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SOME INTEGRAL INEQUALITIES FOR FUNCTIONS WHOSE ABSOLUTE VALUES OF THE THIRD DERIVATIVE IS CONCAVE AND $r-$CONVEX

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ABSTRACT. In this paper, parameterized integral inequalities of the Hadamard and Simpson type are obtained for concave and $r-$ convex functions. For some values of r, these inequalities are obtained by using special tools for positive real numbers. It should be noted that in special cases, some estimates are in the same order as the estimates existing in the literature.

1. INTRODUCTION

It is known that the theory of convexity occupies an important place in optimization problems. In recent years, much more attention has been paid to the refinements and generalizations of the results obtained for classical convexity.

Hermite–Hadamard's integral inequality has a very important place in the theory of convexity. This double inequality is stated as follows in the literature (see [14]):

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and let $a, b \in I$, with $a < b$. The following double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

For this reason, the absolute majority of studies in convexity theory are devoted to finding the upper bound for this inequality for various classes of convex functions. These estimates were obtained mainly by using the properties of convex functions, the classical Hölder and power mean inequalities, and through fractional integrals (e.g. [2–6, 9, 11, 12, 17, 20, 22, 26, 29, 34] and the references therein).

Along with the inequality Hermite–Hadamard type, well known Simpson type inequality, which is provided in the literature as follows:

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If $f:[a,b]\to\mathbb{R}$ is a four times continuously differentiable function on (a,b) and $|f^{(4)}|_\infty := \sup_{x\in(a,b)}\left|f^{(4)}(x)\right| < \infty$, then

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^4}{2880}\left\| f^{(4)}\right\|_\infty .$$

Few papers are devoted to the refinements and generalizations of Simpson–type inequalities for the different classes convex functions (e.g. [10,16,19,27,28,30,31] and the references therein).

The integral version of Jensen's inequality in the literature (see [23] inequality (7.15) page 10) is given as follows:

$$\phi\left(\frac{\int_a^b f(x)d\sigma(x)}{\int_a^b d\sigma(x)}\right)\int_a^b d\sigma(x) \leq \int_a^b \phi(f(x))d\sigma(x). \tag{1.1}$$

Jensen inequality are true for all convex ϕ and $f\in L(a,b)$ and σ a non–negative measure. If the ϕ function is concave, then the inequality is reversed.

The literature studies various classes of convex functions. One of them is the class of $r-$convex functions. The idea of the $r-$convex function is based on concepts that were introduced independently by Martos in [21] and Avriel in [1].

The following definitions are well known in the literature:

Definition 1.1. [13] Let $\phi:[a,b]\to\mathbb{R}^+$. If for all $t\in[0,1]$ and of positive numbers $x,y \in [a,b]$ inequality

$$\phi(tx+(1-t)y) \leq \begin{cases} [t\phi^r(x)+(1-t)\phi^r(y)]^{\frac{1}{r}}, & \text{if } r\neq 0 \\ \phi^t(x)\phi^{1-t}(y), & \text{if } r=0 \end{cases} \tag{1.2}$$

is true, then we say that ϕ is $r-$convex function.

Remark 1.1. Obviously, for $r=0$, we have $log-$convex functions, and for $r=1$, we have ordinary convex functions. In addition, if ϕ is $r-$ convex in $[a,b]$, then ϕ^r is a convex function $(r>0)$.

The concept of $r-$convexity plays a very important role in mathematical programming [1]. A number of refinements of the estimates of the Hermite–Hadamard inequality for $r-$convex functions can be found in some papers [7,8,15,18,24,25,32,33,35] and the references therein.

In the literature, there are studies in which functions are considered whose absolute values of the third derivatives are convex (for example, [16,17,20,34]).

In [34] S. Wu et al. formulated the lemma:

Lemma 1.1. *Let $v\in\mathbb{R}$ and let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a thrice differentiable function on I° (I° is interior of I) and $a,b\in I$ with $a<b$. If $f'''\in L[a,b]$, then*

$$\begin{aligned}\frac{(4-v)f(a)+(2+v)f(b)}{6}-\frac{1}{b-a}\int_a^b f(x)dx-\frac{b-a}{12}\left[vf'(b)-(2-v)f'(a)\right] \\ =\frac{(b-a)^3}{12}\int_0^1 t(1-t)(2t-v)f'''(ta+(1-t)b)dt.\end{aligned} \tag{1.3}$$

Based on this lemma, parameterized Hermite–Hadamard type inequalities are obtained for quasi–convex functions.

In this study, by introducing a parameter in the integration interval, several new parametric Hermite–Hadamard inequalities and Simpson–type inequalities are obtained for functions, such that the absolute values of their third derivative are concave and r-convex functions.

2. Preliminary Results

The results obtained are based on the lemmas below.

Lemma 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function on I° (I° is interior of I) and $a, b \in I$ with $0 \le a < b < \infty$. If $f''' \in L[a,b]$ and $h \in (0,1]$ then we have*

$$\frac{f(c)+f(b)}{2} - \frac{1}{b-c}\int_c^b f(x)dx + \frac{b-c}{12}\left[f'(b)-f'(c)\right] = \frac{(b-a)^3}{12h}\int_0^h t(h-t)(2t-h)f'''(ta+(1-t)b)dt, \tag{2.1}$$

where $c = ha + (1-h)b$.

Proof. It's obvious that

$$\int_0^h t(h-t)(2t-h)f'''(ta+(1-t)b)dt = \int_0^h (-2t^3+3ht^2-h^2t)f'''(ta+(1-t)b)dt.$$

By integrating this integral by parts thrice, we get:

$$\begin{aligned}
&\int_0^h (-2t^3+3ht^2-h^2t)f'''(ta+(1-t)b)dt \\
&= -\frac{1}{a-b}\int_0^h (-6t^2+6ht-h^2)f''(ta+(1-t)b)dt \\
&= -\frac{h^2}{(a-b)^2}\left[f'(b)-f'(c)\right] - \frac{1}{(a-b)^2}\int_0^h (-12t+6h)\, f'(ta+(1-t)b)dt \\
&= -\frac{h^2}{(a-b)^2}\left[f'(b)-f'(c)\right] + \frac{6h}{(b-a)^3}[f(c)+f(b)] \\
&- \frac{12}{(b-a)^3}\int_0^h f(ta+(1-t)b)dt.
\end{aligned}$$

If we make the change of variables $at+(1-t)b = x$, then we can write:

$$\begin{aligned}
&\int_0^h (-2t^3+3ht^2-h^2t)f'''(ta+(1-t)b)dt \\
&= -\frac{h^2}{(a-b)^2}[f'(b)-f'(c)] + \frac{6h}{(b-a)^3}[f(c)+f(b)] - \frac{12}{(b-a)^4}\int_c^b f(x)dx.
\end{aligned}$$

By dividing both sides of this equation by the expression $\frac{(b-a)^3}{12h}$ and taking into account the fact that $h = \frac{b-c}{b-a}$, we get (2.1). The proof is completed. □

Remark 2.1. If we take the value of v parameters in identity (1.3) and h parameters in identity (2.1) as 1, these two identities are equal.

