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SUBCLASS OF BI-UNIVALENT FUNCTIONS SATISFYING SUBORDINATE CONDITIONS DEFINED BY FRASIN DIFFERENTIAL OPERATOR

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Abstract. In this present paper, we introduce a newly defined subclass $\mathfrak{S}^{\varrho,\rho}_{\mathfrak{E},k}(\lambda, x, y : \alpha, \zeta)$ of bi-univalent functions defined by Frasin differential operator satisfying subordinate conditions in the unit disk $\nabla = \{z \in \mathbb{C} : |z| < 1\}$. Fekete-Szego problem and the coefficient estimates of $|b_2|$ and $|b_3|$ for functions of this new class are established. Results acquired generalized some known results.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{a=2}^{\infty} b_a z^a \tag{1.1}$$

which are analytic in the open unit disk $\nabla = \{z \in \mathbb{C} : |z| < 1\}$. Let S be the subclass of A consisting of functions which are holomorphic and univalent in ∇.

Let the function $f(z)$ and $g(z)$ be analytic in ∇. Given the function $f(z), g(z) \in A$, $f(z)$ is subordinate to $g(z)$ if there exist a Schwarz function $w \in \leftthreetimes$, where

$$\leftthreetimes = \{w : w(0) = 0,\ |w(z)| < 1, \quad z \in \nabla\},$$

such that

$$f(z) = g(w(z)) \qquad (z \in \nabla).$$

We denote this subordination by

$$f(z) \prec g(z) \qquad (z \in \nabla).$$

In particular, if the function $g(z)$ is univalent in ∇, the above subordination is equivalent to

$$f(0) = g(0), \quad f(\nabla) \subset g(\nabla).$$

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A function $f(z) \in S$ has an inverse function $f^{-1}(z)$ satisfying the condition

$$f^{-1}(f(z)) = z$$

and

$$f(f^{-1}(w)) = w, \qquad (|w| < r_0(f); r_0(f) \geq 1/4).$$

Also, the inverse function f^{-1} is given by

$$f^{-1}(w) = g(w) = w - b_2 w^2 + (2b_2^2 - b_3)w^3 - (5b_2^3 - 5b_2 b_3 + b_4)w^4 + \cdots \tag{1.2}$$

An analytic function $f(z)$ is said to be bi-univalent in ∇ if both $f(z)$ and $f^{-1}(z)$ are univalent in ∇. The class of bi-univalent functions is denoted by $\mathfrak{E}$. For more details, see [7, 12, 14, 20, 25, 31, 32].
Also several authors have investigated on the coefficient bounds for various subclasses of bi-univalent function such as [3, 5, 8, 15, 16, 18, 21, 26, 28, 29, 33].
A function $f(z)$ of the form (1.1) is said to be in the class $\mathfrak{S}(\phi, x, y)$ if it satisfies the condition

$$\Re\left(\frac{(x-y)zf'(z)}{f(xz) - f(yz)}\right) > \phi$$

for some $0 \leq \phi < 1$, $x, y \in \mathbb{C}$ with $x \neq y$, $|x| \leq 1$, $|y| \leq 1$ and for all $z \in \nabla$. The class $\mathfrak{S}(\phi, x, y)$ was introduced by Frasin [10]. Note that when $x = 1$ we have the class $\mathfrak{S}(\phi, 1, y)$ studied by Owa et al. [17], also choosing $y = -1$ and $x = 1$ we have the class introduced by Sakaguchi [19].

Frasin [11] (see also [1, 2, 22, 27, 30]) introduced the differential operator $D^{\varrho}_{k,\rho} f(z)$ defined as follows:

$$D^0 f(z) = f(z) \tag{1.3}$$

$$D^1_{k,\rho} f(z) = (1-\rho)^k f(z) + (1 - (1-\rho)^k) z f'(z) = D_{k,\rho} f(z), \quad \rho > 0;\ k \in \mathbb{N}, \tag{1.4}$$

$$D^{\varrho}_{k,\rho} f(z) = D_{k,\rho}(D^{\varrho-1} f(z)) \quad (\varrho \in \mathbb{N}). \tag{1.5}$$

