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SIMPSON TYPE INEQUALITIES FOR CONVEX FUNCTION BASED ON THE GENERALIZED FRACTIONAL INTEGRALS

XU-RAN HAI[1] AND SHU-HONG WANG[1]

ABSTRACT. In the paper, based on the generalized fractional integrals ${}^{\rho}\mathcal{K}^{\alpha}_{a+}f$ and ${}^{\rho}\mathcal{K}^{\alpha}_{b-}f$ with $f \in \mathfrak{X}^p_c(a,b)$, authors establish some Simpson type inequalities for convex function. The obtained inequalities generalize the corresponding results for Riemann-Liouville fractional integrals by taking limits when a parameter $\rho \to 1$.

1. INTRODUCTION

Fractional calculus is a field of applied mathematics and deals with derivatives and integrals of arbitrary orders (including complex orders). Although the definitions for fractional integrals are inconsistent and work in some cases but not in others, there are almost practical applications and profound impact in science, engineering, mathematics, economics, and other fields.

Suppose that (a, b) is a finite or infinite interval of the real line $\mathbb{R}$, where $a < b$ and $a, b \in [-\infty, +\infty]$, and α is a complex number with $\mathcal{R}e(\alpha) > 0$. Let $\Gamma(\cdot)$ be the Euleraŕs gamma function given by

$$\Gamma(\chi) = \int_0^{\infty} \tau^{\chi-1} e^{-\tau} \mathtt{d}\tau.$$

In [16], Podlubny introduced the left-side and right-side Riemann-Liouville fractional integrals of order α of a function f as follows:

$$\mathcal{R}^{\alpha}_{a+}f(\chi) = \frac{1}{\Gamma(\alpha)} \int_a^{\chi} (\chi - \tau)^{\alpha-1} f(\tau) \mathtt{d}\tau \tag{1.1}$$

and

$$\mathcal{R}^{\alpha}_{b-}f(\chi) = \frac{1}{\Gamma(\alpha)} \int_{\chi}^{b} (\tau - \chi)^{\alpha-1} f(\tau) \mathtt{d}\tau, \tag{1.2}$$

respectively.

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In [18], Samko introduced the left-side and right-side Hadamard fractional integrals of order α of a function f as follows

$$\mathcal{H}_{a+}^{\alpha} f(\chi) = \frac{1}{\Gamma(\alpha)} \int_a^{\chi} (\ln \chi - \ln \tau)^{\alpha-1} f(\tau) \frac{\mathtt{d}\tau}{\tau} \tag{1.3}$$

and

$$\mathcal{H}_{b-}^{\alpha} f(\chi) = \frac{1}{\Gamma(\alpha)} \int_{\chi}^{b} (\ln \tau - \ln \chi)^{\alpha-1} f(\tau) \frac{\mathtt{d}\tau}{\tau}, \tag{1.4}$$

respectively.

Suppose that $\mathfrak{X}_c^p(a,b)$ is the space of the complex-valued Lebesgue measurable functions f on $[a,b]$ with $\|f\|_{\mathfrak{X}_c^p} < \infty$, that is

$$\mathfrak{X}_c^p(a,b) = \{f : [a,b] \to \mathbb{C} | \|f\|_{\mathfrak{X}_c^p} < \infty\},$$

where the norm $\|f\|_{\mathfrak{X}_c^p}$ is

$$\|f\|_{\mathfrak{X}_c^p} = \left(\int_a^b |\tau^c f(\tau)|^p \frac{\mathtt{d}\tau}{\tau} \right)^{1/p} \quad for \quad 1 \le p < \infty \quad and \quad c \in \mathbb{R}$$

and

$$\|f\|_{\mathfrak{X}_c^{\infty}} = \mathtt{ess} \sup_{a \le \tau \le b} \left[\tau^c |f(\tau)| \right] \quad for \quad p = \infty \quad and \quad c \in \mathbb{R}.$$

In the sense of the above function space, Katugampola in [13] introduced the left-side and right-side fractional integrals of order α of a function $f \in \mathfrak{X}_c^p(a,b)$ defined by

$${}^{\rho}\mathcal{K}_{a+}^{\alpha} f(\chi) = \frac{1}{\Gamma(\alpha)} \int_a^{\chi} \left(\frac{\chi^{\rho} - \tau^{\rho}}{\rho} \right)^{\alpha-1} f(\tau) \frac{\mathtt{d}\tau}{\tau^{1-\rho}} \quad (\rho > 0) \tag{1.5}$$

and

$${}^{\rho}\mathcal{K}_{b-}^{\alpha} f(\chi) = \frac{1}{\Gamma(\alpha)} \int_{\chi}^{b} \left(\frac{\tau^{\rho} - \chi^{\rho}}{\rho} \right)^{\alpha-1} f(\tau) \frac{\mathtt{d}\tau}{\tau^{1-\rho}} \quad (\rho > 0), \tag{1.6}$$

respectively.

The newly defined fractional operators are known as Katugampola fractional integrals in [13] or $\rho-$Riemann-Liouville fractional integrals in [6], which generalized fractional integrals of Riemann-Liouville and Hadamard, respectively[14]:

$$\lim_{\rho \to 1} \left[{}^{\rho}\mathcal{K}_{a+}^{\alpha} f(\chi) \right] = \mathcal{R}_{a+}^{\alpha} f(\chi) \tag{1.7}$$

and

$$\lim_{\rho \to 0} \left[{}^{\rho}\mathcal{K}_{a+}^{\alpha} f(\chi) \right] = \mathcal{H}_{a+}^{\alpha} f(\chi). \tag{1.8}$$

The similar results for right-sided fractionals integral also hold.

