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EXISTENCE AND STABILITY OF FRACTIONAL PANTOGRAPH DIFFERENTIAL EQUATIONS WITH CAPUTO-HADAMARD TYPE DERIVATIVE

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Abstract. Recently, many studies on fractional pantograph equations, involving different fractional derivatives have appeared during the past several years. In this work, we study existence, uniqueness and Ulam-Hyers stability of solutions of Caputo-Hadamard type fractional pantograph differential equations with nonlocal boundary conditions. The Banach contraction principle is used for proving the existence and uniqueness results. We also derive the Ulam-Hyers stability and the generalized Ulam-Hyers stability of solution. Finally, we give some illustrative examples.

1. Introduction and preliminaries

Fractional calculus and its applications has importance in various areas of engineering sciences, mathematical and physical [4, 5, 17, 20, 22, 23]. The pantograph equations is a kind of delay differential equations and arise in many applications such as electrodynamics, astrophysics, nonlinear dynamical systems, probability theory on algebraic structures, quantum mechanics and cell growth, etc. The name pantograph originated from the work of Ockendon and Taylor [21] on the collection of current by the pantograph head of an electric locomotive. For further information and applications, see [6, 13–15, 19]. Recently, pantograph differential equations of fractional order have been studied by many researchers, for example, we refer the reader to [3, 7, 9, 11] and the references cited therein. Moreover, some authors have established the existence and uniqueness of solutions for some fractional pantograph differential equations with different fractional derivative, for example, see [3, 9, 11, 25, 26]. The Ulam-Hyers stability problems have been attracted by many researchers, see [2, 8, 12, 14, 18] and references therein. The stability of fractional pantograph differential equations has been investigated by many authors, we refer the reader to the papers [10, 25, 27].

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In this work, we study existence, uniqueness and Ulam-Hyers stability of solutions for Caputo-Hadamard type pantograph differential equations of fractional order:

$$\begin{cases} {}_{H}^{C}D^{\alpha}u\left(t\right)=\varphi\left(t,u\left(t\right),u\left(\lambda t\right)\right),t\in[1,T],0<\alpha\leq 1,0<\lambda<1, \\ \\ u\left(1\right)=u_{1}-\theta\left(u\right),\ u_{1}\in\mathbb{R}, \end{cases} \tag{1.1}$$

where ${}_{H}^{C}D^{\alpha}$ denote the Caputo-Hadamard type fractional derivative of order α and $\varphi : [1,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}, \theta : C\left([1,T],\mathbb{R}\right)\to\mathbb{R}$ are given continuous functions. The Hadamard fractional integral [1, 15] of order α for a continuous function $\varphi : [1,+\infty)\to\mathbb{R}$ is defined by

$$ {}_{H}I^{\alpha}\varphi\left(t\right)=\tfrac{1}{\Gamma(\alpha)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{\varphi\left(s\right)}{s}ds,\alpha>0,$$

with $\Gamma\left(\alpha\right)=\int_{0}^{\infty}e^{u}u^{\alpha-1}du$. The Caputo-Hadamard fractional derivative [1, 15] of order α for a continuous function $\varphi : [1,+\infty)\to\mathbb{R}$ is defined by

$$ {}_{H}^{C}D^{\alpha}\varphi(t)=\tfrac{1}{\Gamma(n-\alpha)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{n-\alpha-1}\delta^{n}\varphi(s)\frac{ds}{s}={}_{H}I^{n-\alpha}\left(\delta^{n}\varphi\right)\left(t\right),$$

where $n=[\alpha]+1$ and $\delta^{n}=\left(t\frac{d}{dt}\right)^{n}$.

Lemma 1.1. [15] *Let $u\in AC_{\delta}^{n}\left([a,b],\mathbb{R}\right)$. Then*

$$ {}_{H}I^{\alpha}\left({}_{H}^{C}D^{\alpha}u\right)\left(t\right)=u(t)-\sum_{i=0}^{n-1}c_{i}(\log t)^{i},c_{i}\in\mathbb{R},$$

where $AC_{\delta}^{n}\left([a,b],\mathbb{R}\right)=\left\{h:[a,b]\to\mathbb{R}:\delta^{n-1}h\in AC\left([a,b],\mathbb{R}\right)\right\}$.

