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AN EFFICIENT ITERATIVE METHOD AND ITS APPLICATIONS TO A NONLINEAR INTEGRAL EQUATION AND A DELAY DIFFERENTIAL EQUATION IN BANACH SPACES

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ABSTRACT. In this paper, a multi-step iterative method is introduced for contraction mappings. We prove that our new iterative method converges at a rate faster than some of the leading iterative schemes in the existing literature which have been used recently to obtain the solutions of a mixed type Volterra–Fredholm functional nonlinear integral equation and a delay differential equation. A numerical example is also used to show that our new iterative scheme converges at a rate faster than a number of existing iterative schemes for contraction mappings. As some applications, we prove that our new iterative method converges strongly to the unique solutions of a mixed type Volterra–Fredholm functional nonlinear integral equation and a delay differential equation. In addition, we give data dependence result for the solution of the nonlinear integral equation we are considering with the help of our new iterative scheme. Our results improve, generalize and unify some well known results in the existing literature.

1. INTRODUCTION

Fixed point theory has fascinated several authors since 1922 with the celebrated Banach fixed point theorem. There exists a vast literature on the topic field and this is very active field of research at present. Fixed point theorems are very useful tools for proving the existence and uniqueness of the solutions of various mathematical models (integral equations, partial differential equations, ordinary differential equations, variational inequalities, etc.,) see [27]. For example, it can be applied to variational inequalities, optimization, approximation theory, successive approximation, game theory, optimal control, economics and several others. The fixed point theory has been continually studied by many authors (see for example, [1, 2, 27, 57] and the references there in). It is well known that the contraction conditions are very indispensable in the study of fixed point theory. The first important result on fixed point for contraction mapping is the celebrated Banach-Caccioppoli theorem which was published in 1922 in [13] and also appeared in [19].

Key words and phrases. Banach space, Contraction map, Data dependence, Strong convergence, AI Iterative scheme, Nonlinear integral equation, Delay differential equation.

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A wide range of problems of applied science and engineering are often transformed into functional equations. Operator equations representing phenomena occurring in different area of studies such as chemical reactions, neutron transport theory, economic theory and epidemics, often require appropriate and adequate solutions. Thus, the process of obtaining solutions to these equations is to locate the fixed point and approximate it value. On the other hand, if the existence of the fixed point is guaranteed, then it is always desirable to construct an efficient method which can be employed to approximate the fixed point operators. Since the failure of Picard iterative method to converge to the fixed point of nonexpansive mappings even when the existence of the unique fixed point is guaranteed in a complete metric space, many authors have come up with different kind of iterative schemes for approximating the fixed point of the class of nonexpansive mappings and other classes of mappings which are more general than the class of nonexpansive mappings. Some well known iterative schemes in the existing literature includes: Picard [54], Kransnosel'kii [44], Mann [46], Ishikawa [41], Argawal et al. [10], Abbas and Nazir [3] and so on.

For some recent literature on iterative algorithms, we refer the reader to [4,5,11,26]. Many problems in science and engineering are modeled by differential and integral equations, in most cases, delay differential equations and Volterra-Fredholm functional nonlinear integral equations.

Delay differential equations play an important role in applied science and have many applications in biological sciences as follows: they have been used in primary infections, epidemiology, tumor growth and neutral network, etc. (see for example [21], [63] and the references there in). Delay differential equations are also used in statistical analysis, ecology data (see [61]) for the effects in the population dynamics of many species.

On the other hand, many problems of mathematical physics, applied mathematics, and engineering are reduced to Volterra-Fredholm integral equations (see for example [6], [7] and the references there in).

There exists several methods in the literature for solving delay differential equations and nonlinear integral equations (see for example, [7, 17, 18, 22, 24, 27, 30, 31, 34, 47] and the references there in).

As part of the beauty of fixed point theory, many researchers in nonlinear analysis have introduced and studied several iteration schemes for solving delay different equations and functional nonlinear integral equations.

Recently, many authors have employed different iterative schemes for solving the following mixed type Volterra-Fredholm functional nonlinear integral equation which was considered by Crăciun and Şerban [22]:

$$\omega(t) = F\left(t, \omega(t), \int_{r_1}^{\nu_1} \dots \int_{r_m}^{\nu_m} K(t,s,\omega(s))ds, \int_{r_1}^{\lambda_1} \dots \int_{r_m}^{\lambda_m} H(t,s,\omega(s))ds\right), \tag{1.1}$$

where $[r_1;\lambda_1] \times \cdots \times [r_m;\lambda_m]$ is an interval in $\Re^m$, $K, H : [r_1;\lambda_1] \times \cdots \times [r_m;\lambda_m] \times [r_1;\lambda_1] \times \cdots \times [r_m;\lambda_m] \times \Re \to \Re$ continuous functions and $F : [r_1;\lambda_1] \times \cdots \times [r_m;\lambda_m] \times \Re^3 \to \Re$ (see for example, [22, 25, 30, 52]).

Let $C([u,v])$ denote the space of all continuous real valued functions on a closed interval $[u,v]$ endowed with Chebyshev norm. Through out this paper, let Γ be a nonempty closed convex subset of a real Banach space E, $\Re$ denotes the set of real numbers and let $\mathbb{N}$ denote the set of natural numbers.

As an application of fixed point theory, many authors have introduced several iterative schemes for solving the following delay differential equations

$$\omega'(\ell) = f(\ell, \omega(\ell), \omega(\ell - \tau)), \ \ell \in [\ell_0, v], \tag{1.2}$$

with initial condition

$$\omega(\ell) = \psi(\ell), \ell \in [\ell_0 - \tau, \ell_0], \tag{1.3}$$

where $\ell_0, v \in \Re, \tau > 0$, $f \in C([\ell_0, v] \times \Re^2, \Re)$; and $\psi \in C([\ell_0 - \tau, v], \Re)$ (see for example [20,26,31,37] and the references there in). Specifically, the following iterative schemes which are known as Normal-S iterative scheme [56], M iterative scheme [62], Gordian and Uddin iterative scheme [25], Picard-S iterative scheme [31] respectively , have been used by Gursoy [30], Okeke and Abbas [52], Gordian and Uddin [25], Gursoy and Karakaya [31] respectively, to approximate the unique solutions of delay differential equations (1.2)-(1.3) and the mixed type Volterra-Fredholm functional nonlinear integral equation (1.1):

$$\begin{cases} a_0 \in \Gamma, \\ b_n = (1-\mu_n)a_n + \mu_n G a_n, & \forall n \geq 1; \\ a_{n+1} = G b_n, \end{cases} \tag{1.4}$$

$$\begin{cases} m_0 \in \Gamma, \\ c_n = (1-\mu_n)m_n + \mu_n G m_n, \\ \delta_n = G c_n, & \forall n \geq 1; \\ m_{n+1} = G \delta_n, \end{cases} \tag{1.5}$$

$$\begin{cases} d_0 \in \Gamma, \\ u_n = G d_n, \\ v_n = (1-\mu_n)u_n + \mu_n G u_n, & \forall n \geq 1; \\ d_{n+1} = G v_n, \end{cases} \tag{1.6}$$

$$\begin{cases} \eta_0 \in \Gamma, \\ \varrho_n = (1-\mu_n)\eta_n + \mu_n G \eta_n, \\ \gamma_n = (1-\sigma_n)G\eta_n + \sigma_n G \varrho_n, & \forall n \geq 1. \\ \eta_{n+1} = G \gamma_n, \end{cases} \tag{1.7}$$

where μ_n and σ_n are sequences in (0,1).

It has been shown by several authors that multi-steps iteration processes perform better than single step and two steps iteration processes respectively. Glowinski and Le-Tallec [28] used a multi step iterative process to solve elasto-viscoplasticity, liquid crystal and eigenvalue problems. They established that three-step iterative scheme performs better than one-step (Mann) and two-step (Ishikawa) iterative schemes. Haubruge et al. [39] studied the convergence analysis of the three-step iterative processes of Glowinski and Le-Tallec [28] and used the three-step iteration to obtain some new splitting type algorithms for solving variational inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-steps iteration processes also lead to highly parallelized algorithms under certain conditions.

Many researchers have recently been active in constructing multi-steps iteration schemes to obtain faster rate of convergence (see [11, 26, 30, 49–51, 62] and the references there in). Hence, we see that multi-steps iteration processes play pivotal role in nonlinear analysis and gives faster convergence rate.

Motivated by the above results, we introduce the following four steps iterative scheme, called the AI iterative scheme, for approximating the solutions of the delay differential equation (1.2)-(1.3) and the mixed type Volterra-Fredholm functional nonlinear integral equation (1.1):

$$\begin{cases} \omega_0 \in \Gamma, \\ \zeta_n = (1-\mu_n)\omega_n + \mu_n G\omega_n, \\ q_n = G\zeta_n, \\ p_n = Gq_n, \\ \omega_{n+1} = Gp_n, \end{cases} \qquad \forall n \geq 1. \tag{1.8}$$

It is our purpose in this paper to prove analytically that AI iterative scheme (1.8) converges faster than the iterative schemes (1.4)-(1.6) for contraction mappings. We also show with a numerical example that AI iterative scheme has a better speed of convergence than (1.4)-(1.6). Furthermore, we prove that AI iterative scheme (1.8) converges strongly to the unique solutions of the delay differential equation (1.2)-(1.3) and the mixed type Volterra-Fredholm functional nonlinear integral equation (1.1). In addition, we give the data dependence result for the solution of the equation (1.1) via AI iterative scheme (1.8). Since the iterative schemes (1.4)-(1.6) have recently been employed to solve the delay differential equation (1.2)-(1.3) and the mixed type Volterra Fredholm functional nonlinear integral equations (1.1), hence, our results improve and unify the corresponding results in [20, 22, 25, 30, 31, 52], and several others in the existing literature.

2. Preliminaries

The following definitions and Lemmas will be useful in proving our main results.