If $f(z)$ is given by (1.1), then from (1.4) and (1.5) we see that

$$D^{\varrho}_{k,\rho} f(z) = z + \sum_{a=2}^{\infty} \left(1 + (a-1) \sum_{d=1}^{k} \binom{k}{d} (-1)^{d+1} \rho^d \right)^{\varrho} b_a z^a, \quad \varrho \in \mathbb{N}_0 \tag{1.6}$$

Using the relation in (1.6), we have

$$C^k_d(\rho) z (D^{\varrho}_{k,\rho} f(z))' = D^{\varrho+1}_{k,\rho} f(z) - (1 - C^k_d(\rho)) D^{\varrho}_{k,\rho} f(z)$$

where $C^k_d(\rho) := \sum_{d=1}^{k} \binom{k}{d} (-1)^{d+1} \rho^d$.

Remark 1.1. We observe that

(a) When $k = 1$, we obtain the Al-Oboudi differential operator [4].
(b) When $k = \rho = 1$, we obtain the Salagean differential operator [23].

In [6], Babalola defined the class of λ-pseudo starlike functions of order β and prove that all Pseudo-starlike functions are Bazelivic of type $\left(1-\frac{1}{\lambda}\right)$, order $\beta^{\frac{1}{\lambda}}$ are univalent in ∇.

In this present paper, we introduce a new subclass of bi-univalent functions satisfying subordinate conditions and defined by Frasin differential operator. Also, we obtain the coefficient estimates for $|b_2|$ and $|b_3|$ and the Fekete-Szego problem for functions of the new class.

Definition 1.1. A function $f(z) \in \mathfrak{E}$ is said to be in the class $\mathfrak{S}^{\varrho,\rho}_{\mathfrak{E},k}(\lambda, x, y : \alpha, \zeta)$ if the following subordination hold

$$(1-\alpha)\frac{(x-y)z[(D^{\varrho}_{k,\rho}f(z))']^{\lambda}}{D^{\varrho}_{k,\rho}f(xz)-D^{\varrho}_{k,\rho}f(yz)}+\alpha\frac{(x-y)[(z(D^{\varrho}_{k,\rho}f(z))')']^{\lambda}}{(D^{\varrho}_{k,\rho}f(xz)-D^{\varrho}_{k,\rho}f(yz))'}\prec\zeta(z)$$

$$(1-\alpha)\frac{(x-y)w[(D^{\varrho}_{k,\rho}g(w))']^{\lambda}}{D^{\varrho}_{k,\rho}g(xw)-D^{\varrho}_{k,\rho}g(yw)}+\alpha\frac{(x-y)[(w(D^{\varrho}_{k,\rho}g(w))')']^{\lambda}}{(D^{\varrho}_{k,\rho}g(xw)-D^{\varrho}_{k,\rho}g(yw))'}\prec\zeta(w),$$

where $0 \le \alpha \le 1$, $x, y \in \mathbb{C}$ with $x \ne y$, $|x| \le 1$, $|y| \le 1$, $|\lambda| > 0$ and $g(w) = f^{-1}(w)$.

Remark 1.2. We have the following remarks:

(a) Putting $\varrho = 0$ we get the class $\mathfrak{S}_{\mathfrak{E}}(\lambda, x, y : \alpha, \zeta)$ of function $f(z) \in \mathfrak{E}$ which was studied by Emeka and Opoola [9].
(b) Putting $\varrho = 0$, $x = 1$ and $y = -1$, we get the class $\mathfrak{S}_{\mathfrak{E}}(\lambda, 1, -1 : \alpha, \zeta)$ of function $f(z) \in \mathfrak{E}$ which was studied by Eker and Seker [24].
(c) Putting $\varrho = 0$, $x = 1$, $y = 0$ and $\lambda = 1$, we get the class $\mathfrak{S}_{\mathfrak{E}}(1, 1, 0 : \alpha, \zeta)$ of function $f(z) \in \mathfrak{E}$ which was studied by Ali et al. [5].
(d) Putting $\varrho = 0$ and $\alpha = 0$ we get the class $\mathfrak{S}_{\mathfrak{E}}\left(\lambda, x, y : \left(\frac{1+z}{1-z}\right)^{\alpha}\right)$ of function $f(z) \in \mathfrak{E}$ which was studied by Emeka and Opoola [8].
(e) Putting $\varrho = 0$ and $\alpha = 0$ we get the class $\mathfrak{S}_{\mathfrak{E}}\left(\lambda, x, y : \left(\frac{1+(1-2\beta)z}{1-z}\right)\right)$ of function $f(z) \in \mathfrak{E}$ which was studied by Emeka and Opoola [8].
(f) Putting $\varrho = 0$, $\alpha = 0$, $x = 1$ and $y = 0$ we get the class $\mathfrak{S}_{\mathfrak{E}}\left(\lambda, 1, 0 : \left(\frac{1+z}{1-z}\right)^{\alpha}\right)$ of function $f(z) \in \mathfrak{E}$ which was studied by Joshi et al [13].
(g) Putting $\varrho = 0$, $\alpha = 0$, $x = 1$ and $y = 0$ we get the class $\mathfrak{S}_{\mathfrak{E}}\left(\lambda, 1, 0 : \left(\frac{1+(1-2\beta)z}{1-z}\right)\right)$ of function $f(z) \in \mathfrak{E}$ which was studied by Joshi et al [13].
(h) Putting $\varrho = 0$, $x = 1$, $y = 0$, $\alpha = 0$ and $\lambda = 1$, we get the class $\mathfrak{S}_{\mathfrak{E}}(1, 1, 0 : 0, \zeta)$ of function $f(z) \in \mathfrak{E}$ which was studied by Ali et al. [5].

2. Coefficient Estimates

Let

$$\zeta(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \quad (B_1 > 0), \tag{2.1}$$

be an analytic function with positive real part in ∇ with $\zeta(0) = 1$ and $\zeta'(0) > 0$. Also, let $\zeta(\nabla)$ be starlike with respect to 1 and symmetric with respect to the axis.

Theorem 2.1. *Let the function $f(z)$ of the form (1.1) be in the class $\mathfrak{S}^{\varrho,\rho}_{\mathfrak{E},k}(\lambda, x, y : \alpha, \zeta)$. Then*

$$|b_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\left|(3\lambda - x^2 - xy - y^2)(1+2\alpha)(1+2C_d^k(\rho))^{\varrho}B_1^2 + (1+C_d^k(\rho))^{2\varrho}\left[(1+3\alpha)\left((x+y)^2 - 2\lambda(x+y-\lambda+1)\right)B_1^2 - (B_2 - B_1)(1+\alpha)^2(2\lambda - x - y)^2\right]\right|}} \tag{2.2}$$

and

$$|b_3| \leq \frac{B_1^2}{(1+C_d^k(\rho))^{2\varrho}(1+\alpha)^2(2\lambda - x - y)^2} + \frac{B_1}{(1+2C_d^k(\rho))^{\varrho}(1+2\alpha)(3\lambda - x^2 - y^2 - xy)} \tag{2.3}$$

Proof. Let $f(z) \in \mathfrak{S}^{\varrho,\rho}_{\mathfrak{E},k}(\lambda, x, y : \alpha, \zeta)$. Then there are analytic functions $s, t : \nabla \longrightarrow \nabla$ with $s(0) = t(0) = 0$, satisfying

$$(1-\alpha)\frac{(x-y)z[(D^{\varrho}_{k,\rho}f(z))']^{\lambda}}{D^{\varrho}_{k,\rho}f(xz) - D^{\varrho}_{k,\rho}f(yz)} + \alpha\frac{(x-y)[(z(D^{\varrho}_{k,\rho}f(z))')']^{\lambda}}{(D^{\varrho}_{k,\rho}f(xz) - D^{\varrho}_{k,\rho}f(yz))'} = \zeta(s(z)) \tag{2.4}$$

$$(1-\alpha)\frac{(x-y)w[(D^{\varrho}_{k,\rho}g(w))']^{\lambda}}{D^{\varrho}_{k,\rho}g(xw) - D^{\varrho}_{k,\rho}g(yw)} + \alpha\frac{(x-y)[(w(D^{\varrho}_{k,\rho}g(w))')']^{\lambda}}{(D^{\varrho}_{k,\rho}g(xw) - D^{\varrho}_{k,\rho}g(yw))'} = \zeta(t(w)) \tag{2.5}$$

