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REFINEMENT OF SOME BEREZIN NUMBER INEQUALITIES

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Abstract. In this work, we obtain a generalization and refinement of some Berezin number inequalities obtained by A. Taghavi *et al.*, in this paper [*Some upper bounds for the Berezin number of Hilbert space operators,* Filomat 33(14) (2019), 4353–4360]. Among other things, some inequalities for f-connection of operators are also provided.

1. Introduction

Let $B(\mathcal{H})$ denote the $\mathbb{C}^*-$Algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. An operator $A \in B(\mathcal{H})$ is called positive, denoted as $A \geq 0$ if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. The set of all positive operators is denoted by $B(\mathcal{H})^+$, and it is called positive definite denoted as $A > 0$ if $\langle Ax, x\rangle > 0$ for all nonzero $x \in \mathcal{H}$. The numerical range of A is defined by

$$W(A) := \{\langle Ax, x\rangle :\ x \in \mathcal{H},\ ||x|| = 1\}.$$

The numerical radius of A is defined as

$$w(A) = \sup\{|z|,\ z \in W(A)\}.$$

It is well-known that $w(.)$ defines a norm on $\mathcal{H}$, and is equivalent to the usual operator norm $||A|| = sup\{||Ax||,\ x \in \mathcal{H}, ||x|| = 1\}$. And for every $A \in B(\mathcal{H})$ we have

$$\frac{1}{2}||A|| \leq w(A) \leq ||A||.$$

Let Ω be a nonempty set. A functional Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ is a Hilbert space of complex valued functions, which has the property that point evaluations are continuous i.e. for each $\lambda \in \Omega$ the map $f \mapsto f(\lambda)$ is a continuous linear functional on $\mathcal{H}$. The Riesz representation theorem ensues that for each $\lambda \in \Omega$ there exists a unique element $k_\lambda \in \mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda\rangle$ for all $f \in \mathcal{H}$. The collection $\{k_\lambda\} : \lambda \in \Omega$ is called the reproducing kernel of $\mathcal{H}$. If $\{e_n\}$ is an orthonormal basis for a functional Hilbert space $\mathcal{H}$, then the reproducing kernel of $\mathcal{H}$ is given by $k_\lambda(z) = \sum_n \bar{e_n}(\lambda)e_n(z)$; (we can see [5, Problem 37]). For $\lambda \in \Omega$, let $\tilde{k}_\lambda := \frac{k_\lambda}{||k_\lambda||}$ be the normalized reproducing kernel of $\mathcal{H}$. For a bounded

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linear operator A on $\mathcal{H}$, the function $\tilde{A}$, defined on Ω by $\tilde{A}(\lambda) := \langle A\tilde{k}_\lambda, \tilde{k}_\lambda\rangle$, is the Berezin symbol of A, which firstly have been introduced by Berezin. ([2, 3]). The Berezin set and the Berezin number of the operator A are defined by

$$Ber(A) := \{\tilde{A}(\lambda), \lambda \in \Omega\}, \ \ and \ \ ber(A) := \sup\{|\tilde{A}(\lambda)|, \lambda \in \Omega\}.$$

Clearly, the Berezin symbol $\tilde{A}$ is a bounded function on Ω whose values lie in the numerical range of the operator A, and hence

$$Ber(A) \subseteq W(A), \ \ and \ \ ber(A) \leq w(A).$$

The Berezin norm of an operator A is given by

$$||A||_{ber} = \sup\{|\langle A\tilde{k}_{\lambda_1}, \tilde{k}_{\lambda_2}\rangle| : \lambda_1, \lambda_2 \in \Omega\}.$$

Let $A, B \in B(\mathcal{H})$ be positive invertible operators and $\alpha \in [0, 1]$. The $\alpha-$weighted operators geometric mean of A and B, denoted by $A\sharp_\alpha B$, is defined as

$$A\sharp_\alpha B = A^{1/2}\Big(A^{-1/2}BA^{-1/2}\Big)^\alpha A^{1/2}.$$

2. Prerequisites

To prove our Berezin number inequalities, we need the following results concerning Young's inequality.

The well-known Young's inequality, for scalars asserts that for all positive real numbers a, b and $0 \leq \alpha \leq 1$,

$$a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha)b.$$

F. Kittaneh, and Y. Al- Manasrah [12], gave the following refinement of Young inequality as

$$a^\alpha b^{1-\alpha} + r_0(\sqrt{a} - \sqrt{b})^2 \leq \alpha a + (1-\alpha)b, \tag{2.1}$$

where $r_0 = \min\{\alpha, 1-\alpha\}$.

Recently, Manasrah and Kittaneh gave the following generalization refinement of Young's inequality as follows

Theorem 2.1. *Let a and b be two positive numbers and $0 \leq \alpha \leq 1$. Then for all positive integer m, we have*

$$\Big(a^\alpha b^{1-\alpha}\Big)^m + r_0^m\Big(a^{\frac{m}{2}} - b^{\frac{m}{2}}\Big)^2 \leq \Big(\alpha a + (1-\alpha)b\Big)^m, \tag{2.2}$$

where $r_0 = \min\{\alpha, 1-\alpha\}$.