Definition 2.1. A mapping $G : \Gamma \to \Gamma$ is called contraction if there exists a constant $\vartheta \in (0, 1)$ such that $\|G\omega - Gp\| \leq \vartheta\|\omega - p\|$, $\forall\, \omega, p \in \Gamma$.

Definition 2.2 (see Berinde [15])**.** Let $\{l_n\}_{n=0}^{\infty}$ and $\{g_n\}_{n=0}^{\infty}$ be two sequences of real numbers converging to l and g respectively. Then we say that $\{l_n\}_{n=0}^{\infty}$ converges faster than $\{g_n\}_{n=0}^{\infty}$ if

$$\lim_{n\to\infty} \frac{\|l_n - l\|}{\|g_n - g\|} = 0. \tag{2.1}$$

Definition 2.3 (see Berinde [15])**.** Let $\{w_n\}_{n=0}^{\infty}$ and $\{\kappa_n\}_{n=0}^{\infty}$ be two fixed point iteration procedure sequences that converge to the same point p. If $\|w_n - z\| \leq l_n$ and $\|\kappa_n - z\| \leq g_n$ for all $n \in \mathbb{N}$, where $\{l_n\}_{n=0}^{\infty}$ and $\{\kappa_n\}_{n=0}^{\infty}$ are two sequences of positive numbers (converging to zero). Then we say that $\{w_n\}_{n=0}^{\infty}$ converges faster than $\{\kappa_n\}_{n=0}^{\infty}$ to z if $\{l_n\}_{n=0}^{\infty}$ converges faster than $\{g_n\}_{n=0}^{\infty}$.

Lemma 2.1 (see [60]). *Let $\{\rho_n\}$ be nonnegative real sequences satisfying the following inequalities:*

$$\rho_{n+1} \leq (1-\tau_n)\rho_n, \tag{2.2}$$

where $\tau_n \in (0,1)$ for all $n \in \mathbb{N}$, $\sum\limits_{n=0}^{\infty} \tau_n = \infty$, then $\lim\limits_{n\to\infty} \rho_n = 0$.

Lemma 2.2 (see [59]). *Let $\{\rho_n\}$ and $\{\varPsi_n\}$ be two nonnegative real sequences satisfying the following inequalities:*

$$\rho_{n+1} \leq (1-\tau_n)\rho_n + \tau_n \varPsi_n, \tag{2.3}$$

where $\tau_n \in (0,1)$ for all $n \in \mathbb{N}$, $\sum\limits_{n=0}^{\infty} \tau_n = \infty$ and $\varPsi_n \geq 0$ for all $n \in \mathbb{N}$, then

$$0 \leq \limsup_{n\to\infty} \rho_n \leq \limsup_{n\to\infty} \varPsi_n. \tag{2.4}$$

3. Rate of Convergence

In this section, we prove analytically and numerically that the AI iterative process (1.8) converges at a rate faster than all of Normal-S iterative process (1.4), M iterative process (1.5), Garodia and Uddin iterative process (1.6) and Picard-S iterative process [31].

Theorem 3.1. *Let Γ be a nonempty closed convex subset of a Banach space E and $G : \Gamma \to \Gamma$ be a contraction mapping with contraction constant $\vartheta \in (0,1)$ such that $F(G) \neq \emptyset$. If $\{\omega_n\}$ is the sequence defined by (1.8), then $\{\omega_n\}$ converges faster than all the other four processes.*

Proof. For any $z \in F(G)$, from (1.8) we have

$$\begin{aligned}
\|\zeta_n - z\| &= \|(1-\mu_n)\omega_n + \mu_n G\omega_n - z\| \\
&\leq (1-\mu_n)\|\omega_n - z\| + \mu_n\|G\omega_n - z\| \\
&\leq (1-\mu_n)\|\omega_n - z\| + \mu_n\vartheta\|\omega_n - z\| \\
&= (1-(1-\vartheta)\mu_n)\|\omega_n - z\|.
\end{aligned} \tag{3.1}$$

Again, from (1.8) and (3.1), we obtain

$$\begin{aligned}
\|q_n - z\| &= \|G\zeta_n - z\| \\
&\leq \vartheta\|\zeta_n - z\| \\
&\leq \vartheta(1-(1-\vartheta)\mu_n)\|\omega_n - z\|.
\end{aligned} \tag{3.2}$$

Also, from (1.8) and (3.2), we get

$$\begin{aligned}
\|p_n - z\| &= \|Gq_n - z\| \\
&\leq \vartheta\|q_n - z\| \\
&\leq \vartheta^2(1-(1-\vartheta)\mu_n)\|\omega_n - z\|.
\end{aligned} \tag{3.3}$$

So, from (1.8) and (3.3), we have

$$\begin{aligned} \|\omega_{n+1} - z\| &= \|Gp_n - z\| \\ &\leq \vartheta\|p_n - z\| \\ &\leq \vartheta^3(1 - (1 - \vartheta)\mu_n)\|\omega_n - z\| \\ &\leq \vartheta^{3n}(1 - (1 - \vartheta)\mu)^n\|\omega_1 - z\|. \end{aligned}$$

Let

$$h_n = \vartheta^{3n}(1 - (1 - \vartheta)\mu)^n\|\omega_1 - z\|. \tag{3.4}$$

Now, from (1.4), we have

$$\begin{aligned} \|b_n - z\| &= \|(1 - \mu_n)a_n + \mu_n Ga_n - z\| \\ &\leq (1 - \mu_n)\|a_n - z\| + \mu_n\|Ga_n - z\| \\ &\leq (1 - \mu_n)\|a_n - z\| + \mu_n\vartheta\|a_n - z\| \\ &= (1 - (1 - \vartheta)\mu_n)\|a_n - z\|. \end{aligned}$$

So,

$$\begin{aligned} \|a_{n+1} - z\| &= \|Gb_n - z\| \\ &\leq \vartheta\|b_n - z\| \\ &\leq \vartheta(1 - (1 - \vartheta)\mu_n)\|a_n - z\| \\ &\leq \vartheta^n(1 - (1 - \vartheta)\mu)^n\|a_1 - z\|. \end{aligned}$$

Let

$$u_n = \vartheta^n(1 - (1 - \vartheta)\mu)^n\|a_1 - z\|. \tag{3.5}$$

Again, from (1.5), we get

$$\begin{aligned} \|c_n - z\| &= \|(1 - \mu_n)m_n + \mu_n Gm_n - z\| \\ &\leq (1 - \mu_n)\|m_n - z\| + \mu_n\|Gm_n - z\| \\ &\leq (1 - \mu_n)\|m_n - z\| + \mu_n\vartheta\|m_n - z\| \\ &= (1 - (1 - \vartheta)\mu_n)\|m_n - z\|. \end{aligned}$$

And

$$\begin{aligned} \|\delta_n - z\| &= \|Gc_n - z\| \\ &\leq \vartheta\|c_n - z\| \\ &\leq \vartheta(1 - (1 - \vartheta)\mu_n)\|m_n - z\|. \end{aligned}$$

So,

$$\begin{aligned} \|m_{n+1} - z\| &= \|G\delta_n - z\| \\ &\leq \vartheta\|\delta_n - z\| \\ &\leq \vartheta^2(1 - (1 - \vartheta)\mu_n)\|m_n - z\| \\ &\leq \vartheta^{2n}(1 - (1 - \vartheta)\mu)^n\|m_1 - z\|. \end{aligned}$$

Set

$$t_n = \vartheta^{2n}(1-(1-\vartheta)\mu)^n\|m_1 - z\|. \tag{3.6}$$

Now, using (1.6), we get

$$\begin{aligned}
\|u_n - z\| &= \|Gd_n - z\| \\
&\le \vartheta\|d_n - z\|.
\end{aligned}$$

And

$$\begin{aligned}
\|v_n - z\| &= \|(1-\mu_n)u_n + \mu_n G u_n - z\| \\
&\le (1-\mu_n)\|u_n - z\| + \mu_n\|Gu_n - z\| \\
&\le (1-\mu_n)\|u_n - z\| + \mu_n\vartheta\|u_n - z\| \\
&= (1-(1-\vartheta)\mu_n)\|u_n - z\| \\
&\le \vartheta(1-(1-\vartheta)\mu_n)\|d_n - z\|.
\end{aligned}$$

So,

$$\begin{aligned}
\|d_{n+1} - z\| &= \|Gv_n - z\| \\
&\le \vartheta\|v_n - z\| \\
&\le \vartheta^2(1-(1-\vartheta)\mu_n)\|d_n - z\| \\
&\le \vartheta^{2n}(1-(1-\vartheta)\mu)^n\|d_1 - z\|.
\end{aligned}$$

Set

$$\varpi_n = \vartheta^{2n}(1-(1-\vartheta)\mu)^n\|d_1 - z\|.$$

Lastly, from (1.7) we have

$$\begin{aligned}
\|\varrho_n - p\| &= \|(1-\mu_n)\eta_n + \mu_n G\eta_n - z\| \\
&\le (1-\mu_n)\|\eta_n - z\| + \mu_n\|G\eta_n - z\| \\
&\le (1-\mu_n)\|\eta_n - z\| + \mu_n\vartheta\|\eta_n - z\| \\
&= (1-(1-\vartheta)\mu_n)\|\eta_n - z\|.
\end{aligned}$$

And

$$\begin{aligned}
\|\gamma_n - z\| &= \|(1-s_n)G\eta_n + s_n G\varrho_n - z\| \\
&\le (1-\sigma_n)\|G\eta_n - z\| + \sigma_n\|G\varrho_n - z\| \\
&\le (1-\sigma_n)\vartheta\|\eta_n - z\| + \sigma_n\vartheta\|\varrho_n - z\| \\
&\le (1-\sigma_n)\vartheta\|\eta_n - z\| + \sigma_n\vartheta(1-(1-\vartheta)\mu_n)\|\eta_n - z\| \\
&= \vartheta(1-(1-\vartheta)\sigma_n\mu_n)\|\eta_n - z\|.
\end{aligned}$$

So,

$$\begin{aligned} \|\eta_{n+1} - z\| &= \|G\gamma_n - z\| \\ &\leq \vartheta\|\gamma_n - z\| \\ &\leq \vartheta^2(1-(1-\vartheta)\sigma_n\mu_n)\|\eta_n - z\| \\ &\leq \vartheta^{2n}(1-(1-\vartheta)\sigma\mu)^n\|\eta_1 - z\|. \end{aligned}$$

Put

$$\epsilon_n = \vartheta^{2n}(1-(1-\vartheta)\sigma\mu)^n\|\eta_1 - z\|.$$

Now we compute the rate of convergence of AI iterative scheme (1.8) as follows:

(i) Observe that

$$\frac{h_n}{u_n} = \frac{\vartheta^{3n}(1-(1-\vartheta)\mu)^n\|\omega_1 - z\|}{\vartheta^n(1-(1-\vartheta)\mu)^n\|a_1 - z\|} = \vartheta^{2n}\frac{\|\omega_1 - z\|}{\|a_1 - z\|} \to 0 \text{ as } n \to \infty.$$

Thus, $\{\omega_n\}$ converges faster to z than $\{a_n\}$. This implies that, the AI iterative process (1.8) converges faster to z than the normal S-iterative process (1.4).

