mathbb{U}), \tag{1.3}$$

and also, let $\mathcal{G}(\beta_1, \beta_1, \ldots, \beta_k; \alpha)$ be the subclass of $\mathcal{A}$ satisfying the analytic criteria

$$\Re\left\{f'(z) + \beta_1 z f''(z) + \beta_2 z^2 f'''(z) + \cdots + \beta_k z^k f^{(k+1)}(z)\right\} > \alpha \qquad (z \in \mathbb{U}). \tag{1.4}$$

Also denote $\mathcal{H}_{\mathcal{T}}(\beta_1, \beta_1, \ldots, \beta_k; \alpha) = \mathcal{H}(\beta_1, \beta_1, \ldots, \beta_k; \alpha) \cap \mathcal{T}$ and $\mathcal{G}_{\mathcal{T}}(\beta_1, \beta_1, \ldots, \beta_k; \alpha) = \mathcal{G}(\beta_1, \beta_1, \ldots, \beta_k; \alpha) \cap \mathcal{T}$ the subclasses of $\mathcal{T}$.

The classes $\mathcal{H}(\beta_1, \beta_1, \ldots, \beta_k; \alpha)$ and $\mathcal{G}(\beta_1, \beta_1, \ldots, \beta_k; \alpha)$ were introduced by Frasin [7]. In particular, the class $\mathcal{H}(0, 0, \ldots, 0; \alpha) = \mathcal{B}(\alpha)$ was studied by Chen [2,3] and Goal [11], and

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the class $\mathcal{G}(0,0,\ldots,0;\alpha)=\mathcal{C}(\alpha)$ was studied by Sarangi and Uralegaddi [23], Owa and Uralegaddi [21], and Srivastava and Owa [24] (see also, [6]).

A function $f\in\mathcal{A}$ is said to be in the class $\mathcal{R}^{\tau}(A,B)$,$\tau\in\mathbb{C}\backslash\{0\}$, $-1\leq B<A\leq 1$, if it satisfies the inequality

$$\left|\frac{f'(z)-1}{(A-B)\tau-B[f'(z)-1]}\right|<1,\qquad z\in\mathbb{U}.$$

This class was introduced by Dixit and Pal [5].

The Poisson distribution, derived in 1837 by a French mathematician Siméon Denis Poisson, is a discrete probability distribution that is used to express the probability of observing a number of events in a given interval of time or space if these events occur with a known average rate and independently of the time since the last event.

A variable X is said to be Poisson distributed if it takes the values $0,1,2,3,\cdots$ with probabilities e^{-m}, $m\frac{e^{-m}}{1!}$, $m^2\frac{e^{-m}}{2!}$, $m^3\frac{e^{-m}}{3!}$, $\cdots$ respectively, where m is called the parameter. Thus

$$P(X=r)=\frac{m^r e^{-m}}{r!},\ r=0,1,2,3,\cdots.$$

In [18], Porwal (see also, [15, 17]) introduced a power series whose coefficients are probabilities of Poisson distribution

$$\mathcal{K}(m,z)=z+\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}z^n,\qquad z\in\mathbb{U},$$

where $m>0$. By ratio test the radius of convergence of above series is infinity. In [18], Porwal also defined the series

$$\mathcal{F}(m,z)=2z-\mathcal{K}(m,z)=z-\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}z^n,\qquad z\in\mathbb{U}.$$

Using the Hadamard product, Porwal and Kumar [20] introduced a new linear operator $\mathcal{I}(m,z):\mathcal{A}\to\mathcal{A}$ defined by

$$\mathcal{I}(m,z)f=\mathcal{K}(m,z)*f(z)=z+\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}a_nz^n,\qquad z\in\mathbb{U},$$

where $*$ denote the convolution or Hadamard product of two series.

Motivated by several earlier results on connections between various subclasses of analytic and univalent functions by using hypergeometric functions (see for example, [4, 10, 14, 22, 25]) and by the recent investigations (see for example, [1, 8, 9], [15]-[20]), we determine the necessary and sufficient condition for $\mathcal{F}(m,z)$ to be in the class $\mathcal{H}_{\mathcal{T}}(\beta_1,\beta_1,\ldots,\beta_k;\alpha)$ and for $\mathcal{I}(m,z)f$ to be in the class $\mathcal{G}_{\mathcal{T}}(\beta_1,\beta_1,\ldots,\beta_k;\alpha)$ where $f\in\mathcal{R}^{\tau}(A,B)$. Finally, we give condition for the integral operator $\mathcal{G}(m,z)=\int_0^z\frac{\mathcal{F}(m,t)}{t}dt$ to be in the class $\mathcal{G}_{\mathcal{T}}(\beta_1,\beta_1,\ldots,\beta_k;\alpha)$.