Corollary 2.1. *Under the conditions of Lemma 2.1, the following inequality holds:*

$$\left|\frac{f(c)+f(b)}{2}-\frac{1}{b-c}\int_c^b f(x)dx\right| \leq \frac{b-c}{12}\left|f'(b)-f'(c)\right| + \frac{(b-a)^3}{12h}(|I_1|+|I_2|), \tag{2.2}$$

where

$$|I_1| = \int_0^{\frac{h}{2}} \left(2t^3-3ht^2+h^2t\right)\left|f'''(ta+(1-t)b)\right| dt,$$
$$|I_2| = \int_{\frac{h}{2}}^{h} (-2t^3+3ht^2-h^2t)|f'''(ta+(1-t)b)|dt.$$

Proof. From Lemma 2.1 and triangle inequality, we obtain

$$\left|\frac{f(c)+f(b)}{2}-\frac{1}{b-c}\int_c^b f(x)dx\right| \leq \frac{b-c}{12}\left|f'(b)-f'(c)\right| \tag{2.3}$$
$$+\frac{(b-a)^3}{12h}\int_0^h |t(h-t)(2t-h)|\left|f'''(ta+(1-t)b)\right| dt.$$

Since the expression $t(t-h)(2t-h)$ is positive for $t \in \left(0, \frac{h}{2}\right)$ and negative for $t \in (\frac{h}{2}, h)$, the last integral in (2.3) will have the form:

$$\int_0^h |t(t-h)(2t-h)|\left|f'''(ta+(1-t)b)\right| dt = I_1 + I_2,$$

$$\text{where}\quad |I_1| = \int_0^{\frac{h}{2}} \left(2t^3-3ht^2+h^2t\right)\left|f'''(ta+(1-t)b)\right| dt,$$
$$|I_2| = -\int_{\frac{h}{2}}^{h} \left(2t^3-3ht^2+h^2t\right)|f'''(ta+(1-t)b)|dt.$$

The proof is completed. □

Lemma 2.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function on I° and $a, b \in I$ with $0 \leq a < b < \infty$. If $f''' \in L[a,b]$ and $h \in (0,1]$, then we have*

$$\frac{1}{b-c}\int_c^b f(x)dx - \frac{1}{3}\left[\frac{f(c)+f(b)}{2}+2f\left(\frac{c+b}{2}\right)\right] = \frac{(b-a)^3}{12h}(I_1+I_2), \tag{2.4}$$

where

$$c = ha+(1-h)b,$$
$$I_1 = \int_0^{\frac{h}{2}} t^2(2t-h)f'''(ta+(1-t)b)dt,$$
$$I_2 = \int_{\frac{h}{2}}^{h} (2t-h)(h-t)^2 f'''(ta+(1-t)b)dt.$$

Proof. By integrating the first integral by parts thrice, we get:

$$\begin{aligned}
I_1 &= \int_0^{\frac{h}{2}} t^2(2t-h)f'''(ta+(1-t)b)dt = -\frac{1}{a-b}\int_0^{\frac{h}{2}}\left(6t^2-2ht\right)f''(ta+(1-t)b)dt \\
&= -\frac{h^2}{2(a-b)^2}f'\left(b-\frac{b-a}{2}h\right)+\frac{1}{(a-b)^2}\int_0^{\frac{h}{2}}(12t-2h)\,f'(ta+(1-t)b)dt \\
&= -\frac{h^2}{2(a-b)^2}f'\left(b-\frac{b-a}{2}h\right)+\frac{4h}{(a-b)^3}f\left(b-\frac{b-a}{2}h\right)+\frac{2h}{(a-b)^3}f(b) \\
&- \frac{12}{(a-b)^3}\int_0^{\frac{h}{2}} f(ta+(1-t)b)dt.
\end{aligned}$$

So as

$$-\frac{12}{(a-b)^3}\int_0^{\frac{h}{2}} f(ta+(1-t)b)dt = \frac{12}{(b-a)^4}\int_{b-\frac{b-a}{2}h}^{b} f(x)dx,$$

we get

$$\begin{aligned}
I_1 = &-\frac{h^2}{2(a-b)^2}f'\left(b-\frac{b-a}{2}h\right)+\frac{4h}{(a-b)^3}f\left(b-\frac{b-a}{2}h\right) \\
&+\frac{2h}{(a-b)^3}f(b)+\frac{12}{(b-a)^4}\int_{b-\frac{b-a}{2}h}^{b} f(x)dx.
\end{aligned}$$

Similarly, for the second integral, we get

$$\begin{aligned}
I_2 = &\frac{h^2}{2(a-b)^2}f'\left(b-\frac{b-a}{2}h\right)+\frac{2h}{(a-b)^3}f(c)+\frac{4h}{(a-b)^3}f\left(b-\frac{b-a}{2}h\right) \\
&+\frac{12}{(b-a)^4}\int_c^{b-\frac{b-a}{2}h} f(x)dx.
\end{aligned}$$

By summing these two integrals and taking into account that $b-\frac{b-a}{2}h=\frac{c+b}{2}$, we obtain

$$I_1+I_2 = -\frac{2h}{(b-a)^3}\left[f(c)+f(b)+4f\left(\frac{c+b}{2}\right)\right]+\frac{12}{(b-a)^4}\int_c^b f(x)dx.$$

By dividing both sides of this equation by the expression $\frac{(b-a)^3}{6h}$ and taking into account the fact that $h=\frac{b-c}{b-a}$, we get (2.4). The proof is completed. □

Corollary 2.2. *Under the conditions of Lemma 2.2, the following inequality holds:*

$$\left|\frac{1}{b-c}\int_c^b f(x)dx-\frac{1}{3}\left[\frac{f(c)+f(b)}{2}+2f\left(\frac{c+b}{2}\right)\right]\right| \le \frac{(b-a)^3}{12h}\left(|I_1|+|I_2|\right), \tag{2.5}$$

where

$$\begin{aligned}
|I_1| &\le \int_0^{\frac{h}{2}} t^2\,(h-2t)\,|f'''(ta+(1-t)b)|dt, \\
|I_2| &\le \int_{\frac{h}{2}}^{h} (2t-h)\,(h-t)^2\,|f'''(ta+(1-t)b)|\,dt.
\end{aligned}$$

Proof. The proof follows from Lemma 2.2 and the inequality of triangles. □

3. Some New Results for Hadamard Type Inequalities

Theorem 3.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function on I° and $a, b \in I$ with $0 \le a < b < \infty$. If $f''' \in L[a,b]$ and $|f'''|$ is a $r-$convex function on $[a,b]$, then for all $h \in (0,1]$ and for all $r \in \mathbb{R}\backslash\{-1,0,1\}$, we have*

$$\left|\frac{f(c)+f(b)}{2} - \frac{1}{b-c}\int_c^b f(x)dx\right| \le \frac{b-c}{12}\left|f'(b)-f'(c)\right| \tag{3.1}$$
$$+\frac{(b-a)^3}{12h} \times \left\{ \begin{array}{c} h^2 P_2(r)\left[\varphi^{2+\frac{1}{r}}(0) + \varphi^{2+\frac{1}{r}}\left(\frac{h}{2}\right) + \varphi^{2+\frac{1}{r}}(h)\right] \\ +6hP_3(r)\left[\varphi^{3+\frac{1}{r}}(0) - \varphi^{3+\frac{1}{r}}(h)\right] \\ +12P_4(r)\left[\varphi^{4+\frac{1}{r}}(0) - 2\varphi^{4+\frac{1}{r}}\left(\frac{h}{2}\right) + \varphi^{4+\frac{1}{r}}(h)\right] \end{array} \right\},$$

where

$$c = ah + (1-h)b,$$
$$\varphi(t) = |f'''(b)|^r + \left(|f'''(a)|^r - |f'''(b)|^r\right)t,$$
$$P_k(r) = \frac{r^k}{[\varphi(0)-\varphi(1)]^k \prod_{i=1}^k (1+ir)}.$$