(ii) Also,

$$\frac{h_n}{t_n} = \frac{\vartheta^{3n}(1-(1-\vartheta)\mu)^n\|\omega_1 - z\|}{\vartheta^{2n}(1-(1-\vartheta)\mu)^n\|m_1 - z\|} = \vartheta^{n}\frac{\|\omega_1 - z\|}{\|m_1 - z\|} \to 0 \text{ as } n \to \infty.$$

Thus, $\{\omega_n\}$ converges faster to z than $\{m_n\}$. This implies that, the AI iterative process (1.8) converges faster to z than the M iterative process (1.5).

(iii) Also, we see that

$$\frac{h_n}{\varpi_n} = \frac{\vartheta^{3n}(1-(1-\vartheta)\mu)^n\|\omega_n - z\|}{\vartheta^{2n}(1-(1-\vartheta)\mu)^n\|d_1 - z\|} = \vartheta^{n}\frac{\|\omega_1 - z\|}{\|d_1 - z\|} \to 0 \text{ as } n \to \infty.$$

Thus, $\{\omega_n\}$ converges faster to z than $\{d_n\}$. This implies that, the AI iterative process (1.8) converges faster to z than Garodia and Uddin iterative process (1.6).

(iv) Finally, we have that

$$\begin{aligned} \frac{h_n}{\epsilon_n} &= \frac{\vartheta^{3n}(1-(1-\vartheta)\mu)^n\|\omega_n - z\|}{\vartheta^{2n}(1-(1-\vartheta)\sigma\mu)^n\|\eta_1 - z\|} \\ &= \frac{\vartheta^{n}(1-(1-\vartheta)\mu)^n\|\omega_n - z\|}{(1-(1-\vartheta)\sigma\mu)^n\|\eta_1 - z\|} \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus, $\{\omega_n\}$ converges faster to z than $\{\eta_n\}$. This implies that, the AI iterative process (1.8) converges faster to z than the Picard-S iterative process (1.7). Hence, our iterative method converges at a rate faster than all of (1.4)-(1.7) in the sense of Berinde [15]. This completes the prove.

□

By supporting the analytical proof of Theorem 3.1 and to illustrate the efficiency of AI iterative scheme (1.8), we will consider the following numerical example.

Example 3.1. Let $E = \Re$ and $\Gamma = [1, 50]$. Let $G : \Gamma \to \Gamma$ be a mapping defined by $G\omega = \sqrt[3]{2\omega + 4}$ for all $\omega \in \Gamma$. Clearly, G is a contraction with contractive constant $\vartheta = \frac{1}{\sqrt[3]{4}}$ and $\omega = 2$ is a fixed point of G. Take $\mu_n = \sigma_n = \frac{1}{2}$, with an initial value of 30.

By using the above example, we will show that AI iterative scheme (1.8) has better speed of convergence than iterative schemes (1.4)-(1.7).

TABLE 1

n	AI-ITERATION	PICARD-S	NORMAL-S
1	30.000000000	30.000000000	30.000000000
2	2.0336342615	2.2481120233	3.3619754068
3	2.0000906203	2.0053436106	2.1228934171
4	2.0000002447	2.0001174560	2.0118603109
5	2.0000000007	2.0000025829	2.0011522593
6	2.0000000000	2.0000000568	2.0001120174
7	2.0000000000	2.0000000012	2.0000108905
8	2.0000000000	2.0000000000	2.0000010588
9	2.0000000000	2.0000000000	2.0000001029
10	2.0000000000	2.0000000000	2.0000000100
11	2.0000000000	2.0000000000	2.0000000010
12	2.0000000000	2.0000000000	2.0000000001
13	2.0000000000	2.0000000000	2.0000000000

TABLE 1 CONTD.

n	M-ITERATION	GARODIA-UDDIN
1	30.000000000	30.000000000
2	2.2052183845	2.2052183845
3	2.0032795388	2.0032795388
4	2.0000531287	2.0000531287
5	2.0000008609	2.0000008609
6	2.0000000139	2.0000000139
7	2.0000000002	2.0000000002
8	2.0000000000	2.0000000000
9	2.0000000000	2.0000000000
10	2.0000000000	2.0000000000
11	2.0000000000	2.0000000000
12	2.0000000000	2.0000000000
13	2.0000000000	2.0000000000

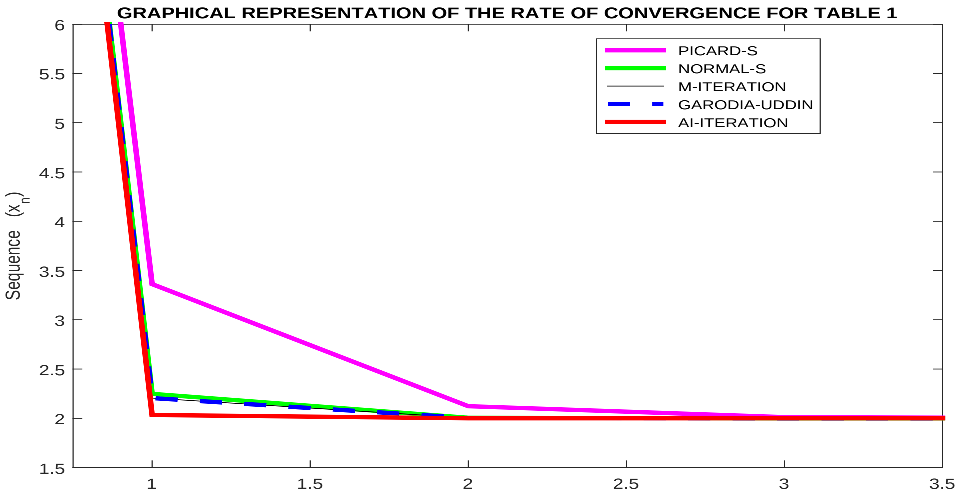

FIGURE 1

It is evident from the table and graph that AI-iteration process (1.8) converges at a better speed than the iteration processes (1.4)-(1.7).

4. Application of AI iterative method to a nonlinear integral equation

In this section, we prove strong convergence theorem of a sequence generated by AI iteration process for the mixed type Volterra-Fredholm functional nonlinear integral equation defined by (1.1) in a real Banach space. And also, we give data dependence result for the solution of the mixed type Volterra-Fredholm functional nonlinear integral equation (1.1) with the help of our new iterative scheme (1.8).

The following result will be very useful in proving our main results.

Theorem 4.1 (see [22])**.** *We assume that the following conditions are satisfied:*

(B_1) $K, H \in C([r_1;\lambda_1] \times \cdots \times [r_m;\lambda_m] \times [r_1;\lambda_1] \times \cdots \times [r_m;\lambda_m] \times \Re)$;
(B_2) $F \in ([r_1;\lambda_1] \times \cdots \times [r_m;\lambda_m] \times \Re^3)$;

(B_3) *there exists nonnegative constants* α, β, γ *such that*

$$|F(t, f_1, \varepsilon_1, h_1) - F(t, f_2, \varepsilon_2, h_2)| \leq \alpha|f_1 - f_2| + \beta|\varepsilon_1 - \varepsilon_2| + \gamma|h_1 - h_2|,$$

for all $t \in [r_1; \lambda_1] \times \cdots \times [r_m; \lambda_m]$, $f_1, \varepsilon_1, h_1, f_2, \varepsilon_2, h_2 \in \Re$;
(B_4) *there exist nonnegative constants* L_K *and* L_H *such that*

$$\begin{aligned} |K(t, s, f) - K(t, s, \varepsilon)| &\leq L_K|f - \varepsilon|, \\ |H(t, s, f) - H(t, s, \varepsilon)| &\leq L_H|f - \varepsilon|, \end{aligned}$$

for all $t, s \in [r_1; \lambda_1] \times \cdots \times [r_m; \lambda_m], f, \varepsilon \in \Re$;
(B_5) $\alpha + (\beta L_K + \gamma L_H)(\lambda_1 - r_1) \cdots (\lambda_m - r_m) < 1$.
Then, the nonlinear integral equation (1.1) *has a unique solution* $z \in C([r_1; \lambda_1] \times \cdots \times [r_m; \lambda_m])$.

Now, we are ready to prove our main results.

Theorem 4.2. *Assume that all the conditions* $(B_1) - (B_5)$ *in Theorem* (4.1) *are satisfied. Let* $\{\omega_n\}$ *be defined by AI iteration process* (1.8) *with real sequence* $\mu_n \in [0, 1]$, *satisfying* $\sum_{n=1}^{\infty} \mu_n = \infty$. *Then* (1.1) *has a unique solution and the AI iteration process* (1.8) *converges strongly to the unique solution of the mixed type Volterra–Fredholm functional nonlinear integral equation* (1.1), *say* $z \in C([r_1; \lambda_1] \times \cdots \times [r_m; \lambda_m])$.

