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HERMITE-HADAMARD-FEJÉR TYPE INEQUALITIES FOR $(s, m)-$PREINVEX FUNCTIONS AND APPLICATIONS

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Abstract. Several new weighted inequalities of Hermite-Hadamard and Fejér type inequalities are established for those functions whose absolute value of first derivative are preinvex. The results given in this paper are extensions of earlier work.

1. Introduction

Convex functions have key role in many fields of pure and applied Mathematics. Researchers are working in theory of convex function and they are drawing their results by using different approaches. Hermite Hadamard Fejér inequalities are the most well known inequalities. Hermite Hadamard inequalities or its weighted versions are related to the integral mean of a convex function.

Xiao et al. [9], established the weighted Hermite-Hadamard inequalities. A. Latif et al. introduced weighted Hermite-Hadamard Noor type inequalities for differentiable preinvex and quasi preinvex functions [3]. Banyat Sroysang generalized some Hermite-Hadamard type inequalities for differentiable convex functions to weighted means [7].

Definition 1.1. (see [1]) **Invex set:** A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the map $\eta : K \times K \to \mathbb{R}^n$ if for every $x, y \in K$ and $t \in [0, 1]$ if

$$y + t\eta (x, y) \in K$$

The invex set K is also called an η-connected set.

Remark 1.1. If $\eta (x, y) = x - y$ and then invex set becomes convex set. Every convex set is an invex set but its converse is not true.

Definition 1.2. (see [1]) **Preinvex function:** Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}^n$. A function $f : K \to \mathbb{R}$ is said to be preinvex with respect to η for every $x, y \in K$ and $t \in [0, 1]$,

$$f (y + t\eta (x, y)) \le tf (x) + (1 - t) f (y) .$$

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Remark 1.2. If $\eta(x,y) = x - y$, then in classical sense, the pre-invex functions become convex functions.

Remark 1.3. A function f is called preincave iff its negative is preinvex

Definition 1.3. (see [8]) $\alpha-$**preinvex Function:** Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}^n$ Then, the function $f : K \to \mathbb{R}$ is said to be $\alpha-$preinvex with respect to η for $\alpha \in [0,1]$, for every $x, y \in K$ and $t \in [0,1]$, if

$$f(y + t\eta(x,y)) \le t^{\alpha} f(x) + (1 - t^{\alpha}) f(y).$$

Remark 1.4. If $\alpha = 1$, then f an $\alpha-$preinvex function becomes preinvex function.

Remark 1.5. If $\alpha = 1$ and $\eta(x,y) = x - y$ then f an $\alpha-$preinvex function becomes $\alpha-$convex function.

Definition 1.4. (see [5]) $m-$**preinvex Function:** The function f on the invex set $K \subseteq [0, b^*]$, $b^* > 0$, is said to be $m-$preinvex with respect to η if

$$f(y + t\eta(x,y)) \le tf(x) + m(1-t) f\left(\frac{y}{m}\right),$$

holds for all $x, y \in K$, $t \in [0,1]$ and $m \in (0,1]$. The function f is said to be m-preincave if and only if $-f$ is m-preinvex.

Remark 1.6. If $m = 1$, then $m-$preinvex function becomes a preinvex function.

Remark 1.7. If $m = 1$ and $\eta(x,y) = x - y$ then an $m-$preinvex function becomes a $m-$convex function.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x)dx \le \frac{f(a) + f(b)}{2}. \tag{1.1}$$

The inequality is known as **Hermite-Hadamard inequality** for convex functions.

In [2], Fejér gave a weighted generalization of the inequality (1.1) as follows:

$$f\left(\frac{a+b}{2}\right) \int_a^b w(x)dx \le \int_a^b w(x)f(x)dx \le \frac{f(a) + f(b)}{2} \int_a^b w(x)dx, \tag{1.2}$$

holds, where $w : [a; b] \to \mathbb{R}$ is nonnegative, integrable and symmetric about $x = \frac{a+b}{2}$.

2. Main Results

To establish our main results we first give the following essential definitions and lemmas.