Proof. Due to the fact that $|f'''|$ is a positive $r-$convex function, then

$$|f'''(ta+(1-t)b)| \le \left[t|f'''(a)|^r + (1-t)|f'''(b)|^r\right]^{\frac{1}{r}} = \varphi^{\frac{1}{r}}(t)$$

and in inequality (2.2) for integrals we can write

$$|I_1| \le \int_0^{\frac{h}{2}} \left(2t^3 - 3ht^2 + h^2 t\right)\varphi^{\frac{1}{r}}(t)dt, \quad |I_2| \le -\int_{\frac{h}{2}}^{h} \left(2t^3 - 3ht^2 + h^2 t\right)\varphi^{\frac{1}{r}}(t)dt.$$

By calculating first integral by parts thrice, we get:

$$\begin{aligned} |I_1| &\le \int_0^{\frac{h}{2}} \left(2t^3 - 3ht^2 + h^2 t\right)\varphi^{\frac{1}{r}}(t)dt \\ &= -\frac{1}{[\varphi(1)-\varphi(0)]\left(1+\frac{1}{r}\right)}\int_0^{\frac{h}{2}} (6t^2 - 6ht + h^2)\varphi^{1+\frac{1}{r}}(t)dt \\ &= -\frac{h^2 P_1(r)}{[\varphi(1)-\varphi(0)]\left(2+\frac{1}{r}\right)} \times \left[\begin{array}{c} \left(-\frac{1}{2}\varphi^{2+\frac{1}{r}}\left(\frac{h}{2}\right) - \varphi^{2+\frac{1}{r}}(0)\right) \\ -\int_0^{\frac{h}{2}} (12t-6h)\varphi^{2+\frac{1}{r}}(t)dt \end{array} \right] \\ &= h^2 P_2(r)\left[\frac{1}{2}\varphi^{2+\frac{1}{r}}\left(\frac{h}{2}\right) + \varphi^{2+\frac{1}{r}}(0)\right] + 6hP_3(r)\varphi^{3+\frac{1}{r}}(0) \\ &\quad - 12P_4(r)\left[\varphi^{4+\frac{1}{r}}\left(\frac{h}{2}\right) - \varphi^{4+\frac{1}{r}}(0)\right]. \end{aligned}$$

Similarly enrolling, for the second integral, we can write:

$$\begin{aligned}|I_2| \leq h^2 P_2(r)\left[\varphi^{2+\frac{1}{r}}(h)+\frac{1}{2}\varphi^{2+\frac{1}{r}}\left(\frac{h}{2}\right)\right]-6hP_3(r)\varphi^{3+\frac{1}{r}}(h)\\ +12P_4(r)\left[\varphi^{4+\frac{1}{r}}(h)-\varphi^{4+\frac{1}{r}}\left(\frac{h}{2}\right)\right].\end{aligned}$$

By summing the last two inequalities, we get:

$$\begin{aligned}|I_1|+|I_2| &\leq h^2 P_2(r)\left[\varphi^{2+\frac{1}{r}}(0)+\varphi^{2+\frac{1}{r}}\left(\frac{h}{2}\right)+\varphi^{2+\frac{1}{r}}(h)\right]\\ &+6hP_3(r)\left[\varphi^{3+\frac{1}{r}}(0)-\varphi^{3+\frac{1}{r}}(h)\right]\\ &+12P_4(r)\left[\varphi^{4+\frac{1}{r}}(0)-2\varphi^{4+\frac{1}{r}}\left(\frac{h}{2}\right)+\varphi^{4+\frac{1}{r}}(h)\right].\end{aligned} \tag{3.2}$$

By taking into account inequalities (3.2) from (2.2), we obtain inequality (3.1). The proof is completed. □

Theorem 3.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function on I° and $a, b \in I$ with $0 \leq a < b < \infty$. If $f''' \in L[a,b]$ and $|f'''|$ is a $r-$convex function on $[a,b]$, then with $r = 0$ and $|f'''(b)| \neq |f'''(a)|$ for all $h \in (0,1]$, we have*

$$\begin{aligned}&\left|\frac{f(c)+f(b)}{2}-\frac{1}{b-c}\int_c^b f(x)dx\right| \\ &\leq \frac{b-c}{12}\left\{|f'(b)-f'(c)|+\frac{(b-a)^2 u^{1-h}}{(u-v)^2}L^2(u,v)\times F\right\},\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}&u=|f'''(b)|, \quad v=|f'''(a)|,\\ &F=2A\left(u^h,v^h\right)-6L\left(u^h,v^h\right)+3L^2\left(u^{\frac{h}{2}},v^{\frac{h}{2}}\right)\end{aligned}$$

and $A(\xi,\tau)=\frac{\xi+\tau}{2}$, $L(\xi,\tau)=\frac{\xi-\tau}{\ln\xi-\ln\tau}$ are respectively the arithmetic and logarithmic mean of two distinct positive numbers.

Proof. Let $R=\frac{|f'''(a)|}{|f'''(b)|}$. Since the $|f'''|$ is a positive log −convex function (see Remark 1.1), we have

$$|f'''(ta+(1-t)b)| \leq |f'''(a)|^t\,|f'''(b)|^{1-t}=|f'''(b)|\,R^t. \tag{3.4}$$

By considering (3.4), for integrals in inequality (2.2), we can write

$$\begin{aligned}|I_1| &\leq |f'''(b)|\int_0^{\frac{h}{2}}\left(2t^3-3ht^2+h^2t\right)R^t dt \text{ and}\\ |I_2| &\leq |f'''(b)|\int_{h/2}^{h}(-2t^3+3ht^2-h^2t)R^t dt.\end{aligned}$$

By calculating both integrals by parts thrice, we get

$$\begin{aligned}
|I_1| &\leq |f'''(b)| \left[\frac{h^2}{\ln^2 R} \left(1 - 0.5R^{h/2}\right) + \frac{6h}{\ln^3 R} - \frac{12}{\ln^4 R} \left(R^{h/2} - 1\right) \right], \\
|I_2| &\leq |f'''(b)| \left[\frac{h^2}{\ln^2 R} \left(R^h + 0.5R^{h/2}\right) - \frac{6h}{\ln^3 R} R^h + \frac{12}{\ln^4 R} \left(R^h - R^{h/2}\right) \right].
\end{aligned}$$

By summing the last two inequalities, we obtain

$$|I_1| + |I_2| \leq \frac{|f'''(b)|}{\ln^2 R} \left[h^2 \left(1 + R^h\right) + \frac{6h\left(1 - R^h\right)}{\ln R} + \frac{12}{\ln^2 R} \left(R^{h/2} - 1\right)^2 \right].$$