We denote by $X=C\left([1,T],\mathbb{R}\right)$ the Banach space of all continuous functions from $[1,T]$ to $\mathbb{R}$ endowed with the norm defined by $\|u\|=\sup\left\{|u\left(t\right)|:t\in[1,T]\right\}$.

In what follows, we present four types of the Ulam stability for the fractional problem (1.1). Let σ a positive real numbers and the function $h\in X$, we consider the following fractional differential inequalities:

$$\left|{}_{H}^{C}D^{\alpha}v\left(t\right)-\varphi\left(t,v\left(t\right),v\left(\lambda t\right)\right)\right| \leq \sigma,\ t\in\left[1,T\right], \tag{1.2}$$

$$\left|{}_{H}^{C}D^{\alpha}y\left(t\right)-f\left(t,y\left(t\right),y\left(\lambda t\right)\right)\right| \leq h\left(t\right),\ t\in\left[1,T\right], \tag{1.3}$$

and

$$\left|{}_{H}^{C}D^{\alpha}y\left(t\right)-f\left(t,y\left(t\right),y\left(\lambda t\right)\right)\right|\leq\sigma h\left(t\right),\ t\in\left[1,T\right]. \tag{1.4}$$

Definition 1.1. The fractional boundary value problem (1.1) is Ulam-Hyers stable if there exists a real number $\tau_{\varphi}>0$ such that for each $\sigma>0$ and for each solution $v\in X$ of the inequality (1.2), there exists a solution $u\in X$ of fractional boundary value problem (1.1) with

$$\left|v\left(t\right)-u\left(t\right)\right|\leq\tau_{\varphi}\sigma,\ t\in\left[1,T\right].$$

Definition 1.2. The fractional boundary value problem (1.1) is generalized Ulam-Hyers stable if there exists $\psi_{\varphi}\in C(\mathbb{R}_{+},\mathbb{R}_{+}),\psi_{\varphi}\left(0\right)=0$, such that for each solution $v\in X$ of the inequality (1.2), there exists a solution $u \in X$ of the fractional boundary value problem (1.1) with

$$|v(t) - u(t)| \leq \psi_{\varphi}(\sigma), \, t \in [1, T].$$

Definition 1.3. The fractional boundary value problem (1.1) is Ulam-Hyers-Rassias stable with respect to $h \in X$ if there exists a real number $\tau_{\varphi} > 0$ such that for each $\sigma > 0$ and for each solution $v \in X$ of the inequality (1.4), there exists a solution $u \in X$ of problem (1.1) with

$$|v(t) - u(t)| \leq \tau_{\varphi} \sigma h(t), \, t \in [1, T].$$

Definition 1.4. The fractional boundary value problem (1.1) is generalized Ulam-Hyers-Rassias stable with respect to $h \in X$ if there exists a real number $\tau_{\varphi,h} > 0$ such that for each solution $v \in X$ of the inequality (1.3), there exists a solution $x \in X$ of problem (1.1) with

$$|v(t) - u(t)| \leq \tau_{\varphi,h} h(t), \, t \in [1, T].$$

Remark 1.1. A function $v \in X$ is a solution of the inequality (1.2) if and only if there exists a function $f : [1, T] \to \mathbb{R}$ such that

(1): : $|f(t)| \leq \sigma, t \in [1, T]$.
(2): : ${}_{H}^{C}D^{\alpha} v(t) = \varphi(t, v(t), v(\lambda t)) + f(t)$, $t \in [1, T], 0 < \lambda < 1$.

2. Existence and uniqueness of solution

Lemma 2.1. *Suppose that $g(t) \in C([1, T], \mathbb{R})$ and consider the fractional problem*

$${}_{H}^{C}D^{\alpha} u(t) = g(t), t \in [1, T], 0 < \alpha < 1, \tag{2.1}$$

with the condition

$$u(1) = u_1 - \theta(u). \tag{2.2}$$

Then, we have

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha - 1} g(s) \frac{ds}{s} + \theta(u) - u_1. \tag{2.3}$$

Proof. Using Lemma 1.1, we obtain

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha - 1} g(s) \frac{ds}{s} - c_0 \tag{2.4}$$

where $c_0 \in \mathbb{R}$.

Thanks to (2.2), we get $c_0 = \theta(u) - u_1$.