Definition 2.1. Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}^n$. A function $f : K \to \mathbb{R}$ is said to be $(s, m)-$preinvex with respect to η for every $x, y \in K$, $t \in [0,1]$ and $m \in (0,1]$.

$$f(a + t\eta(b,a)) \le m(1-t)^s f\left(\frac{a}{m}\right) + t^s f(b). \tag{2.1}$$

Lemma 2.1. *(see [4])Let $K \subseteq \mathbb{R}$, be an open invex subset with respect to and $a, b \in K$ with $a < a + \eta(b,a)$ where $\eta(b,a) \neq 0$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K^o such that $f' \in L([a, a+\eta(b,a)])$. If $w : [a, a+\eta(b,a)] \to [0,\infty)$ is an integrable mapping, then the following equality holds:*

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) w(x) dx - \frac{1}{\eta(b,a)} f\left(a + \frac{\eta(b,a)}{2}\right) \int_a^{a+\eta(b,a)} w(x) dx \tag{2.2}$$
$$= \eta(b,a) \int_0^1 k(t) f'(a + t\eta(b,a)) dt,$$

where

$$k(t) = \begin{cases} \int_0^t w(a + u\eta(b,a)) du, & t \in [0, \frac{1}{2}) \\ -\int_t^1 w(a + u\eta(b,a)) du, & t \in [\frac{1}{2}, 1]. \end{cases}$$

Lemma 2.2. *(see [4]) Let $K \subseteq \mathbb{R}$, be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b,a)$ where $\eta(b,a) \neq 0$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K^o such that $f' \in L([a, a+\eta(b,a)])$. If $w : [a, a+\eta(b,a)] \to [0,\infty)$ is an integrable mapping, then the following equality holds:*

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) w(x) dx - \left(\frac{f(a) + f(a+\eta(b,a))}{2\eta(b,a)}\right) \int_a^{a+\eta(b,a)} w(x) dx \tag{2.3}$$
$$= \frac{\eta(b,a)}{2} \int_0^1 p(t) f'(a + t\eta(b,a)) dt,$$

where

$$p(t) = \int_t^1 w(a + u\eta(b,a)) du - \int_0^t w(a + u\eta(b,a)) du, t \in [0,1].$$

Theorem 2.1. *Let $K \subseteq \mathbb{R}$, be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b,a)$ where $\eta(b,a) \neq 0$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K^o such that $f' \in L([a, a+\eta(b,a)])$. If $w : [a, a+\eta(b,a)] \to [0,\infty)$ is an integrable mapping and symmetric to $a + \frac{1}{2}\eta(b,a)$. If $\left|f'\right|$ is (s,m) preinvex on K , then the following inequality holds:*

$$\left|\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) w(x) dx - \frac{1}{\eta(b,a)} f\left(a + \frac{\eta(b,a)}{2}\right) \int_a^{a+\eta(b,a)} w(x) dx\right| \tag{2.4}$$
$$\leq \frac{m\left|f'\left(\frac{a}{m}\right)\right| + |f'(b)|}{(s+1)(s+2)} \|w\|_\infty \left(1 - \frac{1}{2^{s+1}}\right).$$

where $\|w\|_\infty = \sup_{x \in [a, a+\eta(b,a)]} |w(x)|$, for the continuous function $w : [a, a+\eta(b,a)] \to \mathbb{R}$.

Proof. From Lemma 2.1 and the (s,m) preinvexity of $|f'|$ on K, we have

$$\left|\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) w(x) dx - \frac{1}{\eta(b,a)} f\left(a + \frac{\eta(b,a)}{2}\right) \int_a^{a+\eta(b,a)} w(x) dx\right| \tag{2.5}$$
$$\leq \eta(b,a) \int_0^{1/2} \left(\int_0^t |w(a + u\eta(b,a))| du\right) \left[m(1-t)^s \left|f'\left(\frac{a}{m}\right)\right| + t^s |f'(b)|\right] dt$$
$$+\eta(b,a) \int_{1/2}^1 \left(\int_t^1 |w(a + u\eta(b,a))| du\right) \left[m(1-t)^s \left|f'\left(\frac{a}{m}\right)\right| + t^s |f'(b)|\right] dt.$$