Ighachane and Akkouchi [7], gave a new generalization refinement of Young's inequality as

Theorem 2.2. *Let a and b be two positive numbers and $0 \leq \alpha \leq 1$. Then for all positive integer m, we have*

$$r_0^m\Big(a^{\frac{m}{2}} - b^{\frac{m}{2}}\Big)^2 \leq r_0^m\Big(\frac{b^{m+1} - a^{m+1}}{b-a} - (m+1)(ab)^{\frac{m}{2}}\Big) \leq \Big(\alpha a + (1-\alpha)b\Big)^m - \Big(a^\alpha b^{1-\alpha}\Big)^m,$$

where $r_0 = \min\{\alpha, 1-\alpha\}$.

Later, the same authors in [8], gave a further refinement of Young's inequality as follows

Theorem 2.3. *Let a and b be two positive numbers and $0 \leq \alpha \leq 1$. Then for all positive integer m we have*

$$\begin{aligned}\Big(a^{\alpha}b^{1-\alpha}\Big)^{m} &+ r_0^m\Big(\frac{b^{m+1}-a^{m+1}}{b-a} - (m+1)(ab)^{\frac{m}{2}}\Big)\\ &+ r_m\Big(((ab)^{\frac{m}{4}} - b^{\frac{m}{2}})^2\chi_{(0,\frac{1}{2}]}(\alpha) + ((ab)^{\frac{m}{4}} - a^{\frac{m}{2}})^2\chi_{(\frac{1}{2},1]}(\alpha)\Big)\\ &\leq \Big(\alpha a + (1-\alpha)b\Big)^{m},\end{aligned}$$

where $r_0 = \min\{\alpha, 1-\alpha\}$, $r_m = \min\{(m+1)r_0^m, (1-r_0)^m - r_0^m\}$ and $\chi_I(\alpha)$ the characteristic function.

We know from [6] that for $\alpha \in [0,1]$ and $r \geq 1$,

$$\alpha a + (1-\alpha)b \leq (\alpha a^r + (1-\alpha)b^r)^{1/r}. \tag{2.3}$$

It follows from (2.3) and Theorem 2.2 that

$$\Big(a^{\alpha}b^{1-\alpha}\Big)^{m} + r_0^m\Big(\frac{b^{m+1}-a^{m+1}}{b-a} - (m+1)(ab)^{\frac{m}{2}}\Big) \leq \Big(\alpha a^r + (1-\alpha)b^r\Big)^{\frac{m}{r}}, \tag{2.4}$$

where $r_0 = \min\{\alpha, 1-\alpha\}$. In particular, for $\alpha = \frac{1}{2}$, we get

$$\Big(a^{1/2}b^{1/2}\Big)^{m} \leq \frac{1}{2^{\frac{m}{r}}}\Big(a^r + b^r\Big)^{\frac{m}{r}} - \frac{1}{2^m}\Big(\frac{b^{m+1}-a^{m+1}}{b-a} - (m+1)(ab)^{\frac{m}{2}}\Big). \tag{2.5}$$

We need also the following basic lemmas:

In 1952, Kato [9] showed the mixed Schwarz inequality, which asserts

Lemma 2.1. *Let $A \in \mathcal{B}(\mathcal{H})$ and $\alpha \in (0,1)$. Then*

$$|\langle Ax, y\rangle|^2 \leq \langle |A|^{2\alpha}x, x\rangle\langle |A^*|^{2(1-\alpha)}y, y\rangle, \tag{2.6}$$

for all $x, y \in \mathcal{H}$.

The next lemma is a generalization of the mixed Schwarz inequality, this lemma is proved by F. Kittaneh [11].

Lemma 2.2. *Let $A \in \mathcal{B}(\mathcal{H})$ and let f and g be non-negative continuous functions on $[0, +\infty)$ such that $f(t)g(t) = t$ for all $t \in [0, +\infty)$. Then*

$$|\langle Ax, y\rangle|^2 \leq ||f(|A|)x||||g(|A^*|)y||, \tag{2.7}$$

for all $x, y \in \mathcal{H}$.

The third Lemma follows from spectral theorem for positive operators and Jensen's inequality, this lemma is proved in [13].

Lemma 2.3. *(McCarthy inequality) Let $A \in \mathcal{B}(\mathcal{H})$ $A \geq 0$ and let $x \in \mathcal{H}$ be any unit vector. Then*

(a) $\langle Ax, x\rangle^p \leq \langle A^p x, x\rangle \quad for \;\; p \geq 1,$

(b) $\langle A^p x, x\rangle \leq \langle Ax, x\rangle^p \quad for \;\; 0 < p \leq 1.$

Dragomir in [4] obtained an useful extension for four operators of the Schwarz inequality as following.

Theorem 2.4. [4] *Let $A, B, C, D \in B(\mathcal{H})$, then for all $x, y \in \mathcal{H}$ we have the following inequality*

$$|\langle DCBAx, y\rangle|^2 \leq \langle A^*|B|^2Ax, x\rangle\langle D|C^*|^2D^*y, y\rangle.$$

3. Refinement of some Berezin number inequalities

In this section we provide some improvements to some inequalities for the Berezin number due to Ali Taghavi, Tahere Azimi Roushan, and Vahid Darvish [15].

Our first main result in this section is the following Theorem.