Substituting the value of c_0 in (2.4) yields the solution (2.3).This completes the proof. □

In view of Lemma 2.1, we define an operator $O : X \to X$ as

$$Ou(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha - 1} \varphi(s, u(s), u(\lambda s)) \frac{ds}{s} + \theta(u) - u_1. \tag{2.5}$$

Observe that the existence of a fixed point for the operator O implies the existence of a solution for the fractional boundary value problem (1.1).

Theorem 2.1. *Let $\varphi : [1,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that:*

(H_1) : *There exists a constant $\omega > 0$ such that*

$$|\varphi(t,u_1,u_2) - \varphi(t,v_1,v_2)| \le \omega(|u_1 - v_1| + |u_2 - v_2|), t \in [1,T], u_i, v_i \in \mathbb{R}, i = 1,2.$$

(H_2) : *There exists a constant $\varpi > 0$ such that*

$$|\theta(u) - \theta(v)| \le \varpi |u - v|, \ u, v \in C([1,T], \mathbb{R}).$$

If the inequality

$$2\omega(\log t)^{\alpha} < (1 - \varpi)\Gamma(\alpha + 1), \tag{2.6}$$

is valid, then problem (1.1) *has a unique solution on* $[1,T]$.

Proof. Let us define $M = \sup_{t\in[0,T]} |\varphi(t,0,0)|$ and $N = \|\theta(0)\|$. Setting

$$r \ge \frac{M + N + |u_1|}{1 - (2\omega + \varpi)},$$

we show that $OB_r \subset B_r$, where $B_r = \{u \in X : \|u\| \le r\}$.

For $x \in B_r$, we find the following estimates based on the hypothesis (H_1) and (H_2) :

$$\begin{aligned} |\varphi(t,u(t),u(\lambda t))| &\le |\varphi(t,u(t),u(\lambda t)) - \varphi(s,0,0)| + |\varphi(s,0,0)| \\ &\le 2\omega\|u\| + M \le 2\omega r + M, \end{aligned} \tag{2.7}$$

and

$$|\theta(u)| \le ||\theta(u)| - \theta(0)| + |\theta(0)| \le \varpi\|u\| + \|\theta(0)\| \le \varpi r + N. \tag{2.8}$$

Using these estimates, we get

$$\begin{aligned} |Ou(t)| &\le \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |\varphi(s,u(s),u(\lambda s))| \frac{ds}{s} + |\theta(u)| + |u_1| \\ &\le 2\omega r + M + \varpi r + N + |x_1| = (2\omega + \varpi)r + M + N + |u_1| \le r, \end{aligned} \tag{2.9}$$

which implies that $OB_r \subset B_r$. Now, for $u, v \in B_r$ and for any $t \in J$, we get

$$\begin{aligned} &|Ou(t) - Ov(t)| \\ &\le \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |\varphi(s,u(s),u(\lambda s)) - \varphi(s,v(s),v(\lambda s))| \frac{ds}{s} + |\theta(u) - \theta(v)| \\ &\le \left(2\omega\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)} + \varpi\right)\|u - v\|. \end{aligned} \tag{2.10}$$

Since $t \in [1,T]$, then

$$\|Ou - Ov\| \le \left(2\omega\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} + \varpi\right)\|u - v\|. \tag{2.11}$$

By (2.9), we see that O is a contractive operator. Consequently, by the Banach fixed point theorem, has a fixed point which is a solution of (1.1). □

3. Ulam-Hyers stability

In this section, we will study Ulam's type stability of the fractional boundary value problem (1.1).

Theorem 3.1. *Assume that $\varphi : [1,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying (H_1). If*

$$2\omega\left(\log T\right)^{\alpha}<\Gamma\left(\alpha+1\right), \tag{3.1}$$

then the fractional boundary value problem (1.1) is Ulam-Hyers stable and consequently, generalized Ulam-Hyers stable.

Proof. Let $v\in X$ be a solution of the inequality (1.2) and let us denote by $u\in X$ the unique solution of the problem

$$\begin{cases} {}_{H}^{C}D^{\alpha}u\left(t\right)=f\left(t,u\left(t\right),u\left(\lambda t\right)\right),t\in J,0<\alpha<1,0<\lambda<1,\\ u\left(1\right)=v\left(1\right). \end{cases} \tag{3.2}$$

By using Lemma 2.1, we have

$$u(t)=\frac{1}{\Gamma\left(\alpha\right)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}g\left(s\right)\frac{ds}{s}+c_{0}={}_{H}I^{\alpha}g_{u}\left(t\right)+c_{0},$$

and by integration of the inequality (1.2), we obtain

$$\begin{aligned} \left|v\left(t\right)-{}_{H}I^{\alpha}g_{v}\left(t\right)-c_{1}\right| &\leq \frac{\sigma}{\Gamma\left(\alpha+1\right)}\left(\log t\right)^{\alpha}\\ &\leq \frac{\sigma}{\Gamma\left(\alpha+1\right)}\left(\log T\right)^{\alpha}. \end{aligned} \tag{3.3}$$

On the other hand, if $u\left(1\right)=v\left(1\right)$, then $c_{0}=c_{1}$.