Unless otherwise mentioned, we shall assume in this paper that $0 \leq \alpha < 1$ and $\beta_j \geqslant 0,\ j = 1, 2, \cdots, k$.

2. Preliminary lemmas

To establish our main results, we need the following Lemmas.

Lemma 2.1. [7] *A function $f \in \mathcal{T}$ of the form (1.2) is in the class $\mathcal{H}_{\mathcal{T}}(\beta_1, \beta_1, \ldots, \beta_k; \alpha)$ if and only if*

$$\sum_{n=2}^{\infty} [1 + \beta_1(n-1) + \beta_2(n-1)(n-2) + \cdots + \beta_k(n-1)(n-2)\cdots(n-k)]\, |a_n| \leq 1 - \alpha\ . \quad (2.1)$$

The result (2.1) is sharp.

Lemma 2.2. [7] *A function $f \in \mathcal{T}$ of the form (1.2) is in the class $\mathcal{G}_{\mathcal{T}}(\beta_1, \beta_1, \ldots, \beta_k; \alpha)$ if and only if*

$$\sum_{n=2}^{\infty} n[1 + \beta_1(n-1) + \beta_2(n-1)(n-2) + \cdots + \beta_k(n-1)(n-2)\cdots(n-k)]\, |a_n| \leq 1 - \alpha\ . \quad (2.2)$$

The result (2.2) is sharp.

Lemma 2.3. [5] *If $f \in \mathcal{R}^{\tau}(A, B)$ is of the form (1.1), then*

$$|a_n| \leq (A - B)\frac{|\tau|}{n}, \qquad n \in \mathbb{N} - \{1\}.$$

The result is sharp for the function

$$f(z) = \int_0^z (1 + (A - B)\frac{\tau t^{n-1}}{1 + Bt^{n-1}})dt, \qquad (z \in \mathbb{U}; n \in \mathbb{N} - \{1\}).$$

3. The necessary and sufficient condition

In this section, we obtain the necessary and sufficient conditions for $\mathcal{F}(m, z)$ to be in $\mathcal{H}_{\mathcal{T}}(\beta_1, \beta_1, \ldots, \beta_k; \alpha)$.

Theorem 3.1. *If $m > 0$, then $\mathcal{F}(m, z)$ is in $\mathcal{H}_{\mathcal{T}}(\beta_1, \beta_1, \ldots, \beta_k; \alpha)$ if and only if*

$$\sum_{j=1}^{k} \beta_j m^j \leq e^{-m} - \alpha. \quad (3.1)$$

Proof. Since

$$\mathcal{F}(m, z) = z - \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n \quad (3.2)$$

in view of Lemma 2.1, it suffices to show that

$$\sum_{n=2}^{\infty} [1 + \beta_1(n-1) + \beta_2(n-1)(n-2) + \cdots + \beta_k(n-1)(n-2)\cdots(n-k)] \frac{m^{n-1}}{(n-1)!} e^{-m}$$
$$\leq \quad 1 - \alpha. \quad (3.3)$$

Making use of the facts that

$$\sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} = e^m - 1 \tag{3.4}$$

and

$$\sum_{n=s}^{\infty} \frac{m^{n-1}}{(n-s)!} = m^{s-1}e^m, \;\; s \geq 2, \tag{3.5}$$

we have

$$\begin{aligned} &\sum_{n=2}^{\infty}[1+\beta_1(n-1)+\beta_2(n-1)(n-2)+\cdots+\beta_k(n-1)(n-2)\cdots(n-k)]\frac{m^{n-1}}{(n-1)!}e^{-m} \\ = \;& e^{-m}\left[\beta_1\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-2)!}+\beta_2\sum_{n=3}^{\infty}\frac{m^{n-1}}{(n-3)!}+\cdots+\beta_k\sum_{n=k+1}^{\infty}\frac{m^{n-1}}{(n-(k+1))!}+\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}\right] \\ = \;& \beta_1 m \; + \beta_2 m^2 + \cdots + \beta_k \; m^k + \; 1 - e^{-m}. \end{aligned}$$

But this last expression is bounded above by $1-\alpha$ if and only if (3.1) holds. □

4. Inclusion Properties

Making use of Lemma 2.3, we will study the action of the Poisson distribution series on the class $\mathcal{G}_{\mathcal{T}}(\beta_1, \beta_1, \ldots, \beta_k; \alpha)$.