Given the fact that

$$\begin{aligned}
& h^2\left(1 + R^h\right) = h^2 \left[1 + \left(\frac{v}{u}\right)^h \right] = \frac{2h^2}{u^h} A\left(u^h, v^h\right), \\
& \frac{6h\left(1 - R^h\right)}{\ln R} = \frac{6h^2}{u^h} \frac{u^h - v^h}{\ln v^h - \ln u^h} = -\frac{6h^2}{u^h} L\left(u^h, v^h\right), \\
& \frac{12}{\ln^2 R}\left(R^{h/2} - 1\right)^2 = 12 \left(\frac{R^{h/2} - 1}{\ln R}\right)^2 = \frac{3h^2}{u^h}\left[L\left(u^{h/2}, v^{h/2}\right)\right]^2 \quad \textit{and} \\
& \frac{1}{[\ln u - \ln v]^2} = \frac{1}{(u - v)^2} L^2(u, v),
\end{aligned}$$

we can write

$$|I_1| + |I_2| \leq \frac{h^2 u^{1-h}}{(u - v)^2} L^2(u, v) \times \left[2A\left(u^h, v^h\right) - 6L\left(u^h, v^h\right) + 3L^2\left(u^{h/2}, v^{h/2}\right) \right]. \tag{3.5}$$

By taking into account (3.5), from (2.2) we get (3.3). The proof is completed. □

Theorem 3.3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function on I° and $a, b \in I$ with $0 \leq a < b < \infty$. If $f''' \in L[a, b]$ is a $r-$convex on $[a, b]$, then with $r = 1$ for all $h \in (0, 1]$, we have*

$$\begin{aligned}
\left| \frac{f(c) + f(b)}{2} - \frac{1}{b - c} \int_c^b f(x)dx \right| &\leq \frac{b - c}{12} |f'(b) - f'(c)| \\
&\quad + \frac{(b - c)^3}{384} \left[h|f'''(a)| + (2 - h)|f'''(b)| \right],
\end{aligned} \tag{3.6}$$

where $c = ha + (1 - h)b$.

Proof. Since the $|f'''|$ is an ordinary convex function (see Remark 1.1) on $[a, b]$, for integrals in (2.2), we can write

$$\begin{aligned}
|I_1| &\leq |f'''(a)| \int_0^{h/2} t^2(t - h)(2t - h)dt + |f'''(b)| \int_0^{h/2} t(t - h)(2t - h)(1 - t)dt, \\
|I_2| &\leq |f'''(a)| \int_{h/2}^h t^2(h - t)(2t - h)dt + |f'''(b)| \int_{h/2}^h t(h - t)(2t - h)(1 - t)dt.
\end{aligned}$$

By calculating all the integrals in these inequalities, we get

$$|I_1| \le \frac{7h^5}{960}|f'''(a)| + \frac{(30-7h)h^4}{960}|f'''(b)|, \quad |I_2| \le \frac{23h^5}{960}|f'''(a)| + \frac{(30-23h)h^4}{960}|f'''(b)|.$$

By summing the last two inequalties, we obtain

$$|I_1| + |I_2| \le \frac{h^4}{32}\left[h|f'''(a)| + (2-h)|f'''(b)|\right]. \tag{3.7}$$

By taking into account (3.7) from (2.2), we get (3.6). The proof is completed. □

Remark 3.1. If $h=1$, then from (3.6), we get

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_c^b f(x)dx\right| \le \frac{b-a}{12}\left|f'(b)-f'(a)\right| \tag{3.8}$$
$$+ \frac{(b-a)^3}{384}\left[|f'''(a)| + |f'''(b)|\right].$$

Remark 3.2. If $\|f'''\|_\infty := \sup_{x\in[a,b]}|f'''(x)| < \infty$, then from (3.8), we get

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{12}\left|f'(b)-f'(a)\right| + \frac{(b-a)^3}{192}|f'''|_\infty.$$

An estimate of the same order was obtained by J. Materano, et al. in [20] (Theorem 2.2).

Theorem 3.4. *Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function on I° and $a,b \in I$ with $0 \le a < b < \infty$. If $f''' \in L[a,b]$ is a $r-$convex on $[a,b]$, then with $r=-1$ and $|f'''(a)| \ne |f'''(b)|$ for all $h \in (0,1]$, we have*

$$\left|\frac{f(c)+f(b)}{2} - \frac{1}{b-c}\int_c^b f(x)dx\right| \le \frac{b-c}{12}\left|f'(b)-f'(c)\right| \tag{3.9}$$
$$+\frac{(b-a)^3}{96h}\cdot\frac{G^2(u,v)A(v,\mu)}{A^2(u,v)-G^2(u,v)}\left[h^2 + \frac{G^2(v,\mu)}{A^2(u,v)-G^2(u,v)}\ln\frac{H(v,\mu)}{A(v,\mu)}\right],$$

where

$$v = |f'''(a)|, \quad u = |f'''(b)|, \quad \mu = hu + (1-h)v, \quad \tau = \frac{v}{(u-v)^2}$$

and $A(u,v) = \frac{v+u}{2}$, $G(u,v) = \sqrt{vu}$, $H(u,v) = \frac{2vu}{v+u}$ are respectively the arithmetic, geometric and harmonic means of two distinct positive numbers.

Proof. Due to the fact that $|f'''|$ is a positive $r-$convex function, then with $r=-1$, we can write

$$|f'''(ta+(1-t)b)| \le \left(\frac{t}{|f'''(a)|} + \frac{1-t}{|f'''(b)|}\right)^{-1} = \frac{vu}{v+(u-v)t}.$$

Let $\psi(t) = v + (u-v)t$, then for integrals in (2.2), we can write:

$$\frac{|I_1|}{vu} \le \int_0^{\frac{h}{2}} \frac{2t^3 - 3ht^2 + h^2t}{\psi(t)}dt, \quad \frac{|I_2|}{vu} \le -\int_{\frac{h}{2}}^h \frac{2t^3 - 3ht^2 + h^2t}{\psi(t)}dt.$$

By dividing a polynomial by a polynomial, we calculate both integrals, and then we sum up the resulting inequalities. After simplification we get:

$$|I_1|+|I_2| \le \frac{vu\left[hu+(2-h)v\right]}{4(u-v)^2}\left[h^2+\frac{4v\left[hu+(1-h)v\right]}{(u-v)^2}\ln\frac{4v\left[hu+(1-h)v\right]}{\left[hu+(2-h)v\right]^2}\right]$$
$$=\frac{vu(v+\mu)}{4(u-v)^2}\left[h^2+4\frac{v\mu}{(u-v)^2}\ln\frac{4v\mu}{(v+\mu)^2}\right].$$

Given the fact that

$$v+\mu=2A(v,\mu),$$
$$\frac{vu(v+\mu)}{4(u-v)^2}=\frac{1}{2}\frac{vuA(v,\mu)}{(u-v)^2}=\frac{1}{8}\frac{vuA(v,\mu)}{\left(\frac{v+u}{2}\right)^2-vu}=\frac{1}{8}\frac{G^2(u,v)A(v,\mu)}{A^2(u,v)-G^2(u,v)},$$
$$\frac{4v\mu}{(u-v)^2}=\frac{4v\mu}{(u-v)^2}=\frac{G^2(v,\mu)}{A^2(u,v)-G^2(u,v)} \quad \text{and}$$
$$\ln\frac{4v\mu}{(\mu+v)^2}=\ln\left(\frac{2v\mu}{\mu+v}\cdot\frac{2}{\mu+v}\right)=\ln\frac{H(v,\mu)}{A(v,\mu)}$$

we get:

$$|I_1|+|I_2|\le\frac{1}{8}\frac{G^2(u,v)A(v,\mu)}{A^2(u,v)-G^2(u,v)}\left[h^2+\frac{G^2(v,\mu)}{A^2(u,v)-G^2(u,v)}\ln\frac{H(v,\mu)}{A(v,\mu)}\right]. \tag{3.10}$$

By taking into account (3.10) from (2.2), we get (3.9). The proof is completed. □

Corollary 3.1. *If the conditions of Theorem 3.4 are fulfilled, then with $h=1$, we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \tag{3.11}$$
$$\le\frac{b-a}{12}\left|f'(b)-f'(a)\right|+\frac{(b-a)^3}{96}\times\frac{G^2(u,v)A(u,v)}{A^2(u,v)-G^2(u,v)}$$
$$\times\left\{1+\frac{G^2(u,v)}{A^2(u,v)-G^2(u,v)}\ln\frac{H(u,v)}{A(u,v)}\right\},$$

where $v=|f'''(a)|$, $u=|f'''(b)|$.