For any $t\in\left[1,T\right]$, we have

$$v\left(t\right)-u\left(t\right)=v\left(t\right)-{}_{H}I^{\alpha}g_{u}\left(t\right)-c_{1}+{}_{H}I^{\alpha}\left(g_{v}\left(t\right)-g_{u}\left(t\right)\right),$$

where,

$$g_{u}\left(t\right)=\varphi\left(t,u\left(t\right),u\left(\lambda t\right)\right)\ \text{and}\ g_{v}\left(t\right)=\varphi\left(t,v\left(t\right),v\left(\lambda t\right)\right),$$

then

$$\begin{aligned} {}_{H}I^{\alpha}\left(g_{v}\left(t\right)-g_{u}\left(t\right)\right) &= {}_{H}I^{\alpha}[\varphi(s,v(s),v\left(\lambda t\right))-\varphi(s,u(s),u\left(\lambda t\right))]\\ &=\frac{1}{\Gamma(\alpha)}\int_{1}^{t}(\log\frac{t}{s})^{\alpha-1}[\varphi(s,v(s),v\left(\lambda t\right))-\varphi(s,u(s),u\left(\lambda t\right))]\frac{ds}{s} \end{aligned}$$

Using (H_1), we get

$$\left|{}_{H}I^{\alpha}\left(g_{v}\left(t\right)-g_{u}\left(t\right)\right)\right|\leq\frac{2\omega}{\Gamma(\alpha)}\int_{1}^{t}(\log\frac{t}{s})^{\alpha-1}|v-u|\frac{ds}{s}.$$

This yields that

$$|v(t)-u(t)| \leq |v(t)- {}_{H}I^{\alpha}g_{v}(t)-c_{1}| + \frac{2\omega}{\Gamma(\alpha)}\int_{1}^{t}(\log\frac{t}{s})^{\alpha-1}|v-u|\frac{ds}{s},$$

which implies that

$$|v(t)-u(t)| \leq \frac{\sigma}{\Gamma(\alpha+1)} + \frac{2\omega}{\Gamma(\alpha)}\int_{1}^{t}(\log\frac{t}{s})^{\alpha-1}\|v(s)-u(s)\|\frac{ds}{s}.$$

Thus

$$|v(t)-u(t)| \leq \frac{\sigma}{\Gamma(\alpha+1)} + \frac{2\omega(\log T)^{\alpha}}{\Gamma(\alpha+1)}\|v(s)-u(s)\|,$$

Hence

$$\|v(s)-u(s)\|\left(1-\frac{2\omega(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right) \leq \frac{\sigma}{\Gamma(\alpha+1)}. \tag{3.4}$$

Then, for each $t\in[1,T]$

$$|u(t)-v(t)| \leq \frac{1}{\Gamma(\alpha+1)-2\omega(\log T)^{\alpha}}\sigma = \tau_{\varphi}\sigma. \tag{3.5}$$

So, the fractional boundary value problem (1.1) is Ulam-Hyers stable. By putting $h(\sigma)=\gamma\sigma, h(0)=0$ yields that the fractional boundary value problem (1.1) generalized Ulam-Hyers stable. □

Theorem 3.2. *Let $\varphi:[1,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous function and suppose that (H_1) holds and (3.1). In addition, the following hypothesis holds*

(H_3) : There exists an function $h\in C([1,T],\mathbb{R}_{+})$ and there exists $\eta_h>0$ such that for any $t\in[1,T]$

$$\frac{1}{\Gamma(\alpha)}\int_{1}^{t}(\log\frac{t}{s})^{\alpha-1}h(s)\frac{ds}{s} \leq \eta_{h}h(t). \tag{3.6}$$

Then the fractional boundary value problem (1.1) is Ulam-Hyers-Rassias stable.