Theorem 3.1. *Let $A, B, X \in B(\mathcal{H})$ be such that A, B are positive definite, and $\alpha \in (0,1)$. Then for $m = 1, 2, \ldots,$ and $p \geq 2m$, we have*

$$\begin{aligned} ber^p\Big((A\sharp_\alpha B)X\Big) &\leq ber\Big(\alpha(X^*AX)^{\frac{p}{2m\alpha}} + (1-\alpha)(A\sharp_{2\alpha}B)^{\frac{p}{2m(1-\alpha)}}\Big)^m \\ &- \inf_{\lambda\in\Omega}(\zeta_1(\lambda)+\zeta_2(\lambda)), \end{aligned}$$

where

$$\begin{aligned} \zeta_1(\lambda) &= r_0^m\Big(\frac{\langle (A\sharp_{2\alpha}B)^{\frac{p}{2(1-\alpha)m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{m+1} - \langle (X^*AX)^{\frac{p}{2\alpha m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{m+1}}{\langle (A\sharp_{2\alpha}B)^{\frac{p}{2(1-\alpha)m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle - \langle (X^*AX)^{\frac{p}{2\alpha m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle} \\ &\qquad -(m+1)[\langle (X^*AX)^{\frac{p}{2\alpha m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle (A\sharp_{2\alpha}B)^{\frac{p}{2m(1-\alpha)}}\hat{k}_\lambda, \hat{k}_\lambda\rangle]^{\frac{m}{2}}\Big), \end{aligned}$$

and

$$\begin{aligned} \zeta_2(\lambda) &= \\ & r_m\Big[\Big([\langle (X^*AX)^{\frac{p}{2\alpha m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle (A\sharp_{2\alpha}B)^{\frac{p}{2(1-\alpha)m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle]^{\frac{m}{4}} - \langle (A\sharp_{2\alpha}B)^{\frac{p}{2m(1-\alpha)}}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{m}{2}}\Big)^2 \\ &\qquad \times\chi_{(0,\frac{1}{2}]}(\alpha) \\ &+\Big([\langle (X^*AX)^{\frac{p}{2\alpha m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle (A\sharp_{2\alpha}B)^{\frac{p}{2(1-\alpha)m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle]^{\frac{m}{4}} - \langle (X^*AX)^{\frac{p}{2\alpha m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{m}{2}}\Big)^2 \\ &\qquad \times\chi_{(\frac{1}{2},1]}(\alpha)\Big], \end{aligned}$$

where $r_0 = \min\{\alpha, 1-\alpha\}$, $r_m = \min\{(m+1)r_0^m, (1-r_0)^m - r_0^m\}$.

Proof. We have

$$\begin{aligned}
|\langle (A\sharp_\alpha B)X\hat{k}_\lambda,\hat{k}_\lambda\rangle|^p &= |\langle A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2}X\hat{k}_\lambda,\hat{k}_\lambda\rangle|^p\\
&\le \langle X^*AX\hat{k}_\lambda,\hat{k}_\lambda\rangle^{\frac{p}{2}}\langle A^{1/2}(A^{-1/2}BA^{-1/2})^{2\alpha}A^{1/2}\hat{k}_\lambda,\hat{k}_\lambda\rangle^{\frac{p}{2}}\\
&\quad \text{(by Theorem 2.4)}\\
&= \left(\langle X^*AX\hat{k}_\lambda,\hat{k}_\lambda\rangle^{\frac{p}{2m}}\langle A^{1/2}(A^{-1/2}BA^{-1/2})^{2\alpha}A^{1/2}\hat{k}_\lambda,\hat{k}_\lambda\rangle^{\frac{p}{2m}}\right)^m\\
&\le \left(\langle (X^*AX)^{\frac{p}{2m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle^{\frac{\alpha}{\alpha}}\langle (A\sharp_{2\alpha}B)^{\frac{p}{2m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle^{\frac{1-\alpha}{1-\alpha}}\right)^m\\
&\quad \text{(by Lemma 2.3 (a))}\\
&\le \left(\langle (X^*AX)^{\frac{p}{2\alpha m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle^{\alpha}\langle (A\sharp_{2\alpha}B)^{\frac{p}{2m(1-\alpha)}}\hat{k}_\lambda,\hat{k}_\lambda\rangle^{1-\alpha}\right)^m\\
&\quad \text{(by Lemma 2.3 (a)).}
\end{aligned}$$

So,

$$\begin{aligned}
|\langle (A\sharp_\alpha B)X\hat{k}_\lambda,\hat{k}_\lambda\rangle|^p &\le \left(\alpha\langle (X^*AX)^{\frac{p}{2\alpha m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle+(1-\alpha)\langle (A\sharp_{2\alpha}B)^{\frac{p}{2m(1-\alpha)}}\hat{k}_\lambda,\hat{k}_\lambda\rangle\right)^m\\
&\quad -(\zeta_1(\lambda)+\zeta_2(\lambda))\\
&\quad \text{(by Theorem 2.3)}\\
&\le ber\left(\alpha(X^*AX)^{\frac{p}{2m\alpha}}+(1-\alpha)(A\sharp_{2\alpha}B)^{\frac{p}{2m(1-\alpha)}}\right)^m\\
&\quad -(\zeta_1(\lambda)+\zeta_2(\lambda)).
\end{aligned}$$

Taking the supremum over $\lambda\in\Omega$, we deduce the result. □

The following Theorem is proved in [15].