By the change of the order of integration, we have

$$\begin{aligned}
&\eta(b,a)\int_0^{1/2}\left(\int_0^t |w(a+u\eta(b,a))|\,du\right)\left[m(1-t)^s\left|f'\left(\frac{a}{m}\right)\right|+t^s|f'(b)|\right]dt \qquad (2.6)\\
&= \eta(b,a)\int_0^{1/2}|w(a+u\eta(b,a))|\int_u^{1/2}\left[m(1-t)^s\left|f'\left(\frac{a}{m}\right)\right|+t^s|f'(b)|\right]dtdu\\
&= \int_0^{1/2}|w(a+u\eta(b,a))|\left[m\left|f'\left(\frac{a}{m}\right)\right|\left(\frac{(1-u)^{s+1}}{(s+1)}-\frac{1}{(s+1)2^{s+1}}\right)\right.\\
&\quad\left.+|f'(b)|\left(\frac{1}{(s+1)2^{s+1}}-\frac{u^{s+1}}{2^{s+1}(s+1)}\right)\right]du.
\end{aligned}$$

Using the change of variable $x=a+u\eta(b,a)$ for $u\in[0,1]$ we have from (2.6)

$$\begin{aligned}
&\int_0^{1/2}\left(\int_u^{1/2}|w(a+u\eta(b,a))|\,du\right)\left[m(1-t)^s\left|f'\left(\frac{a}{m}\right)\right|+t^s|f'(b)|\right]dtdu \qquad (2.7)\\
&\le \frac{m}{\eta(b,a)}\left|f'\left(\frac{a}{m}\right)\right|\int_a^{a+\frac{1}{2}\eta(b,a)}\left[\left(\frac{\left(1-\frac{x-a}{\eta(b,a)}\right)^{s+1}}{(s+1)}-\frac{1}{(s+1)2^{s+1}}\right)|w(x)|\,dx\right.\\
&\quad\left.+\frac{1}{\eta(b,a)}|f'(b)|\int_a^{a+\frac{1}{2}\eta(b,a)}\left(\frac{1}{(s+1)2^{s+1}}-\frac{\left(\frac{x-a}{\eta(b,a)}\right)^{s+1}}{(s+1)}\right)|w(x)|\,du\right].\\
&\le \frac{m}{\eta(b,a)}\left|f'\left(\frac{a}{m}\right)\right|\|w(x)\|_\infty\int_a^{a+\frac{1}{2}\eta(b,a)}\left[\left(\frac{\left(1-\frac{x-a}{\eta(b,a)}\right)^{s+1}}{(s+1)}-\frac{1}{(s+1)2^{s+1}}\right)dx\right.\\
&\quad\left.+\frac{1}{\eta(b,a)}|f'(b)|\,\|w(x)\|_\infty\int_a^{a+\frac{1}{2}\eta(b,a)}\left(\frac{1}{(s+1)2^{s+1}}-\frac{\left(\frac{x-a}{\eta(b,a)}\right)^{s+1}}{(s+1)}\right)du\right].
\end{aligned}$$

Similarly by change of order of integration and using the fact that w is symmetric to $a+\frac{1}{2}\eta(b,a)$, we obtain

$$\begin{aligned}
&\eta(b,a)\int_{1/2}^{1}\left(\int_t^1|w(a+u\eta(b,a)|)du\right)\left[m(1-t)^s\left|f'\left(\frac{a}{m}\right)\right|+t^s|f'(b)|\right]dt \qquad (2.8)\\
&= \int_{1/2}^{1}\int_{1/2}^{u}|w(a+(1-u)\eta(b,a))|\left[m(1-t)^s\left|f'\left(\frac{a}{m}\right)\right|+t^s|f'(b)|\right]dtdu\\
&= \int_{1/2}^{1}|w(a+(1-u)\eta(b,a))|\left[m\left|f'\left(\frac{a}{m}\right)\right|\left(\frac{1}{(s+1)2^{s+1}}-\frac{(1-u)^{s+1}}{(s+1)}\right)\right.\\
&\quad\left.+|f'(b)|\left(\frac{u^{s+1}}{2^{s+1}(s+1)}-\frac{1}{(s+1)2^{s+1}}\right)\right]du.
\end{aligned}$$