Define the functions q_1 and q_2 by

$$q_1(z) = \frac{1+s(z)}{1-s(z)} = 1 + g_1 z + g_2 z^2 + \cdots$$

and

$$q_2(z) = \frac{1+t(z)}{1-t(z)} = 1 + h_1 z + h_2 z^2 + \cdots$$

which is also equivalent to

$$s(z) = \frac{q_1(z)-1}{q_1(z)+1} = \frac{1}{2}g_1 z + \frac{1}{2}\left(g_2 - \frac{g_1^2}{2}\right)z^2 + \cdots \tag{2.6}$$

$$t(z) = \frac{q_2(z)-1}{q_2(z)+1} = \frac{1}{2}h_1 z + \frac{1}{2}\left(h_2 - \frac{h_1^2}{2}\right)z^2 + \cdots \tag{2.7}$$

Also q_1 and q_2 are both analytic in ∇ and $q_1(0) = q_2(0) = 1$, since $s, t : \nabla \longrightarrow \nabla$, the function q_1 and q_2 have positive real part in ∇ and hence $|h_i| \leq 2$ and $|g_i| \leq 2$. From (2.4), (2.5), (2.6) and (2.7), we have

$$(1-\alpha)\frac{(x-y)z[(D^{\varrho}_{k,\rho}f(z))']^{\lambda}}{D^{\varrho}_{k,\rho}f(xz) - D^{\varrho}_{k,\rho}f(yz)} + \alpha\frac{(x-y)[(z(D^{\varrho}_{k,\rho}f(z))')']^{\lambda}}{(D^{\varrho}_{k,\rho}f(xz) - D^{\varrho}_{k,\rho}f(yz))'}$$
$$= 1 + \frac{1}{2}B_1 g_1 z + \left(\frac{1}{4}B_2 g_1^2 + \frac{1}{2}B_1\left(g_2 - \frac{g_1^2}{2}\right)\right)z^2 + \cdots \tag{2.8}$$

$$(1-\alpha)\frac{(x-y)w[(D^{\varrho}_{k,\rho}g(w))']^{\lambda}}{D^{\varrho}_{k,\rho}g(xw)-D^{\varrho}_{k,\rho}g(yw)}+\alpha\frac{(x-y)[(w(D^{\varrho}_{k,\rho}g(w))')']^{\lambda}}{(D^{\varrho}_{k,\rho}g(xw)-D^{\varrho}_{k,\rho}g(yw))'}$$
$$=1+\frac{1}{2}B_1h_1w+\left(\frac{1}{4}B_2h_1^2+\frac{1}{2}B_1\left(h_2-\frac{h_1^2}{2}\right)\right)w^2+\cdots \quad (2.9)$$

From (2.8) and (2.9), we get

$$(1+C^k_d(\rho))^{\varrho}(1+\alpha)(2\lambda-x-y)b_2=\frac{1}{2}B_1g_1 \quad (2.10)$$

$$(1+3\alpha)\Big((x+y)^2-2\lambda(x+y-\lambda+1)\Big)(1+C^k_d(\rho))^{2\varrho}b_2^2+(3\lambda-x^2-xy-y^2)(2\alpha+1)$$
$$(1+2C^k_d(\rho))^{\varrho}b_3=\frac{1}{4}B_2g_1^2+\frac{1}{2}B_1\left(g_2-\frac{g_1^2}{2}\right) \quad (2.11)$$

$$-(1+C^k_d(\rho))^{\varrho}(1+\alpha)(2\lambda-x-y)b_2=\frac{1}{2}B_1h_1 \quad (2.12)$$

$$\Bigg[\Big((6\lambda-2x^2-2xy-2y^2)(1+2\alpha)\Big)(1+2C^k_d(\rho))^{\varrho}+\Big((x+y)^2-2\lambda(x+y-\lambda+1)\Big)(1+3\alpha)$$
$$(1+C^k_d(\rho))^{2\varrho}\Bigg]b_2^2-(3\lambda-x^2-xy-y^2)(2\alpha+1)(1+2C^k_d(\rho))^{\varrho}b_3=\frac{1}{4}B_2h_1^2+\frac{1}{2}B_1\left(h_2-\frac{h_1^2}{2}\right)$$
$$(2.13)$$