Proof. Let $v\in X$ be a solution of the inequality (1.4) and let us denote by $x\in X$ the unique solution of the problem

$$\begin{cases} {}_{H}^{C}D^{\alpha}u(t)=f(t,u(t),u(\lambda t)), t\in[1,T], 0<\alpha\leq1, 0<\lambda<1, \\ u(1)=v(1). \end{cases}$$

Thanks to Lemma 2.1, we obtain

$$u(t)= {}_{H}I^{\alpha}g(t)+c_{0},$$

and by integration of the inequality (1.4), we obtain

$$|v(t)- {}_{H}I^{\alpha}g_{v}(t)+c_{1}| \leq \frac{\varepsilon}{\Gamma(\alpha)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}h(s)\frac{ds}{s}.$$

By (H_1), we have

$$\begin{aligned}|v(t)-u(t)| &\leq |v(t)- {}_HI^{\alpha}g_v(t)-c_1| \\ &\quad + \frac{2\omega}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}|v-u|\frac{ds}{s}.\end{aligned}$$

Using (H_3), we can write

$$|v(t)-u(t)| \leq \sigma\eta_h h(t) + \frac{2\omega}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}\|v(s)-u(s)\|\frac{ds}{s}. \tag{3.7}$$

Hence, we have

$$|v(t)-u(t)| \leq \sigma\eta_h h(t) + \frac{2\omega(\log T)^{\alpha}}{\Gamma(\alpha+1)}\|v(s)-u(s)\|,$$

which implies that

$$\|v(s)-u(s)\|\left(1-\frac{2\omega(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right) \leq \sigma\eta_h h(t). \tag{3.8}$$

Then, for each $t\in[1,T]$

$$|u(t)-v(t)| \leq \frac{\eta_h}{1-\frac{2\omega(\log T)^{\alpha}}{\Gamma(\alpha+1)}}\sigma h(t). \tag{3.9}$$

So, the fractional boundary value problem (1.1) is Ulam-Hyers-Rassias stable. □

4. Examples

To illustrate our main results, we treat the following examples.

Example 4.1. Consider the Caputo-Hadamard type fractional pantograph equation

$$\begin{cases} {}_H^CD^{\frac{1}{2}}u(t) = \frac{1}{4}+\frac{3}{16e^{t+5}}x(t)+\frac{3}{16e^{t+5}}u\left(\frac{1}{2}t\right),\ t\in[1,e], \\ u(1) = 1-\frac{2}{19}u(\gamma), 1<\gamma<e. \end{cases} \tag{4.1}$$

For this example, we have $\alpha=\frac{1}{2}, \lambda=\frac{1}{2}$ and $T=e$.

On the other hand,

$$\begin{aligned}\varphi(t,u,v) &= \frac{1}{4}+\frac{3}{16e^{t+5}}u+\frac{3}{16e^{t+5}}v, \\ \theta(u) &= \frac{2}{19}u(\gamma),\quad 1<\gamma<e.\end{aligned}$$

For $t\in[1,2]$ and $(u_1,v_1),(u_2,v_2)\in\mathbb{R}^2$, we have

$$|\varphi(t,u_1,v_1)-\varphi(t,u_2,v_2)| \leq \frac{3}{16e^5}(|u_1-u_2|+|v_1-v_2|).$$

Hence the condition (H_1) holds with $\omega=\frac{3}{16e^5}$. Also, for any $u_1,v_1\in C([1,e])$, we have

$$|g(u_1)-g(v_1)| \leq \frac{2}{19}|u_1-v_1|.$$

So, (H_2) is satisfied with $\varpi=\frac{2}{19}$.

Thus conditions

$$2\omega\left(\log T\right)^{\alpha}=2.526\,7\times10^{-3}<\left(1-\varpi\right)\Gamma\left(\alpha+1\right)=0.792\,94,$$

and

$$2\omega\frac{\left(\log T\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}=2.851\,1\times10^{-3}<1,$$

are satisfied. It follows from Theorem 2.1, that the problem (4.1) has a unique solution on $[1,e]$, and from Theorem 3.1, the fractional problem (4.1) is Ulam-Hyers stable.