By the change of variable $x = a + (1-u)\eta(b,a)$

$$\int_{1/2}^{1}\left(\int_{t}^{1}|w(a+u\eta(b,a))|\,du\right)\left[m(1-t)^{s}\left|f'\left(\frac{a}{m}\right)\right| + t^{s}|f'(b)|\right]dudt \tag{2.9}$$

$$= \frac{m}{\eta(b,a)}\left|f'\left(\frac{a}{m}\right)\right|\int_{a}^{a+\frac{1}{2}\eta(b,a)}\left(\frac{1}{(s+1)2^{s+1}} - \frac{\left(\frac{x-a}{\eta(b,a)}\right)^{s+1}}{(s+1)}\right)|w(x)|\,dx$$

$$+\frac{1}{\eta(b,a)}|f'(b)|\int_{a}^{a+\frac{1}{2}\eta(b,a)}\left(\frac{\left(1-\frac{x-a}{\eta(b,a)}\right)^{s+1}}{(s+1)} - \frac{1}{(s+1)2^{s+1}}\right)|w(x)|\,dx$$

$$= \frac{m}{\eta(b,a)}\left|f'\left(\frac{a}{m}\right)\right|\|w\|_{\infty}\int_{a}^{a+\frac{1}{2}\eta(b,a)}\left(\frac{1}{(s+1)2^{s+1}} - \frac{\left(\frac{x-a}{\eta(b,a)}\right)^{s+1}}{(s+1)}\right)dx$$

$$+\frac{1}{\eta(b,a)}|f'(b)|\,\|w\|_{\infty}\int_{a}^{a+\frac{1}{2}\eta(b,a)}\left(\frac{\left(1-\frac{x-a}{\eta(b,a)}\right)^{s+1}}{(s+1)} - \frac{1}{(s+1)2^{s+1}}\right)dx.$$

Substituting (2.7) and (2.9) in (2.5) and simplifying, we get the inequality (2.4). This completes the proof of the theorem □

Corollary 2.1. *If we take $w(x) = 1$, $x \in [a, a+\eta(b,a)]$ in Theorem 2.1 we get*

$$\left|\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)} f(x)\,w(x)dx - f\left(a+\frac{1}{2}\eta(b,a)\right)\right|$$

$$\leq \frac{m\left|f'\left(\frac{a}{m}\right)\right| + |f'(b)|}{s+1}\int_{a}^{a+\frac{1}{2}\eta(b,a)}\left[\left(1-\frac{x-a}{\eta(b,a)}\right)^{s+1} - \left(\frac{x-a}{\eta(b,a)}\right)^{s+1}\right]dx$$

$$= \frac{m\left|f'\left(\frac{a}{m}\right)\right| + |f'(b)|}{(s+1)(s+2)}\left(1-\frac{1}{2^{s+1}}\right).$$

Theorem 2.2. *Let $K \subseteq \mathbb{R}$, be an open invex subset with respect to $\eta : K\times K \to \mathbb{R}$ and $a, b \in K$ with $a < a+\eta(b,a)$ where $\eta(b,a) \neq 0$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K^{o} such that $f^{'} \in L([a, a+\eta(b,a)])$. If $w : [a, a+\eta(b,a)] \to [0,\infty)$ is an integrable mapping and symmetric to $a+\frac{1}{2}\eta(b,a)$. If $\left|f^{'}\right|^{q}$, $q > 1$ is (s,m) preinvex on K , then the following inequality holds:*

$$\left|\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)} f(x)\,w(x)dx - \frac{1}{\eta(b,a)}f\left(a+\frac{\eta(b,a)}{2}\right)\int_{a}^{a+\eta(b,a)} w(x)dx\right| \tag{2.10}$$

$$\leq \eta(b,a)\left(\frac{1}{(\eta(b,a))^{2}}\int_{a}^{a+\frac{1}{2}\eta(b,a)}\left[\frac{\eta(b,a)}{2} - (x-a)\right]w^{p}(x)dx\right)^{1/p}$$

$$\times\left[\left(m\left|f'\left(\frac{a}{m}\right)\right|^{q}\frac{2^{s+2}-s-3}{2^{s+2}(s+1)(s+2)} + |f'(b)|^{q}\frac{1}{2^{s+2}(s+2)}\right)^{1/q}\right.$$