It follows from (2.10) and (2.12) that

$$g_1=-h_1 \qquad \textit{and} \qquad 2(1+C^k_d(\rho))^{2\varrho}(1+\alpha)^2(2\lambda-x-y)^2b_2^2=\frac{1}{4}B_1^2(g_1^2+h_1^2) \quad (2.14)$$

Adding (2.11) and (2.13) we have

$$\Bigg[(6\lambda-2x^2-2xy-2y^2)(1+2\alpha)(1+2C^k_d(\rho))^{\varrho}+2(1+3\alpha)\Big((x+y)^2-2\lambda(x+y-\lambda+1)\Big)$$
$$(1+C^k_d(\rho))^{2\varrho}\Bigg]b_2^2=\frac{1}{4}(g_1^2+h_1^2)(B_2-B_1)+\frac{1}{2}B_1(g_2+h_2) \quad (2.15)$$

By using (2.14) and (2.15), we get

$$b_2^2=\frac{B_1^3(g_2+h_2)}{2(6\lambda-2x^2-2xy-2y^2)(1+2\alpha)(1+2C^k_d(\rho))^{\varrho}B_1^2+(1+C^k_d(\rho))^{2\varrho}\Big[4(1+3\alpha)\Big((x+y)^2-2\lambda(x+y-\lambda+1)\Big)B_1^2-4(B_2-B_1)(1+\alpha)^2(2\lambda-x-y)^2\Big]}$$
$$(2.16)$$

For the well known inequalities $|h_2| \leq 2$ and $|g_2| \leq 2$ for functions with positive real part, we have

$$|b_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\left|(3\lambda - x^2 - xy - y^2)(1+2\alpha)(1+2C_d^k(\rho))^{\varrho}B_1^2 + (1+C_d^k(\rho))^{2\varrho}\Big[(1+3\alpha)\Big((x+y)^2 - 2\lambda(x+y-\lambda+1)\Big)B_1^2 - (B_2-B_1)(1+\alpha)^2(2\lambda-x-y)^2\Big]\right|}} \tag{2.17}$$

which gives us the desired estimate for $|b_2|$, as asserted in (2.2) .
By subtracting (2.13) from (2.11) and applying (2.14), we get

$$b_3 = \frac{B_1^2 g_1^2}{4(1+C_d^k(\rho))^{2\varrho}(1+\alpha)^2(2\lambda-x-y)^2} + \frac{B_1(g_2-h_2)}{4(1+2C_d^k(\rho))^{\varrho}(1+2\alpha)(3\lambda-x^2-y^2-xy)} \tag{2.18}$$

applying $|h_2| \leq 2$ and $|g_2| \leq 2$ again, we have

$$|b_3| \leq \frac{B_1^2}{(1+C_d^k(\rho))^{2\varrho}(1+\alpha)^2(2\lambda-x-y)^2} + \frac{B_1}{(1+2C_d^k(\rho))^{\varrho}(1+2\alpha)(3\lambda-x^2-y^2-xy)} \tag{2.19}$$

□

Remark 2.1. If we put $\varrho = 0$ in Theorem 2.1 we obtain the corresponding result study by Emeka and Opoola[9].

Remark 2.2. If we put $\varrho = 0$, $x = 1$ and $y = -1$ in Theorem 2.1 we obtain the corresponding result study by Eker and Seker[24].

Remark 2.3. If we put $\varrho = 0$, $x = 1$, $\lambda = 1$ and $y = 0$ in Theorem 2.1 we obtain the corresponding result study by Ali et al [5].

Remark 2.4. If we put $\varrho = 0$, $x = 1$, $\alpha = 0$ and $y = 0$ in Theorem 2.1 we obtain the corresponding result study by Eker and Seker[24].