Proof. With $h=1$ from Theorem 3.4, we get (3.11). The proof is completed. □

Theorem 3.5. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a thrice differentiable function on I° and $a,b\in I$ with $0\le a<b<\infty$. If $f'''\in L[a,b]$ is a concave on $[a,b]$, then for all $h\in(0,1]$, we have*

$$\left|\frac{f(c)+f(b)}{2}-\frac{1}{b-c}\int_c^b f(x)dx\right|\le\frac{b-c}{12}\left|f'(b)-f'(c)\right| \tag{3.12}$$
$$+\frac{(b-c)^3}{144}\left[\left|f'''\left(\frac{2ha+(5-2h)b}{5}\right)\right|+\left|f'''\left(\frac{3ha+(5-3h)b}{5}\right)\right|\right],$$

where $c=ha+(1-h)b$.

Proof. From Lemma 2.1 and triangle inequality, we can write

$$\left|\frac{f(c)+f(b)}{2}-\frac{1}{b-c}\int_c^b f(x)dx\right| \tag{3.13}$$
$$\leq \frac{b-c}{12}\left|f'(b)-f'(c)\right|+\frac{(b-a)^3}{12h}\left|\int_0^h t(h-t)(2t-h)f'''(ta+(1-t)b)dt\right|.$$

Considering that $t(h-t)(2t-h) = -t(t-h)^2 - t^2(t-h)$, then for the integral on the right-hand side of last inequality, we can write:

$$\left|\int_0^h t(h-t)(2t-h)f'''(ta+(1-t)b)dt\right| \leq |I_1|+|I_2|,$$

where

$$|I_1| = \left|\int_0^h t(t-h)^2 f'''(ta+(1-t)b)dt\right|, |I_2| = \left|\int_0^h t^2(t-h)f'''(ta+(1-t)b)dt\right|.$$

For each integral, we use the Jensen inequality (1.1) for concave functions f'''on $[a,b]$:

$$|I_1| \leq \left|\int_0^h t(t-h)^2dt\right|\left|f'''\left(\frac{\int_0^h[at+(1-t)b]t(t-h)^2dt}{\int_0^h t(t-h)^2dt}\right)\right| \tag{3.14}$$

and calculate the integrals:

$$\left|\int_0^h t(t-h)^2dt\right| = \frac{h^4}{12} \quad \text{and}$$
$$\int_0^h [at+(1-t)b]t(t-h)^2dt = \int_0^h \left[\begin{array}{l}(a-b)t^4+(b-2ha+3hb)\,t^3\\ +\left(h^2a-h^2b-2hb\right)t^2+bh^2t\end{array}\right]dt$$
$$= \frac{h^4}{6}\cdot\frac{2ha+(5-2h)\,b}{10}.$$

By substituting the obtained values of the integrals in the inequality (3.14), we get

$$|I_1| \leq \frac{h^4}{12}\left|f'''\left(\frac{2ha+(5-3h)b}{5}\right)\right|. \tag{3.15}$$

Similarly, for the second integral, we can get

$$|I_2| \leq \left|\int_0^h t^2(t-h)dt\right|\left|f'''\left(\frac{\int_0^h[at+(1-t)b]t^2(t-h)dt}{\int_0^h t^2(t-h)dt}\right)\right| \tag{3.16}$$
$$= \frac{h^4}{12}\left|f'''\left(\frac{(3ha+(5-3h)b}{5}\right)\right|.$$

By taking into account (3.15) and (3.16) from (3.13), we obtain (3.12). The proof is completed. □

Corollary 3.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function on I° and $a, b \in I$ with $0 \leq a < b < \infty$. If $f''' \in L[a,b]$ and is concave on $[a,b]$, then we have*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \tag{3.17}$$
$$\leq \frac{b-a}{12} \left| f'(b) - f'(a) \right| + \frac{(b-a)^3}{144} \left[\left| f''' \left(\frac{2a+3b}{5} \right) \right| + \left| f''' \left(\frac{3a+2b}{5} \right) \right| \right].$$

Proof. With $h = 1$ from (3.12), we obtain (3.17). The proof is completed. $\square$

Remark 3.3. If $\|f'''\|_{\infty} := \sup\limits_{x \in (a,b)} |f'''(x)| < \infty$, then from (3.17), we get

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{12} \left| f'(b) - f'(a) \right| + \frac{(b-a)^3}{72} \|f'''\|_{\infty}.$$

4. Some New Results for Simpson Type Inequalities

Theorem 4.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function on I° and $a, b \in I$ with $0 \leq a < b < \infty$. If $f''' \in L[a,b]$ and $|f'''|$ is a $r-$convex function on $[a,b]$, then for all $h \in (0,1]$ and for all $r \in \mathbb{R} \backslash \{-1,0,1\}$, we have*

$$\left| \frac{1}{b-c} \int_c^b f(x)dx - \frac{1}{3} \left[\frac{f(c)+f(b)}{2} + 2f\left(\frac{c+b}{2} \right) \right] \right| \tag{4.1}$$
$$\leq \frac{(b-a)^3}{12h} \times \left\{ \begin{array}{l} h^2 P_2(r) \varphi^{2+\frac{1}{r}} \left(\frac{h}{2} \right) \\ +2hP_3(r) \left[\varphi^{3+\frac{1}{r}}(0) - \varphi^{3+\frac{1}{r}}(h) \right] \\ -12P_4(r) \left[\varphi^{4+\frac{1}{r}}(0) - 2\varphi^{4+\frac{1}{r}} \left(\frac{h}{2} \right) + \varphi^{4+\frac{1}{r}}(h) \right] \end{array} \right\},$$

where

$$c = ah + (1-h)b,$$
$$\varphi(t) = |f'''(b)|^r + \left(|f'''(a)|^r - |f'''(b)|^r \right) t,$$
$$P_k(r) = \frac{r^k}{[\varphi(0) - \varphi(1)]^k \prod_{i=1}^k (1+ir)}.$$

Proof. Given the fact that $|f'''|$ is a positive $r-$convex function on $[a,b]$, then

$$|f'''(ta + (1-t)b)| \leq \left[t|f'''(a)|^r + (1-t)|f'''(b)|^r \right]^{\frac{1}{r}} = \varphi^{\frac{1}{r}}(t)$$

and in inequality (2.5) for integrals, we can write

$$|I_1| \leq \int_0^{\frac{h}{2}} t^2 (h-2t) \varphi^{\frac{1}{r}}(t)dt, \quad |I_2| \leq \int_{\frac{h}{2}}^h (2t-h)(h-t)^2 \varphi^{\frac{1}{r}}(t)dt.$$