Theorem 4.1. *If* $f \in \mathcal{R}^{\tau}(A,B)$, *then* $\mathcal{I}(m,z)f$ *is in* $\mathcal{G}_{\mathcal{T}}(\beta_1, \beta_1, \ldots, \beta_k; \alpha)$ *if*

$$(A-B)\left|\tau\right| \left[\sum_{j=1}^{k} \beta_j m^j + \; 1 - e^{-m}\right] \leq 1-\alpha. \tag{4.1}$$

Proof. In view of Lemma 2.2, it suffices to show that

$$\begin{aligned} &\sum_{n=2}^{\infty} n[1+\beta_1(n-1)+\beta_2(n-1)(n-2)+\cdots+\beta_k(n-1)(n-2)\cdots(n-k)]\frac{m^{n-1}}{(n-1)!}e^{-m}\left|a_n\right| \\ \leq \;& 1-\alpha. \end{aligned}$$

Since $f \in \mathcal{R}^{\tau}(A,B)$, then by Lemma 2.3, we get

$$\left|a_n\right| \leq \frac{(A-B)\left|\tau\right|}{n}. \tag{4.2}$$

Thus, we have

$$\begin{aligned} &\sum_{n=2}^{\infty} n[1+\beta_1(n-1)+\beta_2(n-1)(n-2)+\cdots+\beta_k(n-1)(n-2)\cdots(n-k)]\tfrac{m^{n-1}}{(n-1)!}e^{-m}\left|a_n\right| \\ &\leq (A-B)\left|\tau\right|\Big[\sum_{n=2}^{\infty}[1+\beta_1(n-1)+\beta_2(n-1)(n-2)+\cdots+\beta_k(n-1)(n-2)\cdots(n-k)] \\ &\times\left(\tfrac{m^{n-1}}{(n-1)!}e^{-m}\right)\Big] \\ &= (A-B)\left|\tau\right| e^{-m}\left[\beta_1\sum_{n=2}^{\infty}\tfrac{m^{n-1}}{(n-2)!}+\beta_2\sum_{n=3}^{\infty}\tfrac{m^{n-1}}{(n-3)!}+\cdots+\beta_k\sum_{n=k+1}^{\infty}\tfrac{m^{n-1}}{(n-(k+1))!}+\sum_{n=2}^{\infty}\tfrac{m^{n-1}}{(n-1)!}\right] \\ &= (A-B)\left|\tau\right|\left[\beta_1 m \; + \beta_2 m^2 + \cdots + \beta_k \; m^k + \; 1 - e^{-m}\right]. \end{aligned}$$

But this last expression is bounded by $1-\alpha$, if (4.1) holds. This completes the proof of Theorem 4.1. □

5. An integral operator

Theorem 5.1. *If* $m > 0$, *then the integral operator*

$$\mathcal{G}(m,z) = \int_0^z \frac{\mathcal{F}(m,t)}{t} dt \tag{5.1}$$

is in $\mathcal{G}_{\mathcal{T}}(\beta_1, \beta_1, \ldots, \beta_k; \alpha)$ *if and only if inequality* (3.1) *is satisfied.*

Proof. Since

$$\mathcal{G}(m,z) = z - \sum_{n=2}^{\infty} \frac{e^{-m} m^{n-1}}{n!} z^n,$$

then by Lemma 2.2, we need only to show that

$$\sum_{n=2}^{\infty} n[1 + \beta_1(n-1) + \beta_2(n-1)(n-2) + \cdots + \beta_k(n-1)(n-2)\cdots(n-k)] \frac{m^{n-1}}{n!} e^{-m}$$
$$\leq \quad 1 - \alpha,$$

or, equivalently

$$\sum_{n=2}^{\infty} [1 + \beta_1(n-1) + \beta_2(n-1)(n-2) + \cdots + \beta_k(n-1)(n-2)\cdots(n-k)] \frac{m^{n-1}}{(n-1)!} e^{-m}$$
$$\leq \quad 1 - \alpha.$$

The remaining part of the proof of Theorem 5.1 is similar to that of Theorem 3.1, and so we omit the details. □

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References

[1] R. M. El-Ashwah and W. Y Kota, *Some condition on a Poisson distribution series to be in subclasses of univalent functions*, Acta Universitatis Apulensis, **51** (2017), 89–103.

[2] M.-P. Chen, *On the regular functions satisfying* $\text{Re}\{f(z)/z\} > \alpha$, Bull. Inst. Math. Acad. Sinica, **3** (1975), 65–70.