$$\left.+\left(m\left|f'\left(\frac{a}{m}\right)\right|^{q}\frac{1}{2^{s+2}(s+2)} + |f'(b)|^{q}\frac{2^{s+2}-s-3}{2^{s+2}(s+1)(s+2)}\right)^{1/q}\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1 and change of order of integration, we get

$$\begin{aligned}
&\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,w(x)dx-\frac{1}{\eta(b,a)}f\left(a+\frac{\eta(b,a)}{2}\right)\int_a^{a+\eta(b,a)} w(x)dx\right| \\
\leq\ & \eta(b,a)\int_0^{1/2}\left(\int_0^t w(a+u\eta(b,a))du\right)\left|f'(a+t\eta(b,a))\right|dt \\
&+\eta(b,a)\int_{1/2}^1\left(\int_t^1 w(a+u\eta(b,a))du\right)\left|f'(a+t\eta(b,a))\right|dt \\
=\ & \eta(b,a)\int_0^{1/2}\int_u^{1/2} w(a+u\eta(b,a))\left|f'(a+t\eta(b,a))\right|dtds \\
&+\eta(b,a)\int_{1/2}^1\int_{1/2}^u w(a+u\eta(b,a))\left|f'(a+t\eta(b,a))\right|dtds. \qquad (2.11)
\end{aligned}$$

from the Hölder's inequality, we have

$$\begin{aligned}
&\eta(b,a)\int_0^{1/2}\int_u^{1/2} w(a+u\eta(b,a))\left|f'(a+t\eta(b,a))\right|dtds \qquad (2.12)\\
\leq\ & \eta(b,a)\left(\int_0^{1/2}\int_u^{1/2} w^p(a+u\eta(b,a))dtdu\right)^{1/p} \\
&\times\left(\int_0^{1/2}\int_u^{1/2}\left|f'(a+t\eta(b,a))\right|^q dtdu\right)^{1/q}.
\end{aligned}$$

Since $\left|f'\right|^q$, $q>1$ is (s,m) preinvex on K, then for every $a,b\in K$, $t\in[0,1]$ we have

$$\left|f'(a+t\eta(b,a))\right|^q\leq m(1-t)^s\left|f'\left(\frac{a}{m}\right)\right|^q+t^s\left|f'(b)\right|^q.$$

Hence by solving elementary integrals and using the substitution $x=a+u\eta(b,a)$, $u\in[0,1]$

$$\begin{aligned}
&\eta(b,a)\int_0^{1/2}\int_u^{1/2} w(a+u\eta(b,a))\left|f'(a+t\eta(b,a))\right|dtdu \qquad (2.13)\\
\leq\ & \eta(b,a)\left(\int_0^{1/2}\int_u^{1/2} w^p(a+u\eta(b,a))dtdu\right)^{1/p} \\
&\times\left(\int_0^{1/2}\int_u^{1/2}\left(m(1-t)^s\left|f'\left(\frac{a}{m}\right)\right|^q+t^s\left|f'(b)\right|^q\right)dtdu\right)^{1/q} \\
=\ & \eta(b,a)\left(\frac{1}{(\eta(b,a))^2}\int_a^{a+\frac{1}{2}\eta(b,a)}\left[\frac{\eta(b,a)}{2}-(x-a)\right]w^p(x)dx\right)^{1/p} \\
&\times\left(m\left|f'\left(\frac{a}{m}\right)\right|^q\frac{2^{s+2}-s-3}{2^{s+2}(s+1)(s+2)}+\left|f'(b)\right|^q\frac{1}{2^{s+2}(s+2)}\right)^{1/q}.
\end{aligned}$$