Theorem 3.2. [15] *Let $A_i,B_i,T_i\in B(\mathcal{H})$ for $i=1,2,3,\dots n$, and let f and g be non-negative continuous functions on $[0,+\infty)$ such that $f(t)g(t)=t$, for all $t\in[0,+\infty)$. Then for $p\ge 1$, we have*

$$ber^p\Big(\sum_{i=1}^n A_i^*T_iB_i\Big)\le \frac{n^{p-1}}{2}ber\Big[\sum_{i=1}^n\Big(([B_i^*f^2(|T_i|)B_i]^p+[A_i^*g^2(|T_i^*|)A_i]^p)\Big)\Big].$$

The second main result in this section is generalization and refinement of the above Theorem.

Theorem 3.3. *Let $A_i,B_i,T_i\in B(\mathcal{H})$ for $i=1,2,3,\dots,n$, and let f and g be non-negative continuous functions on $[0,+\infty)$ such that $f(t)g(t)=t$, for all $t\in[0,+\infty)$. Then for $m=1,2,3,\dots,$ and $r\ge m$, $p\ge m$, we have*

$$\begin{aligned}
ber^p\Big(\sum_{i=1}^n A_i^*T_iB_i\Big) &\le \frac{n^{p-m/r}}{2^{m/r}}ber\Big(\sum_{i=1}^n([B_i^*f^2(|T_i|)B_i]^{\frac{pr}{m}}+[A_i^*g^2(|T_i^*|)A_i]^{\frac{pr}{m}})\Big)^{\frac{m}{r}}\\
&\quad -\inf_{\lambda\in\Omega}\xi(\lambda).
\end{aligned}$$

where

$$\xi(\lambda) = \frac{n^{p-1}}{2^m}\sum_{i=1}^{n}\Big(\frac{\langle [B_i^* f^2(|T_i^*|)B_i]^{\frac{p}{m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle^{m+1} - \langle [A_i^* g^2(|T_i|)A_i]^{\frac{p}{m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle^{m+1}}{\langle [B_i^* f^2(|T_i^*|)B_i]^{\frac{p}{m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle - \langle [A_i^* g^2(|T_i|)A_i]^{\frac{p}{m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle}$$
$$-(m+1)[\langle [A_i^* g^2(|T_i|)A_i]^{\frac{p}{m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle\langle [B_i^* f^2(|T_i^*|)B_i]^{\frac{p}{m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle]^{\frac{m}{2}}\Big).$$

Proof. We have,

$$\begin{aligned}
\Big|\Big\langle \sum_{i=1}^{n} A_i^* T_i B_i \hat{k}_\lambda,\hat{k}_\lambda\Big\rangle\Big|^p &= \Big|\sum_{i=1}^{n}\langle T_i B_i\hat{k}_\lambda, A_i\hat{k}_\lambda\rangle\Big|^p \leq \Big[\sum_{i=1}^{n}|\langle T_i B_i\hat{k}_\lambda, A_i\hat{k}_\lambda\rangle|\Big]^p \\
&\leq \Big(\sum_{i=1}^{n}||f(|T_i|)B_i\hat{k}_\lambda||||g(|T_i^*|)A_i\hat{k}_\lambda||\Big)^p \text{ (by Lemma 2.2)} \\
&\leq \Big(\sum_{i=1}^{n}\langle f(|T_i|)B_i\hat{k}_\lambda, f(|T_i|)B_i\hat{k}_\lambda\rangle^{\frac{1}{2}}\langle g(|T_i^*|)A_i\hat{k}_\lambda, g(|T_i^*|)A_i\hat{k}_\lambda\rangle^{\frac{1}{2}}\Big)^p \\
&\leq n^{p-1}\sum_{i=1}^{n}\langle f(|T_i|)B_i\hat{k}_\lambda, f(|T_i|)B_i\hat{k}_\lambda\rangle^{\frac{p}{2}}\langle g(|T_i^*|)A_i\hat{k}_\lambda, g(|T_i^*|)A_i\hat{k}_\lambda\rangle^{\frac{p}{2}} \\
&\qquad \text{by convexity of the function } t \longrightarrow t^p \\
&\leq n^{p-1}\sum_{i=1}^{n}\Big(\langle B_i^* f^2(|T_i|)B_i\hat{k}_\lambda,\hat{k}_\lambda\rangle^{\frac{p}{2m}}\langle A_i^* g^2(|T_i^*|)A_i\hat{k}_\lambda,\hat{k}_\lambda\rangle^{\frac{p}{2m}}\Big)^m \\
&\leq n^{p-1}\sum_{i=1}^{n}\Big(\langle [B_i^* f^2(|T_i|)B_i]^{\frac{p}{m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle^{\frac{1}{2}}\langle [A_i^* g^2(|T_i^*|)A_i]^{\frac{p}{m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle^{\frac{1}{2}}\Big)^m \\
&\qquad \text{(by Lemma 2.3 (a))} \\
&\leq n^{p-1}\sum_{i=1}^{n}\Big[\frac{1}{2}\langle [B_i^* f^2(|T_i|)B_i]^{\frac{p}{m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle^r + \langle [A_i^* g^2(|T_i^*|)A_i]^{\frac{p}{m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle^r\Big]^{\frac{m}{r}} - \xi(\lambda) \\
&\qquad \text{(by inequality (2.5))} \\
&\leq n^{p-1}\frac{n^{1-m/r}}{2^{m/r}}\Big\langle \sum_{i=1}^{n}\Big(([B_i^* f^2(|T_i^*|)B_i]^{\frac{pr}{m}} + [A_i^* g^2(|T_i^*|)A_i]^{\frac{pr}{m}})\Big)\hat{k}_\lambda,\hat{k}_\lambda\Big\rangle^{\frac{m}{r}} - \xi(\lambda)
\end{aligned}$$

by Lemma 2.3 and concavity of the function $t \longrightarrow t^{\frac{m}{r}}$. Taking the supremum over λ, we deduce the result. This completes the proof. □

Remark 3.1. Taking $m = r = 1$, in Theorem 3.4 we obtain a refinement of Theorem 3.2 obtained by Ali Taghavi et al., in [15].