Example 4.2. Let us consider the following Caputo-Hadamard type fractional pantograph equation

$$\begin{cases} {}_{H}^{C}D^{\frac{1}{3}}u\left(t\right)=\frac{1}{2}+\frac{1}{21}\cos\left(t\right)u\left(t\right)+\frac{1}{21}u\left(\frac{1}{3}t\right),\ t\in\left[1,e\right], \\ u\left(1\right)=\frac{2}{5}-\sum_{i=1}^{n}d_{i}u\left(t_{i}\right), \end{cases} \tag{4.2}$$

where $1<t_1<t_2<...<t_n<e, d_i, i=1,2,...,n$ are given positive constants with $\sum_{i=1}^{n}d_i<\frac{1}{5}$.

Consider fractional pantograph equation with $\alpha=\frac{1}{3},\lambda=\frac{1}{3},\varphi\left(t,u,v\right)=\frac{1}{2}+\frac{1}{21}\cos\left(t\right)u+\frac{1}{21}v,\theta\left(u\right)=\sum_{i=1}^{n}d_iu\left(t_i\right).$

For $\left(u_1,v_1\right),\left(u_2,v_2\right)\in\mathbb{R}^2$ and $t\in\left[1,e\right]$, we have

$$\begin{aligned}\left|\varphi(t,u_1,v_1)-\varphi(t,u_2,v_2)\right| &\leq \frac{1}{21}\left|\cos\left(t\right)\right|\left|u_1-u_2\right|+\frac{1}{21}\left|v_1-v_2\right| \\ &\leq \frac{1}{21}\left(\left|u_1-u_2\right|+\left|v_1-v_2\right|\right).\end{aligned}$$

Hence hypothesis (H_1) is satisfied with with $\omega=\frac{5}{21}$. Also, for any $u_1,u_2\in C\left([1,e]\right)$, we have

$$g\left(u_1\right)-g\left(u_2\right)=\left|\sum_{i=1}^{n}d_iu_1\left(t_i\right)-\sum_{i=1}^{n}d_iu_2\left(t_i\right)\right|\leq\sum_{i=1}^{n}d_i\left|u_1-u_2\right|.$$

Hence hypothesis (H_2) is satisfied with $\varpi=\sum_{i=1}^{n}d_i<\frac{1}{5}$.

We can show that

$$\frac{2\omega\left(\log T\right)^{\alpha}}{\left(1-\varpi\right)\Gamma\left(\alpha+1\right)}\simeq0.133\,32<1.$$

Let $h\left(t\right)=t^3$. Then

$$\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}h(s)\frac{ds}{s}\leq\frac{\Gamma\left(4\right)}{\Gamma\left(\frac{13}{3}\right)}t^3=\eta_hh\left(t\right).$$

Thus hypothesis (H_3) is satisfied with $h\left(t\right)=t^3$ and $\eta_h=\frac{\Gamma(4)}{\Gamma(\frac{13}{3})}$.It follows from Theorem 2.1 that the fractional problem (4.2) as a unique solution on $[1,e]$, and from Theorem 3.2 problem (4.2) is Ulam-Hyers-Rassias stable.

5. Conclusion

In this paper, we have discussed the existence and Ulam-type stability of solutions for fractional pantograph differential equations with Caputo-Hadamard derivative. We have establish the existence and uniqueness results applying the Banach fixed point theorem. Moreover, the Ulam-Hyers stability and the generalized Ulam-Hyers stability have been discussed. To illustrate our theoretical results we have given two examples.

References

[1] A. Ardjounia and A. Djoudi, *Existence and uniqueness of solutions for nonlinear implicit Caputo-Hadamard fractional differential equations with nonlocal conditions,* Advances in the Theory of Nonlinear Analysis and its Applications, **3**(1) (2019), 46–52.

[2] Y. Bahous, Z. Dahmani and Z. Bekkouche, *A two-parameter singular fractional differential equation of Lane Emden type,* Turkish J. Ineq., **3**(1) (2019), 35–53.

[3] K. Balachandran, S. Kiruthika and J. J. Trujillo, *Existence of solutions of Nonlinear fractional pantograph equations,* Acta Mathematica Scientia, **33B** (2013), 1–9.

[4] R.L. Bagley, P. J .Torvik, *Fractional calculus in the transient analysis of viscoelastically damped structures,* AIAA. J., **23**(6)(1985), 918–925.

[5] E. Deniz, A. O. Akdemir and E. Yuksel, *New extensions of Chebyshev-Pólya-Szegö type inequalities via conformable integrals,* AIMS Mathematics, **5**(2) (2020), 956–965.