Proof. We now consider the Banach space $E = C([r_1; \lambda_1] \times \cdots \times [r_m; \lambda_m], \|\cdot\|_C)$, where $\|\cdot\|_C$ is the Chebyshev's norm. Let $\{\omega_n\}$ be the iterative sequence generated by AI iterative scheme (1.8) for the operator $A : E \to E$ define by

$$A(\omega)(t) = F\left(t, \omega(t), \int_{r_1}^{\nu_1} \cdots \int_{r_m}^{\nu_m} K(t, s, \omega(s))ds, \int_{r_1}^{\lambda_1} \cdots \int_{r_m}^{\lambda_m} H(t, s, \omega(s))ds\right). \quad (4.1)$$

Our intention is to prove that $\omega_n \to z$ as $n \to \infty$. Now, by using (1.8), (1.1), (4.1) and the assumptions (B_1)–(B_5), we have that

$$
\begin{aligned}
\|\omega_{n+1}-z\| &= \|p_n-z\| \\
&= |A(p_n)(t)-A(z)(t)| \qquad (4.2)\\
&= |F\left(t,p_n(t),\int_{r_1}^{\nu_1}\dots\int_{r_m}^{\nu_m}K(t,s,p_n(s))ds,\int_{r_1}^{\lambda_1}\dots\int_{r_m}^{\lambda_m}H(t,s,p_n(s))ds\right)\\
&\quad -F\left(t,z(t),\int_{r_1}^{\nu_1}\dots\int_{r_m}^{\nu_m}K(t,s,z(s))ds,\int_{r_1}^{\lambda_1}\dots\int_{r_m}^{\lambda_m}H(t,s,z(s))ds\right)|\\
&\le \alpha|p_n(t)-z(t)|+\beta|\int_{r_1}^{\nu_1}\dots\\
&\quad \int_{r_m}^{\nu_m}K(t,s,p_n(s))ds-\int_{r_1}^{\nu_1}\dots\int_{r_m}^{\nu_m}K(t,s,z(s))ds|\\
&\quad +\gamma|\int_{r_1}^{\lambda_1}\dots\int_{r_m}^{\lambda_m}H(t,s,p_n(s))ds-\int_{r_1}^{\lambda_1}\dots\int_{r_m}^{\lambda_m}H(t,s,z(s))ds|\\
&\le \alpha|p_n(t)-z(t)|+\beta\int_{r_1}^{\nu_1}\dots\int_{r_m}^{\nu_m}|K(t,s,p_n(s))-K(t,s,z(s))|ds\\
&\quad +\gamma\int_{r_1}^{\lambda_1}\dots\int_{r_m}^{\lambda_m}|H(t,s,p_n(s))-H(t,s,z(s))|ds\\
&\le \alpha|p_n(t)-z(t)|+\beta\int_{r_1}^{\nu_1}\dots\int_{r_m}^{\nu_m}L_K|p_n(s)-z(s)|ds\\
&\quad +\gamma\int_{r_1}^{\lambda_1}\dots\int_{r_m}^{\lambda_m}L_H|p_n(s)-z(s)|ds\\
&\le \alpha\|p_n-z\|+\beta\prod_{i=1}^{m}(\lambda_i-r_i)L_K\|p_n-z\|\\
&\quad +\gamma\prod_{i=1}^{m}(\lambda_i-r_i)L_H\|p_n-z\|\\
&= [\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(\lambda_i-r_i)]\|p_n-z\|. \qquad (4.3)
\end{aligned}
$$

$$
\begin{aligned}
\|p_n-z\| &= |A(q_n)(t)-A(z)(t)|\\
&= |F\left(t,q_n(t),\int_{r_1}^{\nu_1}\dots\int_{r_m}^{\nu_m}K(t,s,q_n(s))ds,\int_{r_1}^{\lambda_1}\dots\int_{r_m}^{\lambda_m}H(t,s,q_n(s))ds\right)\\
&\quad -F\left(t,z(t),\int_{r_1}^{\nu_1}\dots\int_{r_m}^{\nu_m}K(t,s,z(s))ds,\int_{r_1}^{\lambda_1}\dots\int_{r_m}^{\lambda_m}H(t,s,z(s))ds\right)|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \alpha|q_n(t)-z(t)|+\beta|\int_{r_1}^{\nu_1}\ldots \\
&\quad \int_{r_m}^{\nu_m} K(t,s,q_n(s))ds-\int_{r_1}^{\nu_1}\ldots\int_{r_m}^{\nu_m} K(t,s,z(s))ds| \\
&\quad +\gamma|\int_{r_1}^{\lambda_1}\ldots\int_{r_m}^{\lambda_m} H(t,s,q_n(s))ds-\int_{r_1}^{\lambda_1}\ldots\int_{r_m}^{\lambda_m} H(t,s,z(s))ds| \\
&\leq \alpha|q_n(t)-z(t)|+\beta\int_{r_1}^{\nu_1}\ldots\int_{r_m}^{\nu_m}|K(t,s,q_n(s))-K(t,s,z(s))|ds \\
&\quad +\gamma\int_{r_1}^{\lambda_1}\ldots\int_{r_m}^{\lambda_m}|H(t,s,q_n(s))-H(t,s,z(s))|ds \\
&\leq \alpha|q_n(t)-z(t)|+\beta\int_{r_1}^{\nu_1}\ldots\int_{r_m}^{\nu_m} L_K|q_n(s)-z(s)|ds \\
&\quad +\gamma\int_{r_1}^{\lambda_1}\ldots\int_{r_m}^{\lambda_m} L_H|q_n(s)-z(s)|ds \\
&\leq \alpha\|q_n-z\|+\beta\prod_{i=1}^{m}(\lambda_i-r_i)L_K\|q_n-z\| \\
&\quad +\gamma\prod_{i=1}^{m}(\lambda_i-r_i)L_H\|q_n-z\| \\
&= [\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(\lambda_i-r_i)]\|q_n-z\|. \qquad (4.4)
\end{aligned}
$$

Substituting (4.4) into (4.3) we have

$$
\|\omega_{n+1}-z\| \leq ([\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(\lambda_i-r_i)])^2\|q_n-z\|. \qquad (4.5)
$$

$$
\begin{aligned}
\|q_n-z\| &= |A(\zeta_n)(t)-A(z)(t)| \\
&= |F\left(t,\zeta_n(t),\int_{r_1}^{\nu_1}\ldots\int_{r_m}^{\nu_m} K(t,s,\zeta_n(s))ds,\int_{r_1}^{\lambda_1}\ldots\int_{r_m}^{\lambda_m} H(t,s,\zeta_n(s))ds\right) \\
&\quad -F\left(t,z(t),\int_{r_1}^{\nu_1}\ldots\int_{r_m}^{\nu_m} K(t,s,z(s))ds,\int_{r_1}^{\lambda_1}\ldots\int_{r_m}^{\lambda_m} H(t,s,z(s))ds\right)|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \alpha|\zeta_n(t)-z(t)|+\beta|\int_{r_1}^{\nu_1}\dots \\
&\quad \int_{r_m}^{\nu_m}K(t,s,\zeta_n(s))ds-\int_{r_1}^{\nu_1}\dots\int_{r_m}^{\nu_m}K(t,s,z(s))ds| \\
&\quad +\gamma|\int_{r_1}^{\lambda_1}\dots\int_{r_m}^{\lambda_m}H(t,s,\zeta_n(s))ds-\int_{r_1}^{\lambda_1}\dots\int_{r_m}^{\lambda_m}H(t,s,z(s))ds| \\
&\leq \alpha|\zeta_n(t)-z(t)|+\beta\int_{r_1}^{\nu_1}\dots\int_{r_m}^{\nu_m}|K(t,s,\zeta_n(s))-K(t,s,z(s))|ds \\
&\quad +\gamma\int_{r_1}^{\lambda_1}\dots\int_{r_m}^{\lambda_m}|H(t,s,\zeta_n(s))-H(t,s,z(s))|ds \\
&\leq \alpha|\zeta_n(t)-z(t)|+\beta\int_{r_1}^{\nu_1}\dots\int_{r_m}^{\nu_m}L_K|\zeta_n(s)-z(s)|ds \\
&\quad +\gamma\int_{r_1}^{\lambda_1}\dots\int_{r_m}^{\lambda_m}L_H|\zeta_n(s)-z(s)|ds \\
&\leq \alpha\|\zeta_n-z\|+\beta\prod_{i=1}^{m}(\lambda_i-r_i)L_K\|\zeta_n-z\| \\
&\quad +\gamma\prod_{i=1}^{m}(\lambda_i-r_i)L_H\|\zeta_n-z\| \\
&= [\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(\lambda_i-r_i)]\|\zeta_n-z\|. \qquad (4.6)
\end{aligned}
$$

Substituting (4.6) into (4.5) we have

$$
\|\omega_{n+1}-z\|\leq([\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(\lambda_i-r_i)])^3\|\zeta_n-z\|. \qquad (4.7)
$$

$$
\begin{aligned}
\|\zeta_n-z\| &\leq (1-\mu_n)|\omega_n(t)-z(t)|+\mu_n|A(\omega_n)(t)-A(z)(t)| \\
&= (1-\mu_n)|\omega_n(t)-z(t)| \\
&\quad +\mu_n|F\left(t,\omega_n(t),\int_{r_1}^{\nu_1}\dots\int_{r_m}^{\nu_m}K(t,s,\omega_n(s))ds,\int_{u_1}^{\lambda_1}\dots\int_{r_m}^{\nu_m}H(t,s,\omega_n(s))ds\right) \\
&\quad -F\left(t,z(t),\int_{r_1}^{\nu_1}\dots\int_{r_m}^{\nu_m}K(t,s,z(s))ds,\int_{r_1}^{\lambda_1}\dots\int_{r_m}^{\lambda_m}H(t,s,z(s))ds\right)| \\
&\leq (1-\mu_n)|\omega_n(t)-z(t)|+\mu_n\alpha|\omega_n(t)-z(t)|+\mu_n\beta\int_{r_1}^{\nu_1}\dots\int_{r_m}^{\nu_m}L_K|\omega_n(s)-z(s)|ds \\
&\quad +\mu_n\gamma\int_{r_1}^{\lambda_1}\dots\int_{r_m}^{\lambda_m}L_H|\omega_n(s)-z(s)|ds \\
&\leq \{1-\mu_n(1-[\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(\lambda_i-r_i)])\}\|\omega_n-z\|. \qquad (4.8)
\end{aligned}
$$