Choosing $f(t) = g(t) = \sqrt{t}$, and $T_i = I$ for $i = 1, 2, \ldots, n$, in Theorem 3.4 we obtain the following simpler form.

Corollary 3.1. *Let $A_i, B_i, \in B(\mathcal{H})$ for $(i = 1, 2, 3, \ldots, n)$. Then for $m = 1, 2, 3, \ldots,$ and $r, p \geq m$, we have*

$$\begin{aligned}
ber^p\Big(\sum_{i=1}^{n} A_i^* B_i\Big) &\leq \frac{n^{p-m/r}}{2^{m/r}} ber^{\frac{m}{r}}\Big(\sum_{i=1}^{n}(|B_i|^{\frac{2pr}{m}} + |A_i|^{\frac{2pr}{m}}\Big) \\
&- \inf_{\lambda\in\Omega}\xi(\lambda),
\end{aligned}$$

where

$$\xi(\lambda) = \frac{n^{p-1}}{2^m}\sum_{i=1}^{n}\Big(\frac{\langle |B_i|^{\frac{2p}{m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle^{m+1}-\langle |A_i|^{\frac{2p}{m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle^{m+1}}{\langle |B_i|^{\frac{2p}{m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle-\langle |A_i|^{\frac{2p}{m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle} -(m+1)[\langle |A_i|^{\frac{2p}{m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle\langle |B_i|^{\frac{2p}{m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle]^{\frac{m}{2}}\Big).$$

This theorem is proved in [15].

Theorem 3.4. [15] *Let $A_i, B_i, T_i \in B(\mathcal{H})$ for $(i = 1, 2, 3, \ldots, n)$, and let f and g be non-negative functions on $[0, +\infty)$ which are continuous and satisfy the relation $f(t)g(t) = t$ for all $t \in [0, +\infty)$. Then for $\alpha \in [0, 1]$ and $r \geq 2\max\{\alpha, 1-\alpha\}$*

$$ber^p\Big(\sum_{i=1}^{n} A_i^*T_iB_i\Big) \leq n^{p-1}ber\Big(\sum_{i=1}^{n}(\alpha[B_i^*f^2(|T_i|)B_i]^{\frac{p}{2\alpha}}+(1-\alpha)[A_i^*g^2(|T_i^*|)A_i]^{\frac{p}{2(1-\alpha)}})\Big).$$

The third main result in this section is generalization and refinement of Theorem 3.4.

Theorem 3.5. *Let $A_i, B_i, T_i \in B(\mathcal{H})$ for $i = 1, 2, 3, \ldots, n$, and let f and g be non-negative continuous functions on $[0, +\infty)$ such that $f(t)g(t) = t$, for all $t \in [0, +\infty)$. Then for $m = 1, 2, 3, \ldots,$ and $r \geq 1$, $p \geq 2m\max\{\alpha, 1-\alpha\}$, we have*

$$\begin{aligned} &ber^p\Big(\sum_{i=1}^{n} A_i^*T_iB_i\Big)\\ &\leq n^{p-\frac{m}{r}}ber^{\frac{m}{r}}\Big(\sum_{i=1}^{n}(\alpha[B_i^*f^2(|T_i|)B_i]^{\frac{pr}{\alpha m}}+(1-\alpha)[A_i^*g^2(|T_i^*|)A_i]^{\frac{pr}{m(1-\alpha)}})\Big)\\ &\quad-\inf_{\lambda\in\Omega}\xi(\lambda), \end{aligned}$$

where

$$\xi(\lambda) = \frac{n^{p-1}}{2^m}\sum_{i=1}^{n}\Big(\frac{\langle [B_i^*f^2(|T_i^*|)B_i]^{\frac{p}{2\alpha m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle^{m+1}-\langle [A_i^*g^2(|T_i|)A_i]^{\frac{p}{2m(1-\alpha)}}\hat{k}_\lambda,\hat{k}_\lambda\rangle^{m+1}}{\langle [B_i^*f^2(|T_i^*|)B_i]^{\frac{p}{2\alpha m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle-\langle [A_i^*g^2(|T_i|)A_i]^{\frac{p}{2m(1-\alpha)}}\hat{k}_\lambda,\hat{k}_\lambda\rangle} -(m+1)[\langle [A_i^*g^2(|T_i|)A_i]^{\frac{p}{2m(1-\alpha)}}\hat{k}_\lambda,\hat{k}_\lambda\rangle\langle [B_i^*f^2(|T_i^*|)B_i]^{\frac{p}{2\alpha m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle]^{\frac{m}{2}}\Big).$$