3. Fekete-Szego Problem

Theorem 3.1. *Let the function $f(z)$ of the form (1.1) be in $\mathfrak{S}_{\mathfrak{C},k}^{\varrho,\rho}(\lambda, x, y : \alpha, \zeta)$, then*

$$|b_3 - \mu b_2^2| \leq \begin{cases} B_1|h(\mu)|, & |h(\mu)| \geq \frac{1}{(3\lambda-x^2-y^2-xy)(1+2\alpha)(1+2C_d^k(\rho))^{\varrho}} \\ \frac{B_1}{(3\lambda-x^2-y^2-xy)(1+2\alpha)(1+2C_d^k(\rho))^{\varrho}}, & |h(\mu)| \leq \frac{1}{(3\lambda-x^2-y^2-xy)(1+2\alpha)(1+2C_d^k(\rho))^{\varrho}} \end{cases} \tag{3.1}$$

where

$$h(\mu)=\frac{B_1^2(1-\mu)}{(3\lambda-x^2-xy-y^2)(1+2\alpha)(1+2C_d^k(\rho))^{\varrho}B_1^2+(1+C_d^k(\rho))^{2\varrho}\Big[(1+3\alpha)\Big((x+y)^2-2\lambda(x+y-\lambda+1)\Big)B_1^2-(B_2-B_1)(1+\alpha)^2(2\lambda-x-y)^2\Big]} \tag{3.2}$$

Proof. By subtracting (2.13) from (2.11) and applying (2.14), we have

$$b_3=b_2^2+\frac{B_1(g_2-h_2)}{4(1+2C_d^k(\rho))^{\varrho}(1+2\alpha)(3\lambda-x^2-y^2-xy)} \tag{3.3}$$

From (3.3) and (2.16), we get

$$b_3-\mu b_2^2=\frac{B_1}{4}\Bigg[\Bigg(h(\mu)+\frac{1}{(3\lambda-x^2-y^2-xy)(1+2\alpha)(1+2C_d^k(\rho))^{\varrho}}\Bigg)g_2+\Bigg(h(\mu)-\frac{1}{(3\lambda-x^2-y^2-xy)(1+2\alpha)(1+2C_d^k(\rho))^{\varrho}}\Bigg)h_2\Bigg],$$

where

$$h(\mu)=\frac{B_1^2(1-\mu)}{(3\lambda-x^2-xy-y^2)(1+2\alpha)(1+2C_d^k(\rho))^{\varrho}B_1^2+(1+C_d^k(\rho))^{2\varrho}\Big[(1+3\alpha)\Big((x+y)^2-2\lambda(x+y-\lambda+1)\Big)B_1^2-(B_2-B_1)(1+\alpha)^2(2\lambda-x-y)^2\Big]}$$

For all B_i are real and $B_1>0$, claim (3.1) follows. □

Putting $\varrho=0$ in Theorem 3.1, we have the following corollary.

Corollary 3.1. *Let the function $f(z)$ of the form (1.1) be in $\mathfrak{S}_{\mathfrak{E}}(\lambda,x,y:\alpha,\zeta)$, then*

$$|b_3-\mu b_2^2|\leq\begin{cases}B_1|h(\mu)|, & |h(\mu)|\geq\frac{1}{(3\lambda-x^2-y^2-xy)(1+2\alpha)}\\ \frac{B_1}{(3\lambda-x^2-y^2-xy)(1+2\alpha)}, & |h(\mu)|\leq\frac{1}{(3\lambda-x^2-y^2-xy)(1+2\alpha)}\end{cases}$$

where

$$h(\mu)=\frac{B_1^2(1-\mu)}{\Big[(\lambda-2\lambda(x+y-\lambda)-xy)+\alpha((x^2+4sy+y^2)-6\lambda(x+y-\lambda))\Big]B_1^2-(B_2-B_1)(1+\alpha)^2(2\lambda-x-y)^2}$$

which is the results obtain by Emeka and Opoola [9].