Substituting (4.8) into (4.7) we have

$$\begin{aligned}\|\omega_{n+1}-z\| &\leq ([\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(\lambda_i-r_i)])^3\\ &\quad\times\{1-\mu_n(1-[\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(\lambda_i-r_i)])\}\|\omega_n-z\|.\end{aligned} \tag{4.9}$$

Since from condition (B_5) we have $\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(\lambda_i-r_i)<1$, then it follows that $([\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(\lambda_i-r_i)])^3<1$. Hence from (4.9) we have

$$\|\omega_{n+1}-z\| \leq \{1-\mu_n(1-[\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(\lambda_i-r_i)])\}\|\omega_n-z\|. \tag{4.10}$$

From (4.10), we have the following inequalities:

$$\begin{aligned}\|\omega_{n+1}-z\| &\leq \{1-\mu_n(1-[\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(\lambda_i-r_i)])\}\|\omega_n-z\|\\ \|\omega_n-z\| &\leq \{1-\mu_{n-1}(1-[\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(\lambda_i-r_i)])\}\|\omega_{n-1}-z\|\\ &\vdots\\ \|\omega_1-p\| &\leq \{1-\mu_0(1-[\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(\lambda_i-r_i)])\}\|\omega_0-z\|.\end{aligned} \tag{4.11}$$

From (4.11), we have

$$\|\omega_{n+1}-z\| \leq \|\omega_0-z\|\prod_{k=0}^{n}\left\{1-\mu_k\left(1-[\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(\lambda_i-r_i)]\right)\right\}. \tag{4.12}$$

Since $\mu_k\in[0,1]$ for all $k\in\mathbb{N}$ and recalling from assumption (B_5) that $[\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(\lambda_i-r_i)]<1$, then we have

$$1-\mu_k(1-[\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(\lambda_i-r_i)<1. \tag{4.13}$$

We recall the inequality $1-\omega\leq e^{-\omega}$ for all $\omega\in[0,1]$, thus from (4.12), we have

$$\|\omega_{n+1}-z\| \leq \|\omega_0-z\|e^{-(1-[\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(\lambda_i-r_i)])\sum_{k=0}^{n}\mu_k}. \tag{4.14}$$

Taking the limit of both sides of the above inequalities, we have $\lim_{n\to\infty}\|\omega_n-z\|=0$. Hence, (1.8) converges strongly to the unique solution of the mixed type Volterra-Fredholm functional nonlinear integral equation (1.1). □

We now turn our attention to proving the data dependence of the solution for the integral equation (1.1) with help of AI iteration process (1.8).

Let E be as in the proof of Theorem (4.2) and $G, \tilde{G} : E \to E$ be two operators defined by:

$$G(\omega)(t) = F\left(t, \omega(t), \int_{r_1}^{\nu_1} \dots \int_{r_m}^{\nu_m} K(t,s,\omega(s))ds, \int_{r_1}^{\lambda_1} \dots \int_{r_m}^{\lambda_m} H(t,s,\omega(s))ds\right), \tag{4.15}$$

$$\tilde{G}(\omega)(t) = F\left(t, \omega(t), \int_{r_1}^{\nu_1} \dots \int_{r_m}^{\nu_m} \tilde{K}(t,s,\omega(s))ds, \int_{r_1}^{\lambda_1} \dots \int_{r_m}^{\lambda_m} \tilde{H}(t,s,\omega(s))ds\right), \tag{4.16}$$

where $K, \tilde{K}, H$ and $\tilde{H} \in C([r_1;\lambda_1] \times \cdots \times [r_m;\lambda_m] \times [r_1;\lambda_1] \times \cdots \times [r_m;\lambda_m] \times \Re)$.

Theorem 4.3. *Let F, K and H be as defined in Theorem (4.2). Let $\{\omega_n\}$ be an iterative sequence generated by AI iteration process (1.8) associated with G. Let $\{\tilde{\omega_n}\}$ be the an iterative sequence generated by*

$$\begin{cases} \tilde{\omega_0} \in E, \\ \tilde{\omega}_{n+1} = (1-\mu_n)\tilde{\omega_n} + \mu_n \tilde{G}\tilde{\omega_n}, \\ \tilde{p_n} = \tilde{G}\tilde{q_n}, \\ \tilde{q_n} = \tilde{G}\tilde{\zeta_n}, \\ \tilde{\zeta_n} = \tilde{G}\tilde{\omega_n}, \end{cases} \quad \forall n \geq 1. \tag{4.17}$$

where E is defined as in the proof of Theorem (4.2) and $\mu_n \in [0,1]$ is a real sequence satisfying

(V_1) $\frac{1}{2} \leq \mu_n$, *for all $n \geq 1$;*

(V_2) $\sum\limits_{n=1}^{\infty} \mu_n = \infty$. *In addition, suppose that;*

(V_3) *there exist nonnegative constants Λ_1 and Λ_2 such that $|K(t,s,f) - \tilde{K}(t,s,f)| \leq \Lambda_1$ and $|H(t,s,f) - \tilde{H}(t,s,f)| \leq \Lambda_2$, for all $f \in \Re$ and $t,s \in [r_1;\lambda_1] \times \cdots \times [r_m;\lambda_m]$.*

If z is the solution of (4.15) and also $\tilde{z}$ the solution of (4.16), then we have

$$\|z - \tilde{z}\| \leq \frac{7(\beta\Lambda_1 + \gamma\Lambda_2)\prod\limits_{i=1}^{m}(\lambda_i - r_i)}{1 - [\alpha + (\beta L_K + \gamma L_H)\prod\limits_{i=1}^{m}(\lambda_i - r_i)]}. \tag{4.18}$$

Proof. Using (1.8), (4.15), (4.16), (4.17), conditions $(V_1)-(V_3)$ and assumptions $(B_1)-(B_5)$, we obtain

$$\begin{aligned}
\|\omega_{n+1}-\tilde{\omega}_{n+1}\| &= \|Gp_n - \tilde{G}\tilde{p_n}\| \\
&= |F\left(t, p_n(t), \int_{r_1}^{\nu_1}\cdots\int_{r_m}^{\nu_m} K(t,s,p_n(s))ds, \int_{r_1}^{\lambda_1}\cdots\int_{r_m}^{\lambda_m} H(t,s,p_n(s))ds\right) \\
&\quad -F\left(t, \tilde{p_n}(t), \int_{r_1}^{\nu_1}\cdots\int_{r_m}^{\nu_m} \tilde{K}(t,s,\tilde{p_n}(s))ds, \int_{r_1}^{\lambda_1}\cdots\int_{r_m}^{\lambda_m} \tilde{H}(t,s,\tilde{p_n}(s))ds\right)| \\
&\leq \alpha|p_n(t)-\tilde{p_n}(t)| + \beta|\int_{r_1}^{\nu_1}\cdots \\
&\quad \int_{r_m}^{\nu_m} K(t,s,p_n(s))ds - \int_{r_1}^{\nu_1}\cdots\int_{r_m}^{\nu_m} \tilde{K}(t,s,\tilde{p}(s))ds| \\
&\quad +\gamma|\int_{r_1}^{\lambda_1}\cdots\int_{r_m}^{\lambda_m} H(t,s,p_n(s))ds - \int_{r_1}^{\lambda_1}\cdots\int_{r_m}^{\lambda_m} \tilde{H}(t,s,\tilde{p_n}(s))ds| \\
&\leq \alpha|p_n(t)-\tilde{p_n}(t)| + \beta\int_{r_1}^{\nu_1}\cdots\int_{u_m}^{\nu_m}(|K(t,s,p_n(s)) - K(t,s,\tilde{p_n}(s))| \\
&\quad +|K(t,s,\tilde{p_n}(s)) - \tilde{K}(t,s,\tilde{p_n}(s))|)ds \\
&\quad +\gamma\int_{r_1}^{\lambda_1}\cdots\int_{r_m}^{\lambda_m}(|H(t,s,p_n(s)) - H(t,s,\tilde{p_n}(s))| + \\
&\quad +|H(t,s,\tilde{p_n}(s)) - \tilde{H}(t,s,\tilde{p_n}(s))|)ds \\
&\leq \alpha|p_n(t)-\tilde{p_n}(t)| + \beta\int_{r_1}^{\nu_1}\cdots\int_{r_m}^{\nu_m}(L_K|p_n(s)-\tilde{p_n}(s)| + \Lambda_1)ds \\
&\quad +\gamma\int_{r_1}^{\lambda_1}\cdots\int_{r_m}^{\lambda_m}(L_H|p_n(s)-\tilde{p_n}(s)| + \Lambda_2)ds \\
&\leq \alpha\|p_n-\tilde{p_n}\| + \beta(L_K\|p_n-\tilde{p_n}\| + \Lambda_1)\prod_{i=1}^{m}(\lambda_i - r_i) \\
&\quad +\gamma(L_H\|p_n-\tilde{p_n}\| + \Lambda_2)\prod_{i=1}^{m}(\lambda_i - r_i) \\
&= [\alpha + (\beta L_K + \gamma L_H)\prod_{i=1}^{m}(\lambda_i - r_i)]\|p_n-\tilde{p_n}\| \\
&\quad +(\beta\Lambda_1 + \gamma\Lambda_2)\prod_{i=1}^{m}(\lambda_i - r_i). \qquad (4.19)
\end{aligned}$$