Using the symmetry of w about $a+\frac{1}{2}\eta(b,a)$, we also have

$$\begin{aligned}
&\eta(b,a)\int_{1/2}^{1}\int_{1/2}^{u} w(a+u\eta(b,a))\left|f'(a+t\eta(b,a))\right|dtdu \qquad (2.14)\\
\leq\ &\eta(b,a)\left(\int_{1/2}^{1}\int_{1/2}^{u} w^{p}(a+u\eta(b,a))dtdu\right)^{1/p}\\
&\times\left(\int_{1/2}^{1}\int_{1/2}^{u}\left(m(1-t)^{s}\left|f'\left(\frac{a}{m}\right)\right|^{q}+t^{s}\left|f'(b)\right|^{q}\right)dtdu\right)^{1/q}\\
\leq\ &\eta(b,a)\left(\frac{1}{(\eta(b,a))^{2}}\int_{a}^{a+\frac{1}{2}\eta(b,a)}\left[\frac{\eta(b,a)}{2}-(x-a)\right]w^{p}(x)dx\right)^{1/p}\\
&\times\left(m\left|f'\left(\frac{a}{m}\right)\right|^{q}\frac{1}{2^{s+2}(s+2)}+\left|f'(b)\right|^{q}\frac{2^{s+2}-s-3}{2^{s+2}(s+1)(s+2)}\right)^{1/q}.
\end{aligned}$$

Substituting (2.13) and (2.14) in (2.11) and simplifying, we get the inequality (2.10). This completes the proof of the theorem. □

Theorem 2.3. *Let $K\subseteq\mathbb{R}$, be an open invex subset with respect to $\eta: K\times K\to\mathbb{R}$ and $a,b\in K$ with $a<a+\eta(b,a)$ where $\eta(b,a)\neq 0$. Suppose $f:K\to\mathbb{R}$ is a differentiable mapping on K^{o} such that $f^{'}\in L([a,a+\eta(b,a)])$. If $w:[a,a+\eta(b,a)]\to[0,\infty)$ is an integrable mapping and symmetric to $a+\frac{1}{2}\eta(b,a)$. If $\left|f^{'}\right|$ is (s,m) preinvex on K , then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)} f(x)w(x)dx\right. \qquad (2.15)\\
&\left.-\frac{1}{\eta(b,a)}\left(\frac{f(a)+f(a+\eta(b,a))}{2\eta(b,a)}\right)\int_{a}^{a+\eta(b,a)} w(x)dx\right|\\
\leq\ &\frac{(\eta(b,a))^{2}}{2(s+1)}\left(m\left|f'\left(\frac{a}{m}\right)\right|+\left|f'(b)\right|\right)\|w\|_{\infty}
\end{aligned}$$

where $\|w\|_{\infty}=\sup_{x\in[a,a+\eta(b,a)]}|w(x)|$, for the continuous function $w:[a,a+\eta(b,a)]\to\mathbb{R}$.

Proof. From Lemma 2.2 and the (s,m) preinvexity of $|f'|$ on K, we have

$$\begin{aligned}
&\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,w(x)dx\right. \qquad (2.16)\\
&\left.-\frac{1}{\eta(b,a)}\left(\frac{f(a)+f(a+\eta(b,a))}{2\eta(b,a)}\right)\int_a^{a+\eta(b,a)} w(x)dx\right|\\
\leq\ & \frac{\eta(b,a)}{2}\int_0^1\left|\int_t^1 w(a+u\eta(b,a))du-\int_0^t w(a+u\eta(b,a))du\right|\\
&\times\left|m(1-t)^s\left|f'\left(\frac{a}{m}\right)\right|+t^s\left|f'(b)\right|\right|dt\\
=\ & \frac{\eta(b,a)}{2}\int_0^1\int_t^1 |w(a+u\eta(b,a))|\,du\left(\left|m(1-t)^s\left|f'\left(\frac{a}{m}\right)\right|+t^s\left|f'(b)\right|\right|\right)dt\\
&-\frac{\eta(b,a)}{2}\int_0^1\int_0^t |w(a+u\eta(b,a))|\,du\left(\left|m(1-t)^s\left|f'\left(\frac{a}{m}\right)\right|+t^s\left|f'(b)\right|\right|\right)dt.
\end{aligned}$$