Proof. Using similar arguments as used in Theorem 3.2, we have,

$$\begin{aligned} &\Big|\Big\langle\sum_{i=1}^{n}A_i^*T_iB_i\hat{k}_\lambda,\hat{k}_\lambda\Big\rangle\Big|^p\\ &\leq n^{p-1}\sum_{i=1}^{n}\langle f(|T_i|)B_i\hat{k}_\lambda, f(|T_i|)B_i\hat{k}_\lambda\rangle^{\frac{p}{2}}\langle g(|T_i^*|)A_i\hat{k}_\lambda, g(|T_i^*|)A_i\hat{k}_\lambda\rangle^{\frac{p}{2}}\\ &\leq n^{p-1}\sum_{i=1}^{n}\Big(\langle [B_i^*f^2(|T_i|)B_i]\hat{k}_\lambda,\hat{k}_\lambda\rangle^{\frac{\alpha p}{2\alpha m}}\langle [A_i^*g^2(|T_i^*|)A_i]\hat{k}_\lambda,\hat{k}_\lambda\rangle^{\frac{p(1-\alpha)}{2m(1-\alpha)}}\Big)^m. \end{aligned}$$

So,

$$
\begin{aligned}
&\Big|\Big\langle \sum_{i=1}^{n} A_i^* T_i B_i \hat{k}_\lambda, \hat{k}_\lambda \Big\rangle\Big|^p \\
&\leq n^{p-1}\sum_{i=1}^{n}\Big(\langle [B_i^* f^2(|T_i|)B_i]^{\frac{p}{2\alpha m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle^{\alpha}\langle [A_i^* g^2(|T_i^*|)A_i]^{\frac{p}{2m(1-\alpha)}}\hat{k}_\lambda,\hat{k}_\lambda\rangle^{1-\alpha}\Big)^m \\
&\qquad\qquad\qquad\qquad\text{(by Lemma 2.3)} \\
&\leq n^{p-1}\sum_{i=1}^{n}\Big[\alpha\langle [B_i^* f^2(|T_i|)B_i]^{\frac{p}{2\alpha m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle^{r} + (1-\alpha)\langle [A_i^* g^2(|T_i^*|)A_i]^{\frac{p}{2m(1-\alpha)}}\hat{k}_\lambda,\hat{k}_\lambda\rangle^{r}\Big]^{\frac{m}{r}} \\
&\qquad\qquad\qquad\qquad -\xi(\lambda) \text{ (by inequality (2.4))} \\
&\leq n^{p-1}n^{1-m/r}\Big\langle \sum_{i=1}^{n}\Big((\alpha[B_i^* f^2(|T_i|)B_i]^{\frac{pr}{2\alpha m}} + (1-\alpha)[A_i^* g^2(|T_i^*|)A_i]^{\frac{pr}{2m(1-\alpha)}})\Big)\hat{k}_\lambda,\hat{k}_\lambda\Big\rangle^{\frac{m}{r}} \\
&\qquad\qquad\qquad\qquad -\xi(\lambda)
\end{aligned}
$$

by Lemma 2.3 and concavity of the function $t \longrightarrow t^{\frac{m}{r}}$. Taking the supremum over $\lambda \in \Omega$, we get the desired inequality. This completes the proof. $\square$

Corollary 3.2. *Let $T_i \in B(\mathcal{H})$ for $i = 1, 2, 3, \dots, n$, and let f and g be non-negative continuous functions on $[0, +\infty)$ such that $f(t)g(t) = t$, for all $t \in [0, +\infty)$. Then for $m = 1, 2, 3, \dots,$ and $r \geq 1$, $p \geq 2m\max\{\alpha, 1-\alpha\}$, we have*

$$
\begin{aligned}
ber^p\Big(\sum_{i=1}^{n} T_i\Big) &\leq n^{p-\frac{m}{r}} ber\Big(\sum_{i=1}^{n}(\alpha[f^2(|T_i|)]^{\frac{pr}{2\alpha m}} + (1-\alpha)[g^2(|T_i^*|)]^{\frac{pr}{2(1-\alpha)m}})\Big)^{\frac{m}{r}} \\
&- \inf_{\lambda\in\Omega}\xi(\lambda),
\end{aligned}
$$

where

$$
\begin{aligned}
\xi(\lambda) &= \frac{n^{p-1}}{2^m}\sum_{i=1}^{n}\Big(\frac{\langle [f^2(|T_i^*|)]^{\frac{p}{2\alpha m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle^{m+1} - \langle [g^2(|T_i|)]^{\frac{p}{2m(1-\alpha)}}\hat{k}_\lambda,\hat{k}_\lambda\rangle^{m+1}}{\langle [f^2(|T_i^*|)]^{\frac{p}{2\alpha m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle - \langle [g^2(|T_i|)]^{\frac{p}{2m(1-\alpha)}}\hat{k}_\lambda,\hat{k}_\lambda\rangle} \\
&\qquad -(m+1)[\langle [g^2(|T_i|)]^{\frac{p}{2m(1-\alpha)}}\hat{k}_\lambda,\hat{k}_\lambda\rangle\langle [f^2(|T_i^*|)]^{\frac{p}{2\alpha m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle]^{\frac{m}{2}}\Big).
\end{aligned}
$$

4. Some results for f-connections of operators

The aim of this section is to give some Berezin number inequalities for the f-connection of operators.