[3] M. -P. Chen, *On functions satisfying* $\text{Re}\{f(z)/z\} > \alpha$, Tamkang J. Math., **5** (1974), 231–234.

[4] N. E. Cho, S. Y. Woo and S. Owa, *Uniform convexity properties for hypergeometric functions*, Fract. Cal. Appl. Anal., **5**(3) (2002), 303–313.

[5] K.K. Dixit and S.K. Pal, *On a class of univalent functions related to complex order*, Indian J. Pure Appl. Math., **26**(9) (1995), 889–896.

[6] B.A. Frasin, *Some properties of certain analytic and univalent functions*, Tamsui Oxford J. Math., **23**(1) (2007), 67–77.

[7] B.A. Frasin, *An application of generalized Bessel functions on two subclasses of analytic functions*, submitted.

[8] B.A. Frasin, *On certain subclasses of analytic functions associated with Poisson distribution series*, Acta Univ. Sapientiae, Mathematica, **11**(1) (2019), 78–86.

[9] B. A. Frasin and Ibtisam Aldawish, *On subclasses of uniformly spirallike functions associated with generalized Bessel functions*, Journal of Function Spaces, **2019**, Article ID 1329462, 6 pages.

[10] B.A. Frasin, Tariq Al-Hawary and Feras Yousef, *Necessary and sufficient conditions for hypergeometric functions to be in a subclass of analytic functions*, Afrika Matematika, **30**(1–2) (2019), 223–230.

[11] R. M. Goel, *On functions satisfying* $\text{Re}\{f(z)/z\} > \alpha$, Pub. Math. Debrecen, **18** (1971), 111–117.

[12] R. M. Goel, *The radius of convexity and starlikeness for certain classes of analytic functions with fixed second coefficient*, Ann. Unit. Mariae Curie-Sklodowska &cl. A., **25** (1971), 33–39.

[13] T. H. Macgregor, *Functions whose derivative has a positive real part*, Trans. Amer. Math. Sot., **104** (1962), 532–537.

[14] E. Merkes and B. T. Scott, *Starlike hypergeometric functions*, Proc. Amer. Math. Soc., **12** (1961), 885–888.

[15] G. Murugusundaramoorthy, *Subclasses of starlike and convex functions involving Poisson distribution series* , Afr. Mat., **28** (2017), 1357–1366.

[16] G. Murugusundaramoorthy, *Parabolic starlike and uniformly convex functions associated with Poisson distribution series*, Bulletin of the Transilvania University of Braşov, **10**(59), No. 2 - 2017.

[17] G. Murugusundaramoorthy, K. Vijaya and S. Porwal, *Some inclusion results of certain subclass of analytic functions associated with Poisson distribution series*, Hacettepe J. Math. Stat., **45**(4) (2016), 1101–1107.

[18] S. Porwal, *An application of a Poisson distribution series on certain analytic functions*, J. Complex Anal., (2014), Art. ID 984135, 1–3.

[19] S. Porwal, *Mapping properties of generalized Bessel functions on some subclasses of univalent functions*, Anal. Univ. Oradea Fasc. Matematica, **20**(2) (2013), 51–60.

[20] S. Porwal and M. Kumar, *A unified study on starlike and convex functions associated with Poisson distribution series*, Afr. Mat., **27**(5)(2016), 1021-1027. Stud. Univ. Babeş-Bolyai Math., **63**(1) (2018), 71–78.

[21] S. Owa and B. A. Uralegaddi, *An application of the fractional calculus*, J. Karnatak Unia. Sci., **30** (1985).

[22] H. Silverman, *Starlike and convexity properties for hypergeometric functions*, J. Math. Anal. Appl., **172** (1993), 574–581.

[23] S. M. Sarangi and B. A. Uralegaddi, *The radius of convexity and starlikeness for certain classes of analytic functions with negative coefficients*, I, Alti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nurur., **65**(8) (1978), 38–42.

[24] H. M. Srivastava and Owa, *Certain classes of analytic functions with varying arguments*, Journal of Mathematical Analysis and Applications, **136** (1988), 217–228.

[25] H. M. Srivastava, G. Murugusundaramoorthy and S. Sivasubramanian, *Hypergeometric functions in the parabolic starlike and uniformly convex domains*, Integr. Transf. Spec. Func., **18** (2007), 511–520.

[26] K. Yamaguchi, *On functions satisfying* $\mathrm{Re}\{f(z)/z\} > 0$, Proc. Amer. Math. Soc., **17** (1966), 588–591.

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