By integrating both integrals by parts thrice, we get:

$$I_1 \leq \frac{h^2 P_2(r)}{2} \varphi^{2+\frac{1}{r}} \left(\frac{h}{2} \right) + 2hP_3(r) \left[2\varphi^{3+\frac{1}{r}} \left(\frac{h}{2} \right) + \varphi^{3+\frac{1}{r}}(0) \right]$$
$$+ 12P_4(r) \left[\varphi^{4+\frac{1}{r}} \left(\frac{h}{2} \right) - \varphi^{4+\frac{1}{r}}(0) \right],$$

$$|I_2| \le \frac{h^2 P_2(r)}{2}\varphi^{2+\frac{1}{r}}\left(\frac{h}{2}\right) - 2hP_3(r)\left[\varphi^{3+\frac{1}{r}}(h) + 2\varphi^{3+\frac{1}{r}}\left(\frac{h}{2}\right)\right]$$
$$- 12P_4(r)\left[\varphi^{4+\frac{1}{r}}(h) - \varphi^{4+\frac{1}{r}}\left(\frac{h}{2}\right)\right].$$

By summing the last two inequalities, we get:

$$|I_1| + |I_2| \le h^2 P_2(r)\varphi^{2+\frac{1}{r}}\left(\frac{h}{2}\right) + 2hP_3(r)\left[\varphi^{3+\frac{1}{r}}(0) - \varphi^{3+\frac{1}{r}}(h)\right] \tag{4.2}$$
$$- 12P_4(r)\left[\varphi^{4+\frac{1}{r}}(h) - 2\varphi^{4+\frac{1}{r}}\left(\frac{h}{2}\right) + \varphi^{4+\frac{1}{r}}(0)\right].$$

By taking into account inequalities (4.2) from (2.5), we obtain inequality (4.1). The proof is completed. ☐

Theorem 4.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function on I° and $a, b \in I$ with $0 \le a < b < \infty$. If $f''' \in L[a,b]$ and $|f'''|$ is a $r-$convex function on $[a,b]$ and $r = 0$, then for all $h \in (0,1]$, we have*

$$\left|\frac{1}{b-c}\int_c^b f(x)dx - \frac{1}{3}\left[\frac{f(c)+f(b)}{2} + 2f\left(\frac{c+b}{2}\right)\right]\right| \tag{4.3}$$
$$\le \frac{(b-a)^3}{12h} \times \frac{hu^{1-h}}{(u-v)^2}L^2(u,v) \times D,$$

where

$$c = ha + (1-h)b,\ u = |f'''(b)|, \quad v = |f'''(a)|,$$
$$D = G\left(u^h, v^h\right) + 2L\left(u^h, v^h\right) - 3L^2\left(u^{\frac{h}{2}}, v^{\frac{h}{2}}\right)$$

and $G(\xi,\tau) = \sqrt{\xi\tau}$, $L(\xi,\tau) = \frac{\xi-\tau}{\ln\xi - \ln\tau}$ are respectively the geometric mean and logarithmic mean of two distinct positive numbers ξ and τ.

Proof. Since the $|f'''|$ is a positive log $-$convex function (see Remark 1.1) by considering (3.4), for integrals in inequality (2.5), we can write

$$|I_1| \le |f'''(b)|\int_0^{\frac{h}{2}}\left(-2t^3 + ht^2\right)R^t dt,$$
$$|I_2| \le |f'''(b)|\int_{\frac{h}{2}}^h\left(2t^3 - 5ht^2 + 4h^2t - h^3\right)R^t dt.$$

By calculating both integrals by parts thrice, we get

$$|I_1| \le |f'''(b)|\left[\frac{h^2}{2\ln^2 R}R^{h/2} - \frac{2h}{\ln^3 R}\left(2R^{h/2}+1\right) + \frac{12}{\ln^4 R}\left(R^{h/2}-1\right)\right],$$
$$|I_2| \le |f'''(b)|\left[\frac{h^2}{2\ln^2 R}R^{h/2} + \frac{2h}{\ln^3 R}\left(R^h + 2R^{h/2}\right) - \frac{12}{\ln^4 R}\left(R^h - R^{h/2}\right)\right].$$

By summing the last two inequalities, we obtain

$$|I_1| + |I_2| \le \frac{|f'''(b)|}{\ln^2 R}\left[h^2R^{h/2} + \frac{2h\left(R^h-1\right)}{\ln R} - \frac{12}{\ln^2 R}\left(R^{h/2}-1\right)^2\right].$$

Because of

$$h^2 R^{h/2} = h^2 \left(\frac{v}{u}\right)^{h/2} = \frac{h^2}{u^h}(u \cdot v)^{h/2} = \frac{h^2}{u^h} G\left(u^h, v^h\right),$$

$$\frac{2h\left(R^h - 1\right)}{\ln R} = \frac{2h^2}{u^h}\frac{v^h - u^h}{\ln v^h - \ln u^h} = \frac{2h^2}{u^h} L\left(u^h, v^h\right),$$

$$\frac{12}{\ln^2 R}\left(R^{h/2} - 1\right)^2 = 12\left(\frac{R^{h/2} - 1}{\ln R}\right)^2 = \frac{3h^2}{u^h}\left[L\left(u^{h/2}, v^{h/2}\right)\right]^2,$$

$$\frac{1}{[\ln u - \ln v]^2} = \frac{1}{(u - v)^2} L^2(u, v)$$

we can write

$$I_1 + I_2 \le \frac{h^2 u^{1-h}}{(u - v)^2} L^2(u, v) \times \left[G\left(u^h, v^h\right) + 2L\left(u^h, v^h\right) - 3L^2\left(u^{h/2}, v^{h/2}\right)\right]. \tag{4.4}$$

By taking into account (4.4) from (2.5), we get (4.3). The proof is completed. □

Theorem 4.3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function on I° and $a, b \in I$ with $0 \le a < b < \infty$. If $f''' \in L[a, b]$ is $r-$convex on $[a, b]$ and $r = 1$, then for all $h \in [0, 1]$ we have*

$$\begin{aligned} &\left|\frac{1}{b - c}\int_c^b f(x)dx - \frac{1}{3}\left[\frac{f(c) + f(b)}{2} + 2f\left(\frac{c + b}{2}\right)\right]\right| \\ &\le \frac{(b - c)^3}{1152}\left(h\left|f'''(a)\right| + (2 - h)\left|f'''(b)\right|\right), \end{aligned} \tag{4.5}$$

where $c = ha + (1 - h)b$.

Proof. Since the $|f'''|$ an ordinary convex function (see Remark 1.1), then in inequality (2.5) for integrals, we can write

$$|I_1| \le |f'''(a)| \int_0^{\frac{h}{2}} t^3 (h - 2t)\, dt + |f'''(b)| \int_0^{\frac{h}{2}} t^2 (h - 2t)(1 - t) dt \;\; \text{and}$$

$$|I_2| \le |f'''(a)| \int_{\frac{h}{2}}^h t(h - t)^2 (h - 2t)\, dt + |f'''(b)| \int_{\frac{h}{2}}^h (h - t)^2 (h - 2t)(1 - t) dt.$$

By calculating all integrals, we get

$$|I_1| \le \frac{h^5}{320}|f'''(a)| + \frac{h^4(10 - 3h)}{960}|f'''(b)|, \quad |I_2| \le \frac{7h^5}{960}|f'''(a)| + \frac{h^4(10 - 7h)}{960}|f'''(b)|.$$

By summing the last two inequalities, we get

$$|I_1| + |I_2| \le \frac{h^4}{96}\left[h|f'''(a)| + (2 - h)|f'''(b)|\right]. \tag{4.6}$$

By taking into account (4.6) from (2.2), we obtain (4.5). The proof is completed. □

Remark 4.1. With $h = 1$ from (4.5), we get

$$\left|\frac{1}{b - a}\int_a^b f(x)dx - \frac{1}{3}\left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a + b}{2}\right)\right]\right| \le \frac{(b - a)^3}{1152}\left(|f'''(a)| + |f'''(b)|\right).$$

This estimate was obtained by S. Hussain and S. Qaisar in [16] (Theorem 2).