Let f be a continuous function defined on the real interval J containing the spectrum of the operator $A^{-1/2}BA^{-1/2}$, where B is a self-adjoint operators and A is a positive invertible operator. By using the continuous functional calculus, Tafazoli et al. [14] defined f-connection σ_f as follows:

$$A\sigma_f B = A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2}.$$

The first main result of this section and reads as follows.

Theorem 4.1. *Let $A, B, X \in B(\mathcal{H})$ be such that A, B are positive definite. Then for all integer m, and $r \geq 1$, we have*

$$ber^m((A\sigma_f B)X) \leq 2^{\frac{-m}{r}} ber^{\frac{m}{r}}\Big((X^* A^{1/2} f^2(A^{-1/2}BA^{-1/2})A^{1/2}X)^r + A^r\Big) - \inf_{\lambda\in\Omega} \xi(\lambda),$$

where

$$\xi(\lambda) = \frac{1}{2^m}\Big(\frac{\langle X^* A^{1/2} f^2(A^{-1/2}BA^{-1/2})A^{1/2}X\hat{k}_\lambda, \hat{k}_\lambda\rangle^{m+1} - \langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle^{m+1}}{\langle X^* A^{1/2} f^2(A^{-1/2}BA^{-1/2})A^{1/2}X\hat{k}_\lambda, \hat{k}_\lambda\rangle - \langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle}$$
$$-(m+1)(\langle X^* A^{1/2} f^2(A^{-1/2}BA^{-1/2})A^{1/2}X\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle)^{\frac{m}{2}}\Big).$$

Proof. We have,

$$\begin{aligned}
|\langle A\sigma_f B)X\hat{k}_\lambda, \hat{k}_\lambda\rangle|^m \;&=\; |\langle A^{1/2} f(A^{-1/2}BA^{-1/2})A^{1/2}X\hat{k}_\lambda, \hat{k}_\lambda\rangle|^m \\
&\leq ||f(A^{-1/2}BA^{-1/2})A^{1/2}X\hat{k}_\lambda||^m ||A^{1/2}\hat{k}_\lambda||^m \\
&\leq \Big(\langle f(A^{-1/2}BA^{-1/2})A^{1/2}X\hat{k}_\lambda, f(A^{-1/2}BA^{-1/2})A^{1/2}X\hat{k}_\lambda\rangle^{\frac{1}{2}}\langle A^{1/2}\hat{k}_\lambda, A^{1/2}\hat{k}_\lambda\rangle^{\frac{1}{2}}\Big)^m \\
&\leq \Big(\langle X^* A^{1/2} f^2(A^{-1/2}BA^{-1/2})A^{1/2}X\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{1}{2}}\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{1}{2}}\Big)^m \\
&\leq 2^{\frac{-m}{r}}\Big(\langle X^* A^{1/2} f^2(A^{-1/2}BA^{-1/2})A^{1/2}X\hat{k}_\lambda, \hat{k}_\lambda\rangle^r + \langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle^r\Big)^{\frac{m}{r}} - \xi(\lambda) \\
&\qquad \text{(by inequality (2.5))} \\
&\leq 2^{\frac{-m}{r}}\Big(\langle((X^* A^{1/2} f^2(A^{-1/2}BA^{-1/2})A^{1/2}X)^r + A^r)\hat{k}_\lambda, \hat{k}_\lambda\rangle\Big)^{\frac{m}{r}} - \xi(\lambda) \\
&\qquad \text{(by Lemma 2.3 (a)).}
\end{aligned}$$

Taking the supremum over $\lambda \in \Omega$, we deduce the result. □

Taking $X = I$, in Theorem 4.1, then we have the following corollary

Corollary 4.1. *Let $A, B \in B(\mathcal{H})$ be such that A, B are positive definite. Then for all integer m, and $r \geq 1$, we have*

$$ber(A\sigma_f B) \leq 2^{\frac{-m}{r}} ber^{\frac{m}{r}}\Big((A^{1/2} f^2(A^{-1/2}BA^{-1/2})A^{1/2})^r + A^r\Big) - \inf_{\lambda\in\Omega} \xi(\lambda).$$

where

$$\xi(\lambda) = \frac{1}{2^m}\Big(\frac{\langle A^{1/2} f^2(A^{-1/2}BA^{-1/2})A^{1/2}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{m+1} - \langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle^{m+1}}{\langle A^{1/2} f^2(A^{-1/2}BA^{-1/2})A^{1/2}\hat{k}_\lambda, \hat{k}_\lambda\rangle - \langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle}$$
$$-(m+1)(\langle A^{1/2} f^2(A^{-1/2}BA^{-1/2})A^{1/2}\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle)^{\frac{m}{2}}\Big)$$

Taking $r = 1$, $X = I$, and $f(t) = \sqrt{t}$, in Theorem 4.1, we have the following simplified form.

Corollary 4.2. *Let $A, B \in B(\mathcal{H})$ be such that A, B are positive definite. Then for all integer m, we have*

$$ber(A\sharp B) \leq 2^{-m} ber^m(A + B) - \inf_{\lambda\in\Omega} \xi(\lambda),$$

where

$$\xi(\lambda)=\frac{1}{2^m}\Big(\frac{\langle B\hat{k}_\lambda,\hat{k}_\lambda\rangle^{m+1}-\langle A\hat{k}_\lambda,\hat{k}_\lambda\rangle^{m+1}}{\langle B\hat{k}_\lambda,\hat{k}_\lambda\rangle-\langle A\hat{k}_\lambda,\hat{k}_\lambda\rangle}-(m+1)(\langle A\hat{k}_\lambda,\hat{k}_\lambda\rangle\langle B\hat{k}_\lambda,\hat{k}_\lambda\rangle)^{\frac{m}{2}}\Big).$$

The second main result of this section and reads as follows.