$$\begin{aligned}
\|p_n-\tilde{p_n}\| &= \|Gq_n - \tilde{G}\tilde{q_n}\| \\
&= |F\left(t, q_n(t), \int_{r_1}^{\nu_1}\cdots\int_{r_m}^{\nu_m} K(t,s,q_n(s))ds, \int_{r_1}^{\lambda_1}\cdots\int_{r_m}^{\lambda_m} H(t,s,q_n(s))ds\right) \\
&\quad -F\left(t, \tilde{q_n}(t), \int_{r_1}^{\nu_1}\cdots\int_{r_m}^{\nu_m} \tilde{K}(t,s,\tilde{q_n}(s))ds, \int_{r_1}^{\lambda_1}\cdots\int_{r_m}^{\lambda_m} \tilde{H}(t,s,\tilde{q_n}(s))ds\right)|
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha|q_n(t) - \tilde{q_n}(t)| + \beta|\int_{r_1}^{\nu_1} \dots \\
&\int_{r_m}^{\nu_m} K(t,s,q_n(s))ds - \int_{r_1}^{\nu_1} \dots \int_{r_m}^{\nu_m} \tilde{K}(t,s,\tilde{q}(s))ds| \\
&+\gamma|\int_{r_1}^{\lambda_1} \dots \int_{r_m}^{\lambda_m} H(t,s,q_n(s))ds - \int_{r_1}^{\lambda_1} \dots \int_{r_m}^{\lambda_m} \tilde{H}(t,s,\tilde{q_n}(s))ds| \\
&\leq \alpha|q_n(t) - \tilde{q_n}(t)| + \beta \int_{r_1}^{\nu_1} \dots \int_{r_m}^{\nu_m} (|K(t,s,q_n(s)) - K(t,s,\tilde{q_n}(s))| \\
&+|K(t,s,\tilde{q}_n(s)) - \tilde{K}(t,s,\tilde{q_n}(s))|)ds \\
&+\gamma \int_{r_1}^{\lambda_1} \dots \int_{r_m}^{\lambda_m} (|H(t,s,q_n(s)) - H(t,s,\tilde{q_n}(s))| + \\
&+|H(t,s,\tilde{q}_n(s)) - \tilde{H}(t,s,\tilde{q_n}(s))|)ds \\
&\leq \alpha|q_n(t) - \tilde{q_n}(t)| + \beta \int_{r_1}^{\nu_1} \dots \int_{r_m}^{\nu_m} (L_K|q_n(s) - \tilde{q_n}(s)| + \Lambda_1)ds \\
&+\gamma \int_{r_1}^{\lambda_1} \dots \int_{r_m}^{\lambda_m} (L_H|q_n(s) - \tilde{q_n}(s)| + \Lambda_2)ds \\
&\leq \alpha\|q_n - \tilde{q_n}\| + \beta(L_K\|q_n - \tilde{q_n}\| + \Lambda_1) \prod_{i=1}^{m} (\lambda_i - r_i) \\
&+\gamma(L_H\|q_n - \tilde{q_n}\| + \Lambda_2) \prod_{i=1}^{m} (\lambda_i - r_i) \\
&= [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^{m} (\lambda_i - r_i)]\|q_n - \tilde{q_n}\| \\
&+(\beta\Lambda_1 + \gamma\Lambda_2) \prod_{i=1}^{m} (\lambda_i - r_i). \qquad (4.20)
\end{aligned}
$$

Substituting (4.20) into (4.19) we have

$$
\begin{aligned}
\|\omega_{n+1} - \tilde{\omega}_{n+1}\| \leq\; & ([\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^{m} (\lambda_i - r_i)])^2 \|q_n - \tilde{q_n}\| \\
&+[\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^{m} (\lambda_i - r_i)](\beta\Lambda_1 + \gamma\Lambda_2) \prod_{i=1}^{m} (\lambda_i - r_i) \\
&+(\beta\Lambda_1 + \gamma\Lambda_2) \prod_{i=1}^{m} (\lambda_i - r_i). \qquad (4.21)
\end{aligned}
$$

$$
\begin{aligned}
\|q_n-\tilde{q}_n\| &= \|G\zeta_n-\tilde{G}\tilde{\zeta_n}\| \\
&= |F\left(t,\zeta_n(t),\int_{r_1}^{\nu_1}\dots\int_{r_m}^{\nu_m}K(t,s,\zeta_n(s))ds,\int_{r_1}^{\lambda_1}\dots\int_{r_m}^{\lambda_m}H(t,s,\zeta_n(s))ds\right) \\
&\quad -F\left(t,\tilde{\zeta_n}(t),\int_{r_1}^{\nu_1}\dots\int_{r_m}^{\nu_m}\tilde{K}(t,s,\tilde{\zeta_n}(s))ds,\int_{r_1}^{\lambda_1}\dots\int_{r_m}^{\lambda_m}\tilde{H}(t,s,\tilde{\zeta_n}(s))ds\right)| \\
&\le \alpha|\zeta_n(t)-\tilde{\zeta_n}(t)|+\beta|\int_{r_1}^{\nu_1}\dots \\
&\quad \int_{r_m}^{\nu_m}K(t,s,\zeta_n(s))ds-\int_{r_1}^{\nu_1}\dots\int_{r_m}^{\nu_m}\tilde{K}(t,s,\tilde{\zeta}(s))ds| \\
&\quad +\gamma|\int_{r_1}^{\lambda_1}\dots\int_{r_m}^{\lambda_m}H(t,s,\zeta_n(s))ds-\int_{r_1}^{\lambda_1}\dots\int_{r_m}^{\lambda_m}\tilde{H}(t,s,\tilde{\zeta_n}(s))ds| \\
&\le \alpha|\zeta_n(t)-\tilde{\zeta_n}(t)|+\beta\int_{r_1}^{\nu_1}\dots\int_{r_m}^{\nu_m}(|K(t,s,\zeta_n(s))-K(t,s,\tilde{\zeta_n}(s))| \\
&\quad +|K(t,s,\tilde{\zeta_n}(s))-\tilde{K}(t,s,\tilde{\zeta_n}(s))|)ds \\
&\quad +\gamma\int_{r_1}^{\lambda_1}\dots\int_{r_m}^{\lambda_m}(|H(t,s,\zeta_n(s))-H(t,s,\tilde{\zeta_n}(s))|+ \\
&\quad +|H(t,s,\tilde{\zeta_n}(s))-\tilde{H}(t,s,\tilde{\zeta_n}(s))|)ds \\
&\le \alpha|\zeta_n(t)-\tilde{\zeta_n}(t)|+\beta\int_{r_1}^{\nu_1}\dots\int_{r_m}^{\nu_m}(L_K|\zeta_n(s)-\tilde{\zeta_n}(s)|+\Lambda_1)ds \\
&\quad +\gamma\int_{r_1}^{\lambda_1}\dots\int_{r_m}^{\lambda_m}(L_H|\zeta_n(s)-\tilde{\zeta_n}(s)|+\Lambda_2)ds \\
&\le \alpha\|\zeta_n-\tilde{\zeta_n}\|+\beta(L_K\|\zeta_n-\tilde{\zeta_n}\|+\Lambda_1)\prod_{i=1}^{m}(\lambda_i-r_i) \\
&\quad +\gamma(L_H\|\zeta_n-\tilde{\zeta_n}\|+\Lambda_2)\prod_{i=1}^{m}(\lambda_i-r_i) \\
&= [\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(\lambda_i-r_i)]\|\zeta_n-\tilde{\zeta_n}\| \\
&\quad +(\beta\Lambda_1+\gamma\Lambda_2)\prod_{i=1}^{m}(\lambda_i-r_i). \qquad (4.22)
\end{aligned}
$$

Substituting (4.22) into (4.21), we have

$$
\begin{aligned}
\|\omega_{n+1}-\tilde{\omega}_{n+1}\| &\le ([\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(\lambda_i-r_i)])^3\|\zeta_n-\tilde{\zeta_n}\| \\
&\quad +([\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(\lambda_i-r_i)])^2(\beta\Lambda_1+\gamma\Lambda_2)\prod_{i=1}^{m}(\lambda_i-r_i)
\end{aligned}
$$

$$+[\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(\lambda_i-r_i)](\beta\Lambda_1+\gamma\Lambda_2)\prod_{i=1}^{m}(\lambda_i-r_i)$$
$$+(\beta\Lambda_1+\gamma\Lambda_2)\prod_{i=1}^{m}(\lambda_i-r_i). \tag{4.23}$$

$$\begin{aligned}
\|\zeta_n-\tilde{\zeta}_n\| &\le (1-\mu_n)|\omega_n(t)-\tilde{\omega_n}(t)|+\mu_n|G(\omega_n)(t)-\tilde{G}(\tilde{\omega_n})(t)| \\
&= (1-\mu_n)|\omega_n(t)-\tilde{\omega_n}(t)| \\
&\quad +\mu_n|F\left(t,\omega_n(t),\int_{r_1}^{\nu_1}\dots\int_{r_m}^{\nu_m}K(t,s,\omega_n(s))ds,\int_{r_1}^{\lambda_1}\dots\int_{r_m}^{\lambda_m}H(t,s,\omega_n(s))ds\right) \\
&\quad -F\left(t,\tilde{\omega_n}(t),\int_{r_1}^{\nu_1}\dots\int_{r_m}^{\nu_m}K(t,s,\tilde{\omega_n}(s))ds,\int_{r_1}^{\lambda_1}\dots\int_{r_m}^{\lambda_m}H(t,s,\tilde{\omega_n}(s))ds\right)| \\
&\le (1-\mu_n)|\omega_n(t)-\tilde{\omega_n}(t)| \\
&\quad +\mu_n\alpha|\omega_n(t)-\tilde{\omega_n}(t)|+\mu_n\beta\int_{r_1}^{\nu_1}\dots\int_{u_m}^{\nu_m}(L_K|\omega_n(s)-\tilde{\omega_n}(s)|+\Lambda_1)ds \\
&\quad +\mu_n\gamma\int_{r_1}^{\lambda_1}\dots\int_{r_m}^{\lambda_m}(L_H|\omega_n(s)-\tilde{\omega_n}(s)|+\Lambda_2)ds \\
&\le \{1-\mu_n(1-[\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(\lambda_i-r_i)])\}\|\omega_n-\tilde{\omega_n}\| \\
&\quad +\mu_n(\beta\Lambda_1+\gamma\Lambda_2)\prod_{i=1}^{m}(\lambda_i-r_i). \qquad (4.24)
\end{aligned}$$