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References

[1] T. Al-Hawary, B.A. Frasin and F. Yousef, *Coefficient estimates for certain classes of analytic function of complex order,* Afr. Mat., **28** (2018), 1265–1271; Available online at https://doi.org/10.1007/s13370-018-0623-z.

[2] I. Aldawish, T. Al-Hawary and B.A. Frasin, *Subclasses of bi-univalent function defined by Frasin differential operator,* E. Mathematics., **8** (2020), 1–11; Available online at https://doi.org/10.3390/math8050783.

[3] S. Altinkaya and S. Yalcm, *Coefficient estimates for two new subclasses of bi-univalent function with respect to symmetric points,* J. Funct. Spaces., (2015), 180–185; Available online at https://doi.org/10.1155/2015/145242.

[4] F.M. Al-Oboudi, *On univalent functions defined by a generalized Salagean operator,* Int. J. Math. Math. Sci., (2004), 1429–1436; Available online at https://doi.org/10.1155/S0161171204108090.

[5] R.M. Ali, S.K. Lee, V. Ravichandran and S. Supramaniam, *Coefficient estimates for bi-univalent function Ma-Minda starlike and convex functions,* Appl. Math. Lett., **25** (2012), 344–351; Available online at https://dx.doi.org/10.1016/j.aml.2011.09.012.

[6] K.O. Babalola, *On λ-pseudo-starlike functions,* J. Class. Anal., **3** (2013), 137–147; Available online at https://doi.org/10.7153/jca-03-12.

[7] D.A. Brannan and T.S. Taha, *On some classes of bi-univalent functions,* Stud. Univ. Babes-Bolyai Math., **31** (1986), 70–77; Available online at http://dx.doi.org/10.1016/B978-0-08-031636-9.50012-7.

[8] E.P. Emeka and T.O. Opoola, *On some subclasses of bi-univalent functions associating pseudo-starlike functions with Sakaguchi type functions,* General Mathematics, **25** (2017), 85–95.

[9] E.P. Emeka and T.O. Opoola, *A newly defined subclass of bi-univalent functions satisfying subordinate conditions,* Mathematica, **61**(84) (2019), 146–155; Available online at http://dx.doi.org/10.24193/mathcluj.2019.2.05.

[10] B.A. Frasin, *Coefficient inequalities for certain classes of Sakaguchi type functions,* Int. J. Nonlinear Sci., **10** (2010), 206–211.

[11] B.A. Frasin, *A new differential operator of analytic functions involving binomial series,* Bol. Soc. Paran. Mat., **38** (2018), 205–213; Available online at https://doi.org/10.5269/bspm.v38i5.40188.

[12] S.G. Hamidi and J.M. Jahangiri, *Faber polynomial coefficients of bi-subordinate function,* C. R. Math. Acad. Sci. Paris., **354** (2016), 365–370; Available online at https://doi.org/10.1016/j.crma.2016.01.013.

[13] S. Joshi, S. Joshi and H. Pawar, *On some subclasses of bi-univalent functions associated with pseudo-starlike functions,* J. Egyptian Math. Soc., **24** (2016), 522–525; Available online at https://doi.org/10.1016/j.joems.2016.03.007.

[14] M. Lewin, *On a coefficient problem for bi-univalent functions,* Proc. Amer. Math. Soc., **31** (1967), 63–68; Available online at http://dx.doi.org/10.2307/2035225.

[15] G. Murugusundaramoothy, N. Magesh and V. Prameela, *Coefficient bounds for certain subclasses of bi-univalent functions,*Abstr. Appl. Anal., (2013); Available online at https://doi.org/10.1155/2013/573017.

[16] S.O. Olatunji and P.T. Ajai, *On subclasses of bi-univalent functions of Bazilevic type involving linear and Salagean operator,* Internat. J. Pure Appl. Math., **92** (2014), 645–656; Available online at https://doi.org/10.12732/ijpam.v92i5.2.

[17] S. Owa, T. Sekine and R. Yamakawa, *On Sakaguchi type functions,* Appl. Math. Comput., **187** (2007), 356–361; Available online at https://doi.org/10.1016/j.amc.2006.08.133.