Theorem 4.4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function on I° and $a, b \in I$ with $0 \leq a < b < \infty$. If $f''' \in L[a,b]$ is a $r-$convex on $[a,b]$, then with $r = -1$ and $|f'''(a)| \neq |f'''(b)|$ for all $h \in [0,1]$, we have*

$$\left| \frac{1}{b-c} \int_c^b f(x)dx - \frac{1}{3}\left[\frac{f(c)+f(b)}{2} + 2f\left(\frac{c+b}{2}\right)\right]\right| \tag{4.7}$$
$$\leq \frac{(b-a)^3}{96h} \cdot \frac{G^2(u,v)A(\mu,v)}{A^2(v,u)-G^2(u,v)}$$
$$\times\left[3h^2 + \frac{1}{A^2(u,v)-G^2(u,v)}\left(v^2 \ln\frac{A(\mu,v)}{v} - \mu^2 \ln\frac{\mu}{A(\mu,v)}\right)\right],$$

where

$$c = ah + (1-h)b, \ \ v = |f'''(a)|, \ \ u = |f'''(b)|, \mu = hu + (1-h)v$$

and $A(u,v) = \frac{v+u}{2}$, $G(u,v) = \sqrt{vu}$ are respectively the arithmetic and geometric means of two distinct positive numbers.

Proof. Due to the fact that $|f'''|$ is a positive $r-$convex function, then with $r = -1$, we can write

$$|f'''(ta + (1-t)b)| \leq \left(\frac{t}{|f'''(a)|} + \frac{1-t}{|f'''(b)|}\right)^{-1} = \frac{vu}{v+(u-v)t}.$$

Let $\psi(t) = v + (u-v)t$, then in inequality (2.5) for integrals, we get

$$\frac{|I_1|}{vu} \leq \int_0^{\frac{h}{2}} \frac{-2t^3 + ht^2}{\psi(t)}dt \text{ and } \frac{|I_2|}{vu} \leq \int_{\frac{h}{2}}^{h} \frac{2t^3 - 5ht^2 + 4h^2t - h^3}{\psi(t)}dt.$$

By dividing a polynomial by $\psi(t)$, we calculate both integrals, and then we sum the results. After simplification, we get:

$$|I_1| + |I_2| \leq \frac{vu\,(v+\mu)}{4(u-v)^2}\left[3h^2 + \frac{4v^2}{(u-v)^2}\ln\left(\frac{v+\mu}{2v}\right) - \frac{4\mu^2}{(u-v)^2}\ln\left(\frac{2\mu}{v+\mu}\right)\right].$$

Given the fact that

$$\frac{vu\,(v+\mu)}{4(u-v)^2} = \frac{1}{8}\frac{G^2(u,v)A(\mu,v)}{A^2(u,v)-G^2(v,u)}, \qquad \frac{4v^2}{(u-v)^2} = \frac{v^2}{A^2(u,v)-G^2(u,v)},$$
$$\frac{v+\mu}{2v} = \frac{A(v,\mu)}{v}, \quad \frac{2\mu}{v+\mu} = \frac{\mu}{A(v,\mu)} \text{ and } \frac{4\mu^2}{(u-v)^2} = \frac{\mu^2}{A^2(u,v)-G^2(u,v)},$$

we can write:

$$|I_1| + |I_2| \leq \frac{1}{8}\frac{G^2(u,v)A(v,\mu)}{A^2(u,v)-G^2(v,u)}\left[\begin{array}{c} 3h^2 + \frac{v^2}{A^2(u,v)-G^2(u,v)}\ln\frac{A(v,\mu)}{v} \\ -\frac{\mu^2}{A^2(u,v)-G^2(u,v)}\ln\frac{\mu}{A(v,\mu)}\end{array}\right]. \tag{4.8}$$

By taking into account (4.8) from (2.5), we get (4.7). The proof is completed. □

Corollary 4.1. *If the conditions of Theorem 4.4 are fulfilled, then with $h = 1$, we get*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{h}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \\
&\leq \frac{(b-a)^3}{96} \times \frac{G^2(u,v)A(u,v)}{A^2(u,v)-G^2(u,v)} \\
&\times \left\{3 + \frac{1}{A^2(u,v)-G^2(u,v)}\left[v^2 \ln\frac{A(u,v)}{v} - u^2 \ln\frac{u}{A(u,v)}\right]\right\},
\end{aligned} \tag{4.9}$$

where $v = |f'''(a)|$, $u = |f'''(b)|$.

Proof. Indeed, with $h = 1$, from Theorem 4.4, we get (4.9). The proof is completed. □

Theorem 4.5. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function on I° and $a, b \in I$ with $0 \leq a < b < \infty$. If $f''' \in L[a,b]$ is a concave on $[a,b]$, then for all $h \in [0,1]$, we have*

$$\begin{aligned}
&\left|\frac{1}{b-c}\int_c^b f(x)dx - \frac{1}{3}\left[\frac{f(c)+f(b)}{2} + 2f\left(\frac{c+b}{2}\right)\right]\right| \\
&\leq \frac{(b-c)^3}{1152}\left(\left|f'''\left(\frac{3ha+(10-3h)b}{10}\right)\right| + \left|f'''\left(\frac{7ha+(10-7h)b}{10}\right)\right|\right),
\end{aligned} \tag{4.10}$$

where $c = ha + (1-h)b$.

Proof. For each integral in inequality (2.5), we use the Jensen inequality (1.1) for concave functions f'''on $[a,b]$:

$$|I_1| \leq \left|\int_0^{\frac{h}{2}} t^2(2t-h)dt\right| \left|f'''\left(\frac{\int_0^{\frac{h}{2}}[at+(1-t)b]t^2(2t-h)dt}{\int_0^{\frac{h}{2}} t^2(2t-h)dt}\right)\right| \tag{4.11}$$

and calculate the integrals:

$$\int_0^{\frac{h}{2}} t^2(2t-h)dt = -\frac{h^4}{96} \text{ and } \int_0^{\frac{h}{2}} [at+(1-t)b]t^2(2t-h)dt = -\frac{3ha+(10-3h)b}{10}.$$

By substituting the obtained values of the integrals in the inequality (4.11), we get:

$$|I_1| \leq \frac{h^4}{96}\left|f'''\left(\frac{3ha+(10-3h)b}{10}\right)\right|. \tag{4.12}$$

Similarly, for the second integral, we get:

$$\begin{aligned}
|I_2| &\leq \left|f'''\left(\frac{\int_{\frac{h}{2}}^h [at+(1-t)b](2t-h)(h-t)^2dt}{\int_{\frac{h}{2}}^h (2t-h)(h-t)^2dt}\right)\right| \\
&\times \left|\int_{\frac{h}{2}}^h (2t-h)(h-t)^2dt\right| \leq \frac{h^4}{96}\left|f'''\left(\frac{7ha+(10-7h)b}{10}\right)\right|.
\end{aligned} \tag{4.13}$$

By taking into account (4.12) and (4.13) from (2.5), we obtain (4.10). The proof is completed. □

Corollary 4.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function on I° and $a, b \in I$ with $0 \le a < b < \infty$. If $f''' \in L[a, b]$ and is concave on $[a, b]$ then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \tag{4.14}$$
$$\le \frac{(b-a)^3}{1152}\left(\left|f'''\left(\frac{3a+7b}{10}\right)\right| + \left|f'''\left(\frac{7a+3b}{10}\right)\right|\right).$$

Proof. With $h = 1$ from (4.10), we obtain (4.14). The proof is completed. □

Remark 4.2. If $\|f'''\|_\infty := \sup_{x\in(a,b)} |f'''(x)| < \infty$, then from (4.14), we obtain

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \le \frac{(b-a)^3}{576}\|f'''\|_\infty .$$

An estimate of the same order was obtained by S. Hussain and S. Qaisar in [16] (Corollary 7).