[6] G A. Derfel, A. Iserles, The pantograph equation in the complex plane, *J Math Anal Appl.*, 213 (1997), 117-132.

[7] E.M. Elsayed, S. Harikrishnan and K. Kanagarajan, *Analysis of nonlinear neutral pantograph differential equations with Hilfer fractional derivative,* MathLAB, **1**(2) (2018), 231–240.

[8] M. Houas and M. Bezziou, *Existence and stability results for fractional differential equations with two Caputo fractional derivatives,* Facta Univ. Ser. Math. Inform., **34**(2) (2019), 341–357.

[9] S. Harikrishnan. K. Kanagarajan and D. Vivek, *Solutions of nonlocal initial value problems for fractional pantograph equation,* Journal Nonlinear Analysis and Application, **2018**(2) (2018), 136–144.

[10] S. Harikrishnan. K. Kanagarajan and D. Vivek, *Existence and stability results for boundary value problem for differential equation with $\psi-$Hilfer fractional derivative,* Journal of Applied Nonlinear Dynamic, **8**(2) (2019), 251–259.

[11] S. Harikrishnan R. Ibrahim and K. Kanagarajan, *Establishing the existence of Hilfer fractional pantograph equations with impulses,* Fundam. J. Math. Appl., **1**(1) (2018), 36–42.

[12] R.W. Ibrahim, *Ulam stability of boundary value problem,* Kragujevac. J. Math., **37**(2) (2013), 287–297.

[13] A. Iserles,On the generalized pantograph functional-differential equation, Eur. J. Appl. Math., **4** (1993), 1–38.

[14] A. Iserles, *Exact and discretized stability of the pantograph equation*, Appl. Numer. Math., **24** (1997), 295–308.

[15] A Iserles, Y. Liu, *On pantograph integro-differential equations*, J. Integral Equations Appl., **6** (1994), 213–237.

[16] F. Jarad, T. Abdeljawad, D. Baleanu, *Caputo-type modification of the Hadamard fractional derivatives*, Adv. Differ. Equ., **2012**(142) (2012), 1–8.

[17] V.V. Kulish, J.L. Lage, *Application of fractional calculus to fluid mechanics,* J. Fluids Eng., **124**(3) (2002), 803–806.

[18] N. Lungu, D. Popa, *Hyers-Ulam stability of a first order partial differential equation,* J. Math. Anal. Appl., **385** (2012), 86–91.

[19] D. Li, M Z. Liu, *Runge-Kutta methods for the multi-pantograph delay equation*, Appl. Math. Comput., **163** (2005), 383–395.

[20] D. Nie , S.Rashid, A.O. Akdemir, D. Baleanu and J. B Liu, *On some new weighted inequalities for differentiable exponentially convex and exponentially quasi-convex functions with applications,* Mathematics, **7** (2019), 1–12.

[21] J R. Ockendon, A B. Taylor, *The dynamics of a current collection system for an electric locomotive*, Proc. RSoc London, Ser. A., **322**(1971), 447–468.

[22] K. Oldham, *Fractional differential equations in electrochemistry,* Adv. Eng. Softw., **41**(1) (2010), 9–12.
[23] S. Rashid, F. Safdar, A. O. Akdemir, M. A. Noor and K. I. Noor, *Some new fractional integral inequalities for exponentially m-convex functions via extended generalized Mittag-Leffler function,* J. Inequal. Appl., **2019**(299) (2019), 1–17.
[24] I. A. Rus, *Ulam stabilities of ordinary differential equations in a Banach space,* Carpathian J. Math., **26** (2010), 103–107.
[25] D. Vivek K. Kanagarajan and S. Sivasundaram, *Dynamics and stability of pantograph equations via Hilfer fractional derivative,* Nonlinear Studies, **23**(4) (2016), 685–698.
[26] D. Vivek K. Kanagarajan and S. Sivasundaram, *On the behavior of solutions of Hilfer-Hadamard type fractional neutral pantograph equations with boundary conditions,* Communications in Applied Analysis, **22**(2) (2018), 211–232.
[27] D. Vivek K. Kanagarajan and S. Sivasundaram, *Dynamics and stability of* $q-$*fractional order pantograph equations with nonlocal condition,* Journal of Mathematics and Statistics, **14**(1) (2018), 64–71.

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