Substituting (4.24) into (4.23), we obtain

$$\begin{aligned}
\|\omega_{n+1}-\tilde{\omega}_{n+1}\| &\le ([\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(\lambda_i-r_i)])^3 \\
&\quad \times\{1-\mu_n(1-[\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(\lambda_i-r_i)])\}\|\omega_n-\tilde{\omega_n}\| \\
&\quad +\mu_n(\beta\Lambda_1+\gamma\Lambda_2)\prod_{i=1}^{m}(\lambda_i-r_i)([\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(\lambda_i-r_i)])^3 \\
&\quad +([\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(\lambda_i-r_i)])^2(\beta\Lambda_1+\gamma\Lambda_2)\prod_{i=1}^{m}(\lambda_i-r_i) \\
&\quad +[\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(\lambda_i-r_i)](\beta\Lambda_1+\gamma\Lambda_2)\Pi_{i=1}^{m}(\lambda_i-r_i) \\
&\quad +(\beta\Lambda_1+\gamma\Lambda_2)\prod_{i=1}^{m}(\lambda_i-r_i). \qquad (4.25)
\end{aligned}$$

Recalling from assumption (B_5) that $[\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^{m} (\lambda_i - r_i)] < 1$ which follows that $([\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^{m} (\lambda_i - r_i)])^2 < 1$ and $([\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^{m} (\lambda_i - r_i)])^3 < 1$. Thus, (4.25) reduces to

$$\begin{aligned}
\|\omega_{n+1} - \tilde{\omega}_{n+1}\| &\leq \{1 - \mu_n(1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^{m} (\lambda_i - r_i)])\} \|\omega_n - \tilde{\omega}_n\| \\
&\quad + \mu_n(\beta\Lambda_1 + \gamma\Lambda_2) \prod_{i=1}^{m} (\lambda_i - r_i) + 3(\beta\Lambda_1 + \gamma\Lambda_2) \prod_{i=1}^{m} (\lambda_i - r_i).
\end{aligned} \tag{4.26}$$

From our assumption $\frac{1}{2} \leq \mu_n$, we have that

$$1 - \mu_n \leq \mu_n \Rightarrow 1 = 1 - \mu_n + \mu_n \leq \mu_n + \mu_n = 2\mu_n.$$

Thus, we have from (4.26) that

$$\begin{aligned}
\|\omega_{n+1} - \tilde{\omega}_{n+1}\| &\leq \{1 - \mu_n(1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^{m} (\lambda_i - r_i)])\} \|\omega_n - \tilde{\omega}_n\| \\
&\quad + \mu_n(\beta\Lambda_1 + \gamma\Lambda_2) \prod_{i=1}^{m} (\lambda_i - r_i) \\
&\quad + 3(1 - \mu_n + \mu_n)(\beta\Lambda_1 + \gamma\Lambda_2) \prod_{i=1}^{m} (\lambda_i - r_i) \\
&\leq \{1 - \mu_n(1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^{m} (\lambda_i - r_i)])\} \|\omega_n - \tilde{\omega}_n\| \\
&\quad + 7\mu_n(\beta\Lambda_1 + \gamma\Lambda_2) \prod_{i=1}^{m} (\lambda_i - r_i) \\
&= \{1 - \mu_n(1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^{m} (\lambda_i - r_i)])\} \|\omega_n - \tilde{\omega_n}\| \\
&\quad + \mu_n(1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^{m} (\lambda_i - r_i)]) \\
&\quad \times \left(\frac{7(\beta\Lambda_1 + \gamma\Lambda_2) \prod_{i=1}^{m} (\lambda_i - r_i)}{1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^{m} (\lambda_i - r_i)} \right).
\end{aligned} \tag{4.27}$$

For all $n \geq 1$, from (4.27) put

$$\begin{aligned} \rho_{\mathfrak{n}} &= \|\omega_n - \tilde{\omega_n}\|, \\ \tau_n &= \mu_n(1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^{m} (\lambda_i - r_i)]) \in (0,1), \\ \Psi_n &= \frac{7(\beta\Lambda_1 + \gamma\Lambda_2) \prod_{i=1}^{m} (\lambda_i - r_i)}{1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^{m} (\lambda_i - r_i)} \geq 0. \end{aligned}$$

Notice that (4.27) takes the form $\rho_{n+1} \leq (1-\tau_n)\rho_n + \tau_n \Psi_n$. Thus, all the conditions of Lemma 2.4 are satisfied. Hence, we obtain that

$$\|\omega_n - \tilde{\omega}_n\| \leq \frac{7(\beta\Lambda_1 + \gamma\Lambda_2) \prod_{i=1}^{m} (\lambda_i - r_i)}{1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^{m} (\lambda_i - r_i)}. \tag{4.28}$$

□

5. Application of AI iterative method to a Delay Differential Equation

Let $C([u,v])$ be endowed with Chebyshev norm $\|\omega - p\|_\infty = \max_{\ell \in [u,v]} |\omega(\ell) - p(\ell)|$, $\forall\ \omega, p \in C([u,v])$. Then the space $(C([u,v]), \|\cdot\|_\infty)$ is generally known to be a Banach space, see [35]. Our interest now is to consider the delay differential equation (1.2)-(1.3).

We assume that the following conditions are satisfied:

(M_1) $\ell_0, v \in \Re, \tau > 0$;
(M_2) $f \in C([\ell_0, v] \times \Re^2, \Re)$;
(M_3) $\psi \in C([\ell_0 - \tau, v], \Re)$;
(M_4) there exists L_f such that

$$|f(\ell, a_1, a_2) - f(\ell, b_1, b_2)| \leq L_f(|a_1 - b_1| + |a_2 - b_2|), \tag{5.1}$$

for all $a_1, a_2, b_1, b_2 \in \Re$ and $\ell \in [\ell_0, v]$;
(M_5) $2L_f(v - \ell_0) < 1$.

The problem (1.2)-(1.3) can be reformulated in the following integral equation:

$$\omega(\ell) = \begin{cases} \psi(\ell), & \ell \in [\ell_0 - \tau, \ell_0], \\ \psi(\ell_0) + \int_{\ell_0}^{\ell} f(s, \omega(s), \omega(s-\tau))ds, & \ell \in [\ell_0, v]. \end{cases} \tag{5.2}$$

The following result which was obtained Coman et al. [20] will be useful in proving our main result in this section.

Theorem 5.1 (see [20]). *Suppose that conditions $(M_1) - (M_5)$ are satisfied. Then the problem (1.2)-(1.3) has unique solution, $z \in C([\ell_0 - \tau, v], \Re) \cap C^1([\ell_0, v], \Re)$ and the Picard iterative method converges to z for any $\wp \in C([\ell_0 - \tau, v], \Re)$.*

Now, we prove the following result using our new iterative scheme (1.8).

Theorem 5.2. *Suppose that conditions* $(M_1)-(M_5)$ *are satisfied. Then the iterative sequence* $\{\omega_n\}$ *generated by AI iterative method* (1.8) *with* $\sum_{n=1}^{\infty}\mu_n=\infty$, *converges strongly to the unique solution of the problem* (1.2)-(1.3), *say* $z\in C([\ell_0-\tau,v],\Re)\cap C^1([\ell_0,v],\Re)$ *for any* $\wp\in C([\ell_0-\tau,v],\Re)$.

Proof. Let $\{\omega_n\}$ be an iterative sequence generated by the IK iterative process (1.8) for an operator defined by:

$$G\omega(\ell)=\begin{cases}\psi(\ell), & \ell\in[\ell_0-\tau,\ell_0],\\ \psi(\ell_0)+\int_{\ell_0}^{\ell}f(s,\omega(s),\omega(s-\tau))ds, & \ell\in[\ell_0,v].\end{cases}\tag{5.3}$$

Let z be the fixed point of G. We will now prove that $\omega_n\to z$ as $n\to\infty$. Apparently, it is easy to see that $\omega_n\to z$ as $n\to\infty$, for $\ell\in[\ell_0-\tau,\ell_0]$.