5. Examples of functions satisfying the conditions of Theorems 3.1 and 4.1

Example 5.1. It is easy to show that a positive definite linear function $\varphi(x) = px + q$, $p > 0$, $q \le 0$ is a $2-$convex function.

Really, for $r = 2$ from inequality

$$\varphi(xt + (1-t)y) \le [t\varphi^r(x) + (1-t)\varphi^r(y)]^{1/r},$$

we get:

$$\begin{aligned} t(px+q) + (1-t)(py+q) &\le [t(px+q)^2 + (1-t)(py+q)^2]^{1/2} \Longrightarrow \\ [t(px+q) + (1-t)(py+q)]^2 &\le t(px+q)^2 + (1-t)(py+q)^2 \Longrightarrow \\ t(1-t)[p(x-y)]^2 &\ge 0, \forall t \in [0,1], \forall x, y \in R. \end{aligned}$$

Given that the third derivative must be $r-$convex, then from $f'''(x) = \varphi(x)$, we get a polynomial function

$$f(x) = \sum_{k=0}^{4} a_k x^k, \quad a_k \in R \ \ and \ \ a_3 \le 0, \ \ a_4 > 0.$$

Example 5.2. Taking into account that the function $\varphi(x) = \sqrt{px+q}$, $p > 0$, $q \le 0$ is $4-$convex (the reader can easily verify this himself) using the equality $f'''(x) = \varphi(x)$,we get:

$$f(x) = \frac{8}{105p^3}(px+q)^{7/2} + \sum_{k=0}^{2} a_k x^k, \ a_k \in R.$$

This function's on any interval from $[0, +\infty)$ satisfies the conditions of Theorem's 3.1 and 4.1, and thus, the inequalities (3.1) and (4.1).

For simplicity of calculations, consider the function $f(x) = \frac{x^4}{24}, x \in [0, 1]$.

The third derivative of this function$f'''(x) = x$ and this function is a $2-$convex.

We calculate the left and right sides of the inequality (3.1) for this function.

Since $a = 0, b = 1, c = ah + (1-h)b = 1-h$, then for the left side of the (3.1) we get:

$$\begin{aligned} L.side &= \left| \frac{f(b)+f(c)}{2} - \frac{1}{b-c}\int_c^b \frac{x^4}{24}dx \right| \\ &= \left| \frac{f(1)+f(1-h)}{2} - \frac{1}{h}\int_{1-h}^1 \frac{x^4}{24}dx \right| = \frac{1}{240h}|(1-h)^4(3h+2)-2|. \end{aligned}$$

For convenience, we write the right-hand side of inequality (3.1) in the form:

$$R.side = R_1 + \frac{(b-a)^2}{12h}[R_2 + R_3 + R_4],$$

where

$$R_1 = \frac{b-c}{12}|f'(b) - f'(c)| = \frac{h}{12}|f'(1) - f'(1-h)| = \frac{h}{12}\left|\frac{1}{6} - \frac{(1-h)^3}{6}\right| = \frac{h^2(h^2-3h+3)}{72}.$$

And, since the $\varphi(t) = 1-t$, then

$$\begin{aligned} \varphi(0) &= 1, \quad \varphi(1) = 0, \quad \varphi\left(\frac{h}{2}\right) = 1 - \frac{h}{2} \; and \; \varphi(h) = 1-h \\ R_2 &= \frac{4h^2}{15}\left[1 + \left(1-\frac{h}{2}\right)^{2,5} + (1-h)^{2.5}\right], \\ R_3 &= \frac{48h}{105}[1-(1-h)^{3.5}], \\ R_4 &= \frac{12\cdot 2^4}{15\cdot 63}\left[1 - 2\left(1-\frac{h}{2}\right)^{4,5} + (1-h)^{4.5}\right]. \end{aligned}$$

The correctness of the inequality is proved by numerical calculations. The calculation results are shown in the table

Calculation table for the inequality (3.1)

h	L.side	R1	R2	R3	R4	R. side	R.side-L.side
0,1	**0,020457**	0,000376	0,007062	0,014099	0,007043	**0,023879**	0,003422277
0,2	**0,01948**	0,001356	0,024969	0,049559	0,024685	**0,042694**	0,023214361
0,3	**0,018107**	0,002738	0,049826	0,097786	0,048427	**0,057193**	0,039085658
0,4	**0,016513**	0,004356	0,078988	0,152263	0,074702	**0,068096**	0,051582409
0,5	**0,014844**	0,006076	0,110928	0,208368	0,100808	**0,076094**	0,061249926
0,6	**0,013213**	0,0078	0,145071	0,263183	0,124836	**0,08184**	0,068626989
0,7	**0,011707**	0,00946	0,181617	0,315268	0,145596	**0,085945**	0,07423839
0,8	**0,01038**	0,011022	0,221311	0,364406	0,162528	**0,088964**	0,078584331
0,9	**0,009257**	0,012488	0,26514	0,411298	0,175605	**0,09138**	0,082123377
1	**0,008333**	0,013889	0,313807	0,457143	0,185216	**0,093569**	0,085236082

Using the function $f(x) = \frac{x^4}{24}$, calculations are made for Theorem 4.1 in a similar way. The results are shown in the table.

Calculation table for the inequality (4.1)

h	L. side	R1	R2	R3	R. side	R.side-L.side
0,1	**3,47222E-08**	0,00234573	0,00469958	-0,00704	**1,69203E-06**	1,65731E-06
0,2	**5,55556E-07**	0,00819662	0,01651972	-0,02468	**1,31721E-05**	1,26165E-05
0,3	**2,8125E-06**	0,01598669	0,03259546	-0,04843	**4,31829E-05**	4,03704E-05
0,4	**8,88889E-06**	0,02442383	0,05075426	-0,0747	**9,92244E-05**	9,03355E-05
0,5	**2,17014E-05**	0,03247595	0,06945613	-0,10081	**0,000187411**	0,000165709
0,6	**4,5E-05**	0,03935649	0,0877278	-0,12484	**0,000312276**	0,000267276
0,7	**8,33681E-05**	0,04450904	0,10508923	-0,1456	**0,000476495**	0,000393127
0,8	**0,000142222**	0,04759122	0,12146862	-0,16253	**0,000680451**	0,000538229
0,9	**0,000227812**	0,04845744	0,13709949	-0,17561	**0,000921474**	0,000693661
1	**0,000347222**	0,04714045	0,15238095	-0,18522	**0,001192089**	0,000844867

As can be seen from the tables (the last column), the difference between the right and left sides of the inequalities (3.1) and (4.1) for all h is positive, i.e. inequalities holds for any $h \in (0, 1]$.

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