[18] A. B. Patil and U. H. Naik, *Bounds on initial coefficients for a new subclass of bi-univalent functions*, New Trends Math. Sci., **6** (1) (2018), 85–90; Available online at https://doi.org/10.20852/ntmsci.2018.248.

[19] K. Sakaguchi, *On a certain univalent mapping,* J. Math. Soc. Japan, **11** (1959), 72–75; Available online at https://doi.org/10.2969/jmsj/01110072.

[20] T.G. Shaba, *On some new subclass of bi-univalent functions associated with Opoola differential operator*, Open J. Math. Anal., **4**(2) (2020), 74–79; Available online at https://doi.org/10.30538/psrp-oma2020.0064.

[21] T. G. Shaba, *Certain new subclasses of analytic and bi-univalent functions using salagean operator*, Asia Pac. J. Math., **7**(29) (2020), 1–11; Available online at https://doi.org/10.28924/apjm/7-29.

[22] T. G. Shaba, A. A. Ibrahim and A. A. Jimoh, *On a new subclass of bi-pseudo-starlike functions defined by frasin differential operator*, Adv. Math., Sci. J. **9**(7) (2020), 4829–4841; Available online at https://doi.org/10.37418/amsj.9.7.49.

[23] G.S. Salagean, *Subclasses of Univalent functions,* Lecture Notes in Math., Spinger Verlag, Berlin., **1013** (1983), 362-372; Available online at https://dx.doi.org/10.1007/BFb0066543.

[24] S. Sumer Eker and B. Seker, *On λ-pseudo bi-starlike and λ-pseudo bi-convex functions with respect symmetrical points,* Tbilisi Math. J., **11** (2018), 49–57; Available online at https://doi.org/10.2478/tmj-2018-0004.

[25] H.M. Srivastava, A.K Mishra and P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions,* Appl. Math. Lett., **23** (2010), 1188–1192; Available online at https://dx.doi.org/10.1016/j.aml.2010.05.009.

[26] T.S. Taha, *Topic in Univalent Function Theory,* Ph.D. Thesis, University of London, 1981.

[27] A.K. Wanas and B.A. Frasin, *Strong differential sandwich results for Frasin differential operator,* Earthline J. Math. Sci., **3**(1) (2020), 95–105; Available online at https://doi.org/10.34198/ejms.3120.95104.

[28] Q.-H. Xu, Y.-C. Gui and H.M. Srivastava, *Coefficient estimates for certain subclass of analytic and bi-univalent functions,* Appl. Math. Lett., **25** (2012), 990–994; Available online at https://dx.doi.org/10.1016/j.aml.2011.11.013.

[29] Q.-H. Xu, H.-G. Xiao and H.M. Srivastava, *A certain general subclass of analytic and bi-univalent functions and associated coefficient estimates problems,* Appl. Math. Comput., **218** (2012), 11461–11465; Available online at https://doi.org/10.1016/j.amc.2012.05.034.

[30] F. Yousef, T. Al-Hawary, and G. Murugusundaramoorthy, *Fekete-Szegö functional problems for some subclasses of bi-univalent functions defined by Frasin differential operator,* Afrika Matematika, **30**(3-4) (2019), 495–503; Available online at https://doi.org/10.1007/s13370-019-00662-7.

[31] F. Yousef, B.A. Frasin and T. Al-Hawary, *Fekete-Szegö inequality for analytic and bi-univalent functions subordinate to Chebyshev polynomials,* Filomat, **32**(9) (2018), 3229–3236; Available online at https://doi.org/10.2298/fil1809229y.

[32] F. Yousef, S. Alroud and M. Illafe, *A comprehensive subclass of bi-univalent functions associated with Chebyshev polynomials of the second kind,* Bolettín de la Sociedad Matemática Mexicana, **26** (2020), 329–339; Available online at https://doi.org/10.1007/s40590-019-00245-3.

[33] P. Zaprawa, *On the Fekete-Szego problem for classes of bi-univalent functions,* Bull. Belg. Math. Soc. Simon Stevin., **21** (2014), 167–178; Available online at https://doi.org/10.36045/bbms/1394544302.

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