For $\ell\in[\ell_0,v]$, we have from (1.8) and (M_4) that

$$\begin{aligned}
\|\omega_{n+1}-z\|_\infty &= \|Gp_n-z\|_\infty\\
&= \|Gp_n-Gz\|_\infty\\
&= \max_{\ell\in[\ell_0-\tau,v]}|Gp_n(\ell)-Gz(\ell)|\\
&= \max_{\ell\in[\ell_0-\tau,v]}\Big|\psi(\ell_o)\\
&\quad+\int_{\ell_0}^{\ell}f(s,p_n(s),p_n(s-\tau))ds-\psi(\ell_o)-\int_{\ell_0}^{\ell}f(s,z(s),z(s-\tau))ds\Big|\\
&= \max_{\ell\in[\ell_0-\tau,v]}\left|\int_{\ell_0}^{\ell}f(s,p_n(s),p_n(s-\tau))ds-\int_{\ell_0}^{\ell}f(s,z(s),z(s-\tau))ds\right|\\
&\le \max_{\ell\in[\ell_0-\tau,v]}\int_{\ell_0}^{\ell}|f(s,p_n(s),p_n(s-\tau))-f(s,z(s),z(s-\tau))|ds\\
&\le \max_{\ell\in[\ell_0-\tau,v]}\int_{\ell_0}^{\ell}L_f(|p_n(s)-z(s)|+|p_n(s-\tau)-z(s-\tau)|)ds\\
&\le \int_{\ell_0}^{\ell}L_f(\max_{\ell\in[\ell_0-\tau,v]}|p_n(s)-z(s)|+\max_{\ell\in[\ell_0-\tau,v]}|p_n(s-\tau)-z(s-\tau)|)ds\\
&\le \int_{\ell_0}^{\ell}L_f(\|p_n-z\|_\infty+\|p_n-z\|_\infty)ds\\
&\le 2L_f(\ell-\ell_0)\|p_n-z\|_\infty\\
&\le 2L_f(v-\ell_0)\|p_n-z\|_\infty.
\end{aligned}\tag{5.4}$$

$$\begin{aligned}
\|p_n - z\|_\infty &= \|Gq_n - z\|_\infty \\
&= \|Gq_n - Gz\|_\infty \\
&= \max_{\ell\in[\ell_0-\tau,v]} |Gq_n(\ell) - Gz(\ell)| \\
&= \max_{\ell\in[\ell_0-\tau,v]} \Big|\psi(\ell_o) \\
&\quad + \int_{\ell_0}^{\ell} f(s,q_n(s),q_n(s-\tau))ds - \psi(\ell_o) - \int_{\ell_0}^{\ell} f(s,z(s),z(s-\tau))ds\Big| \\
&= \max_{\ell\in[\ell_0-\tau,v]} \left|\int_{\ell_0}^{\ell} f(s,q_n(s),q_n(s-\tau))ds - \int_{\ell_0}^{\ell} f(s,z(s),z(s-\tau))ds\right| \\
&\leq \max_{\ell\in[\ell_0-\tau,v]} \int_{\ell_0}^{\ell} |f(s,q_n(s),q_n(s-\tau)) - f(s,z(s),z(s-\tau))|ds \\
&\leq \max_{\ell\in[\ell_0-\tau,v]} \int_{\ell_0}^{\ell} L_f(|q_n(s)-z(s)| + |q_n(s-\tau)-z(s-\tau)|)ds \\
&\leq \int_{\ell_0}^{\ell} L_f(\max_{\ell\in[\ell_0-\tau,v]} |q_n(s)-z(s)| + \max_{\ell\in[\ell_0-\tau,v]} |q_n(s-\tau)-z(s-\tau)|)ds \\
&\leq \int_{\ell_0}^{\ell} L_f(\|q_n-z\|_\infty + \|q_n-z\|_\infty)ds \\
&\leq 2L_f(\ell-\ell_0)\|q_n-z\|_\infty \\
&\leq 2L_f(v-\ell_0)\|q_n-z\|_\infty. \qquad (5.5)
\end{aligned}$$

Putting (5.5) in (5.4), we obtain

$$\|\omega_{n+1} - z\| \leq [2L_f(v-\ell_0)]^2\|q_n - z\|_\infty. \qquad (5.6)$$

$$\begin{aligned}
\|q_n - z\|_\infty &= \|G\zeta_n - z\|_\infty \\
&= \|G\zeta_n - Gz\|_\infty \\
&= \max_{\ell\in[\ell_0-\tau,v]} |G\zeta_n(\ell) - Gz(\ell)| \\
&= \max_{\ell\in[\ell_0-\tau,v]} \Big|\psi(\ell_o) \\
&\quad + \int_{\ell_0}^{\ell} f(s,\zeta_n(s),\zeta_n(s-\tau))ds - \psi(\ell_o) - \int_{\ell_0}^{\ell} f(s,z(s),z(s-\tau))ds\Big| \\
&= \max_{\ell\in[\ell_0-\tau,v]} \left|\int_{\ell_0}^{\ell} f(s,\zeta_n(s),\zeta_n(s-\tau))ds - \int_{\ell_0}^{\ell} f(s,z(s),z(s-\tau))ds\right| \\
&\leq \max_{\ell\in[\ell_0-\tau,v]} \int_{\ell_0}^{\ell} |f(s,\zeta_n(s),\zeta_n(s-\tau)) - f(s,z(s),z(s-\tau))|ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \max_{\ell\in[\ell_0-\tau,v]}\int_{\ell_0}^{\ell} L_f(|\zeta_n(s)-z(s)|+|\zeta_n(s-\tau)-z(s-\tau)|)ds\\
&\leq \int_{\ell_0}^{\ell} L_f(\max_{\ell\in[\ell_0-\tau,v]}|\zeta_n(s)-z(s)|+\max_{\ell\in[\ell_0-\tau,v]}|\zeta_n(s-\tau)-z(s-\tau)|)ds\\
&\leq \int_{\ell_0}^{\ell} L_f(\|\zeta_n-z\|_\infty+\|\zeta_n-z\|_\infty)ds\\
&\leq 2L_f(\ell-\ell_0)\|\zeta_n-z\|_\infty\\
&\leq 2L_f(v-\ell_0)\|\zeta_n-z\|_\infty. \qquad (5.7)
\end{aligned}
$$

Putting (5.7) in (5.6), we obtain

$$
\|\omega_{n+1}-z\| \leq [2L_f(v-\ell_0)]^3\|\zeta_n-z\|_\infty. \qquad (5.8)
$$

$$
\begin{aligned}
\|\zeta_n-z\|_\infty &= \|(1-\mu_n)\omega_n+\mu_n G\omega_n-z\|_\infty\\
&= \|(1-\mu_n)(\omega_n-z)+\mu_n(G\omega_n-z)\|_\infty\\
&\leq (1-\mu_n)\|\omega_n-z\|_\infty+\mu_n\|G\omega_n-Gz\|_\infty\\
&= (1-\mu_n)\|\omega_n-z\|_\infty+\mu_n\max_{\ell\in[\ell_0-\tau,v]}|G\omega_n(\ell)-Gz(\ell)|\\
&= (1-\mu_n)\|\omega_n-z\|_\infty+\mu_n\max_{\ell\in[\ell_0-\tau,v]}\Big|\psi(\ell_o)\\
&\quad +\int_{\ell_0}^{\ell} f(s,\omega_n(s),\omega_n(s-\tau))ds-\psi(\ell_o)-\int_{\ell_0}^{\ell} f(s,z(s),z(s-\tau))ds\Big|\\
&= (1-\mu_n)\|\omega_n-z\|_\infty+\mu_n\max_{\ell\in[\ell_0-\tau,v]}\Big|\int_{\ell_0}^{\ell} f(s,\omega_n(s),\omega_n(s-\tau))ds\\
&\quad -\int_{\ell_0}^{\ell} f(s,z(s),z(s-\tau))ds\Big|\\
&\leq (1-\mu_n)\|\omega_n-z\|_\infty+\mu_n\max_{\ell\in[\ell_0-\tau,v]}\int_{\ell_0}^{\ell}|f(s,\omega_n(s),\omega_n(s-\tau))\\
&\quad -f(s,z(s),z(s-\tau))|ds\\
&\leq (1-\mu_n)\|\omega_n-z\|_\infty+\mu_n\max_{\ell\in[\ell_0-\tau,v]}\int_{\ell_0}^{\ell} L_f(|\omega_n(s)-z(s)|\\
&\quad +|\omega_n(s-\tau)-z(s-\tau)|)ds\\
&\leq (1-\mu_n)\|\omega_n-z\|_\infty+\mu_n\int_{\ell_0}^{\ell} L_f(\max_{\ell\in[\ell_0-\tau,v]}|q_n(s)-z(s)|\\
&\quad +\max_{\ell\in[\ell_0-\tau,v]}|\omega_n(s-\tau)-z(s-\tau)|)ds
\end{aligned}
$$

$$\begin{aligned}
&\leq (1-\mu_n)\|\omega_n - z\|_\infty + \mu_n \int_{\ell_0}^{\ell} L_f(\|\omega_n - z\|_\infty + \|\omega_n - z\|_\infty)ds \\
&\leq (1-\mu_n)\|\omega_n - z\|_\infty + 2\mu_n L_f(\ell - \ell_0)\|\omega_n - z\|_\infty \\
&\leq [1-\mu_n(1-2L_f(v-\ell_0))]\|\omega_n - z\|_\infty. \qquad (5.9)
\end{aligned}$$

Substituting (5.9) into (5.8), we have

$$\|\omega_{n+1} - z\| \leq [2L_f(v-\ell_0)]^3[1-\mu_n(1-2L_f(v-\ell_0))]\|\omega_n - z\|_\infty. \qquad (5.10)$$

Recalling from assumption (M_5) that $2L_f(v-\ell_0) < 1$, it follows that $[2L_f(v-\ell_0)]^3 < 1$. Thus from (5.10), we have

$$\|\omega_{n+1} - z\| \leq [1-\mu_n(1-2L_f(v-\ell_0))]\|\omega_n - z\|_\infty. \qquad (5.11)$$

Since $\mu_n \in [0,1]$, and from assumption (M_5) we set $\tau_n = \mu_n(1-2L_f(v-\ell_0)) < 1$. It follows that $\tau_n \in [0,1]$ such that $\sum_{n=0}^{\infty}\tau_n = \infty$ and also set $\rho_n = \|\omega_n - z\|_\infty$. Notice that (5.11) takes the form

$$\rho_{n+1} \leq (1-\tau_n)\rho_n. \qquad (5.12)$$

Thus all the conditions of Lemma 2.3 are satisfied. Hence, $\lim_{n\to\infty}\|\omega_n - z\|_\infty = 0$.
This completes the proof of Theorem 5.2. □

6. Conclusion

Fixed point theory play important role in applied science and engineering. Part of the beauty and applications of the concept of fixed point theory has been demonstrated in this paper. Owing to the fact that multi-steps iterative methods in most cases performs better in term faster rate of convergence than one-step and two-steps iterative method, hence, the results in this paper generalize, improve and unify the corresponding results in [20, 22, 25, 30, 31, 37, 52].

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