By the change of the order of integration, we have

$$\begin{aligned}
&\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,w(x)dx-\left(\frac{f(a)+f(a+\eta(b,a))}{2\eta(b,a)}\right)\int_a^{a+\eta(b,a)} w(x)dx\right| \qquad (2.17)\\
=\ & \frac{\eta(b,a)}{2}\int_0^1 |w(a+u\eta(b,a))|\int_0^u\left(\left|m(1-t)^s\left|f'\left(\frac{a}{m}\right)\right|+t^s\left|f'(b)\right|\right|\right)dtdu\\
&-\frac{\eta(b,a)}{2}\int_0^1 |w(a+u\eta(b,a))|\int_u^1\left(\left|m(1-t)^s\left|f'\left(\frac{a}{m}\right)\right|+t^s\left|f'(b)\right|\right|\right)dtdu\\
=\ & \frac{\eta(b,a)}{2(s+1)}\left(m\left|f'\left(\frac{a}{m}\right)\right|+\left|f'(b)\right|\right)\int_0^1 |w(a+u\eta(b,a))|\,du\\
=\ & \frac{\eta(b,a)}{2(s+1)}\left(m\left|f'\left(\frac{a}{m}\right)\right|+\left|f'(b)\right|\right)\int_a^{a+\eta(b,a)} |w(x)|\,dx\\
=\ & \frac{(\eta(b,a))^2}{2(s+1)}\left(m\left|f'\left(\frac{a}{m}\right)\right|+\left|f'(b)\right|\right)\|w\|_\infty \qquad (2.18)
\end{aligned}$$

which is the required. □

Theorem 2.4. *Let $K\subseteq\mathbb{R}$, be an open invex subset with respect to $\eta: K\times K\to\mathbb{R}$ and $a,b\in K$ with $a<a+\eta(b,a)$ where $\eta(b,a)\neq 0$. Suppose $f:K\to\mathbb{R}$ is a differentiable mapping on K^o such that $f'\in L([a,a+\eta(b,a)])$. If $w:[a,a+\eta(b,a)]\to[0,\infty)$ is an integrable mapping and symmetric to $a+\frac{1}{2}\eta(b,a)$. If $\left|f'\right|^q$, $q>1$ is (s,m) preinvex on K, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,w(x)dx-\left(\frac{f(a)+f(a+\eta(b,a))}{2\eta(b,a)}\right)\int_a^{a+\eta(b,a)} w(x)dx\right|\\
\leq\ & \frac{1}{2}\left(\int_0^1 g^p(t)dt\right)^{1/p}\left(\frac{m\left|f'\left(\frac{a}{m}\right)\right|^q+\left|f'(b)\right|^q}{s+1}\right)^{1/q} \qquad (2.19)
\end{aligned}$$

where

$$g(t)=\begin{cases}\frac{1}{\eta(b,a)}\int_{a+t\eta(b,a)}^{a+(1-t)\eta(b,a)}w(x)dx, & t\in\left[0,\frac{1}{2}\right]\\ -\frac{1}{\eta(b,a)}\int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)}w(x)dx, & t\in\left[\frac{1}{2},1\right]\end{cases}$$

$t\in[0,1]$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From lemma 2.2 we have

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)w(x)dx-\left(\frac{f(a)+f(a+\eta(b,a))}{2\eta(b,a)}\right)\int_a^{a+\eta(b,a)}w(x)dx\right|$$
$$\leq\frac{\eta(b,a)}{2}\int_0^1\int_t^1 w(a+u\eta(b,a))du-\int_0^t w(a+u\eta(b,a))du\left|f'(a+t\eta(b,a))\right|dt. \tag{2.20}$$

Using the symmetry of w about $a+\frac{1}{2}\eta(b,a)$, we also have

$$\begin{aligned}&\int_t^1 w(a+u\eta(b,a))du-\int_0^t w(a+u\eta(b,a))du \\ =\;&\int_t^1 w(a+u\eta(b,a))du-\int_0^t w(a+(1-u)\eta(b,a))du \\ =\;&\frac{1}{\eta(b,a)}\int_{a+t\eta(b,a)}^{a+\eta(b,a)}w(x)dx+\frac{1}{\eta(b,a)}\int_{a+\eta(b,a)}^{a+(1-t)\eta(b,a)}w(x)dx \\ =\;&g(x)\end{aligned} \tag{2.21}$$

where

$$g(t)=\begin{cases}\frac{1}{\eta(b,a)}\int_{a+t\eta(b,a)}^{a+(1-t)\eta(b,a)}w(x)dx, & t\in\left[0,\frac{1}{2}\right]\\ -\frac{1}{\eta(b,a)}\int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)}w(x)dx, & t\in\left[\frac{1}{2},1\right]\end{cases}$$