Theorem 4.2. *Let $A, B, X \in B(\mathcal{H})$ be such that A, B are positive definite. Then for all integer m, and $r > 1$, we have*

$$||(A\sigma_f B)X||_{ber}^m \le 2^{\frac{-m}{r}}\Big(||(X^*A^{1/2}f^2(A^{-1/2}BA^{-1/2})A^{1/2}X||_{ber}^r+||A||_{ber}^r\Big)^{\frac{m}{r}}-\inf_{\lambda_1,\lambda_2\in\Omega}\xi(\lambda_1,\lambda_2),$$

where

$$\xi(\lambda_1,\lambda_2)=\frac{1}{2^m}\Big(\frac{\langle X^*A^{1/2}f^2(A^{-1/2}BA^{-1/2})A^{1/2}X\hat{k}_{\lambda_1},\hat{k}_{\lambda_1}\rangle^{m+1}-\langle A\hat{k}_{\lambda_2},\hat{k}_{\lambda_2}\rangle^{m+1}}{\langle X^*A^{1/2}f^2(A^{-1/2}BA^{-1/2})A^{1/2}X\hat{k}_{\lambda_1},\hat{k}_{\lambda_1}\rangle-\langle A\hat{k}_{\lambda_2},\hat{k}_{\lambda_2}\rangle}-(m+1)(\langle X^*A^{1/2}f^2(A^{-1/2}BA^{-1/2})A^{1/2}X\hat{k}_{\lambda_1},\hat{k}_{\lambda_1}\rangle\langle A\hat{k}_{\lambda_2},\hat{k}_{\lambda_2}\rangle)^{\frac{m}{2}}\Big).$$

Proof. We have,

$$\begin{aligned}
|\langle A\sigma_f B)X\hat{k}_{\lambda_1},\hat{k}_{\lambda_2}\rangle|^m &= |\langle A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2}X\hat{k}_{\lambda_1},\hat{k}_{\lambda_2}\rangle|^m\\
&\le |\langle f(A^{-1/2}BA^{-1/2})A^{1/2}X\hat{k}_{\lambda_1},A^{1/2}\hat{k}_{\lambda_2}\rangle|^m\\
&\le \Big(\langle f(A^{-1/2}BA^{-1/2})A^{1/2}X\hat{k}_{\lambda_1},f(A^{-1/2}BA^{-1/2})A^{1/2}X\hat{k}_{\lambda_1}\rangle^{\frac{1}{2}}\langle A^{1/2}\hat{k}_{\lambda_2},A^{1/2}\hat{k}_{\lambda_2}\rangle^{\frac{1}{2}}\Big)^m\\
&= \Big(\langle X^*A^{1/2}f^2(A^{-1/2}BA^{-1/2})A^{1/2}X\hat{k}_{\lambda_1},\hat{k}_{\lambda_1}\rangle^{\frac{1}{2}}\langle A\hat{k}_{\lambda_2},\hat{k}_{\lambda_2}\rangle^{\frac{1}{2}}\Big)^m\\
&\le 2^{\frac{-m}{r}}\Big(\langle X^*A^{1/2}f^2(A^{-1/2}BA^{-1/2})A^{1/2}X\hat{k}_{\lambda_1},\hat{k}_{\lambda_1}\rangle^r+\langle A\hat{k}_{\lambda_2},\hat{k}_{\lambda_2}\rangle^r\Big)^{\frac{m}{r}}-\xi(\lambda_1,\lambda_2)\\
&\qquad(\text{ by inequality } (2.5)).
\end{aligned}$$

Taking the supremum over $\lambda_1, \lambda_2 \in \Omega$, we deduce the result. □

Letting $f(t) = \sqrt{t}$, then for $m = 1, 2, \ldots,$ we have the following corollary.

Corollary 4.3. *Let $A, B, X \in B(\mathcal{H})$ be such that A, B are positive definite. Then for all integer m, and $r > 1$, we have*

$$||(A\sharp B)X||_{ber}^m \le 2^{-m}(||X^*BX||_{ber}^r+||A||_{ber}^r)^{\frac{m}{r}}-\inf_{\lambda_1,\lambda_2\in\Omega}\xi(\lambda_1,\lambda_2),$$

where

$$\xi(\lambda_1,\lambda_2)=\frac{1}{2^m}\Big(\frac{\langle X^*BX\hat{k}_{\lambda_1},\hat{k}_{\lambda_1}\rangle^{m+1}-\langle A\hat{k}_{\lambda_2},\hat{k}_{\lambda_2}\rangle^{m+1}}{\langle X^*BX\hat{k}_{\lambda_1},\hat{k}_{\lambda_1}\rangle-\langle A\hat{k}_{\lambda_2},\hat{k}_{\lambda_2}\rangle}-(m+1)(\langle A\hat{k}_{\lambda_2},\hat{k}_{\lambda_2}\rangle\langle X^*BX\hat{k}_{\lambda_1},\hat{k}_{\lambda_1}\rangle)^{\frac{m}{2}}\Big).$$

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