From(2.20), we get

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)w(x)dx-\left(\frac{f(a)+f(a+\eta(b,a))}{2\eta(b,a)}\right)\int_a^{a+\eta(b,a)}w(x)dx\right|$$
$$\leq\frac{1}{2}\int_0^1 g(t)\left|f'(a+t\eta(b,a))\right|dt. \tag{2.22}$$

By Hölder's inequality, it follows from (2.22) that

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)w(x)dx-\left(\frac{f(a)+f(a+\eta(b,a))}{2\eta(b,a)}\right)\int_a^{a+\eta(b,a)}w(x)dx\right|$$
$$\leq\frac{1}{2}\left(\int_0^1 g^p(t)dt\right)^{1/p}\left(\int_0^1\left|f'(a+t\eta(b,a))\right|^q dt\right)^{1/q}. \tag{2.23}$$

Since $\left|f'\right|^q, q>1$ is (s,m) preinvex on K , then for every $a,b\in K$, $t\in[0,1]$ we have

$$\left|f'(a+t\eta(b,a))\right|^q\leq m(1-t)^s\left|f'\left(\frac{a}{m}\right)\right|^q+t^s\left|f'(b)\right|^q$$

and hence from (2.23), we get that

$$\begin{aligned}
&\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,w(x)dx - \left(\frac{f(a)+f(a+\eta(b,a))}{2\eta(b,a)}\right)\int_a^{a+\eta(b,a)} w(x)dx\right| \\
\leq\ & \frac{1}{2}\left(\int_0^1 g^p(t)dt\right)^{1/p}\left(\int_0^1\left[m(1-t)^s\left|f'\left(\frac{a}{m}\right)\right|^q + t^s\left|f'(b)\right|^q\right]dt\right)^{1/q} \\
=\ & \frac{1}{2}\left(\int_0^1 g^p(t)dt\right)^{1/p}\times\left(\frac{m\left|f'\left(\frac{a}{m}\right)\right|^q+\left|f'(b)\right|^q}{s+1}\right)^{1/q} \qquad (2.24)
\end{aligned}$$

which is the require result. □

3. Application for random variables

Suppose that for $0 < a < a+\eta(b,a)$, $g : [a, a+\eta(b,a)] \to [0,1)$ is a continuous probability density function for continuous random variable X. Which is symmetric about $a+\frac{1}{2}\eta(b,a)$. Also for $\tau \in \mathbb{R}$, the $\tau-$moment is given as

$$E_r(x) = \int_a^{a+\eta(b,a)} x^\tau g(x)\,dx \qquad (3.1)$$

is finite.

Using the fact that w is symmetric and $\int_a^{a+\eta(b,a)} w(x)\,dx = 1$, then we have

$$E(x) = \int_a^{a+\eta(b,a)} x^\tau w(x)\,dx = \frac{2a+\eta(b,a)}{2} \qquad (3.2)$$

since

$$\begin{aligned}
\int_a^{a+\eta(b,a)} xw(x)\,dx &= \int_a^{a+\eta(b,a)} (2a+\eta(b,a)-x)\,w(2a+\eta(b,a)-x)\,dx \\
&= \int_a^{a+\eta(b,a)} (2a+\eta(b,a)-x)\,w(x)\,dx
\end{aligned}$$

Proposition 3.1. *For $\tau \geq 2$, we have the following inequality*

$$\begin{aligned}
&\left|E_r(x) - (E(x))^\tau\right| \qquad (3.3) \\
\leq\ & \tau\eta(b,a)\frac{m\left(\frac{a}{m}\right)^{\tau-1}+b^{\tau-1}}{(s+1)(s+2)}\left\|w\right\|_\infty\left(1-\frac{1}{2^{s+1}}\right).
\end{aligned}$$

Proof. Let $f(x) = x^{\tau,}$ for $\tau \geq 2$, the function $|f'(x)| = \tau x^{\tau-1}$ is an $(s,m)-$preinvex function. Using the identities (3.1) and (3.2) in (2.4) we obtain the required inequality. □

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