For more results on the fractional integrals please see [5, 9, 10, 12, 15, 17, 22] and the references therein.

Simpson's inequality is indicated in [7].

Theorem 1.1. *The function $f:[a,b]\to\mathbb{R}$ is supposed to be four times continuously differentiable on the interval (a,b) with $\|f^{(4)}\|_\infty=\sup_{\chi\in(a,b)}|f^{(4)}(\chi)|<\infty$, then*

$$\left|\frac{f(a)+4f(\frac{a+b}{2})+f(b)}{6}-\frac{1}{b-a}\int_a^b f(\tau)\,\mathrm{d}\tau\right|\le\frac{1}{2880}\|f^{(4)}\|_\infty(b-a)^4. \tag{1.9}$$

Simpson's inequality is the most famous on account of its rich geometrical significance and applications. On more recent generalizations for Simpson's inequality, please see previous papers [1–4, 8, 11, 19–21] and the references therein.

In the paper, based on the generalized fractional integrals ${}^\rho\mathcal{K}^\alpha_{a+}f$ and ${}^\rho\mathcal{K}^\alpha_{b-}f$ with $f\in\mathfrak{X}^p_c(a,b)$, authors establish some Simpson type inequalities for convex function. The obtained inequalities generalize the corresponding results for Riemann-Liouville fractional integrals by taking limits when a parameter $\rho\to1$.

2. Main results

Now we give a interest equality.

Lemma 2.1. *Suppose that $f:[a^\rho,b^\rho]\to\mathbb{R}$ is differentiable with $\rho>0$ and $0\le a<b$, and $f\in\mathfrak{X}^p_c(a^\rho,b^\rho)$. If the generalized fractional integrals exist, then for any $\alpha>0$, the equality*

$$\begin{aligned}&\frac{f(a^\rho)+4f(\chi^\rho)+f(b^\rho)}{6}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}^\alpha_{\chi-}f(a^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}^\alpha_{\chi+}f(b^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right]\\ =&\frac{\rho}{6}\int_0^1(1-3\tau^{\rho\alpha})\tau^{\rho-1}\big[(b^\rho-\chi^\rho)f'(\tau^\rho\chi^\rho+(1-\tau^\rho)b^\rho)-(\chi^\rho-a^\rho)f'((1-\tau^\rho)a^\rho+\tau^\rho\chi^\rho)\big]\,\mathrm{d}\tau\end{aligned} \tag{2.1}$$

hold, where the fractional integrals are considered for the function $f(\chi^\rho)$ and evaluated at a and b, respectively.

Proof. Using integration by parts, it easy to follow that

$$\begin{aligned}&\frac{\rho}{6}\int_0^1(1-3\tau^{\rho\alpha})\tau^{\rho-1}\big[(b^\rho-\chi^\rho)f'(\tau^\rho\chi^\rho+(1-\tau^\rho)b^\rho)-(\chi^\rho-a^\rho)f'((1-\tau^\rho)a^\rho+\tau^\rho\chi^\rho)\big]\mathrm{d}\tau\\ =&\frac{2f(\chi^\rho)+f(b^\rho)}{6}-\frac{\rho\alpha}{2}\int_0^1\tau^{\rho\alpha-1}f(\tau^\rho\chi^\rho+(1-\tau^\rho)b^\rho)\mathrm{d}\tau\\ &+\frac{2f(\chi^\rho)+f(a^\rho)}{6}-\frac{\rho\alpha}{2}\int_0^1\tau^{\rho\alpha-1}f((1-\tau^\rho)a^\rho+\tau^\rho\chi^\rho)\mathrm{d}\tau\\ =&\frac{f(a^\rho)+4f(\chi^\rho)+f(b^\rho)}{6}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}^\alpha_{\chi-}f(a^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}^\alpha_{\chi+}f(b^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right],\end{aligned} \tag{2.2}$$

which completes the proof of Lemma 2.1. □

In particular, taking limits when $\chi\to a$ and $\chi\to b$ respectively in the identity (2.1), and using the L'Hospital rule, we have

$$\begin{aligned}&\frac{2f(a^\rho)+f(b^\rho)}{6}-\frac{\rho^\alpha\Gamma(\alpha+1)\big[{}^\rho\mathcal{K}^\alpha_{a+}f(b^\rho)\big]}{2(b^\rho-a^\rho)^\alpha}\\ =&\frac{\rho(b^\rho-a^\rho)}{6}\int_0^1(1-3\tau^{\rho\alpha})\tau^{\rho-1}f'(\tau^\rho a^\rho+(1-\tau^\rho)b^\rho)\mathrm{d}\tau\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}&\frac{2f(b^\rho)+f(a^\rho)}{6}-\frac{\rho^\alpha\Gamma(\alpha+1)\left[{}^\rho\mathcal{K}^\alpha_{b-}f(a^\rho)\right]}{2(b^\rho-a^\rho)^\alpha}\\=&\frac{\rho(b^\rho-a^\rho)}{6}\int_0^1(3\tau^{\rho\alpha}-1)\tau^{\rho-1}f'((1-\tau^\rho)a^\rho+\tau^\rho b^\rho)\mathrm{d}\tau.\end{aligned}\tag{2.4}$$

Summing the identities (2.4) and (2.3) and substituting the integral variable, we obtain

$$\begin{aligned}&\frac{f(a^\rho)+f(b^\rho)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho\mathcal{K}^\alpha_{a+}f(b^\rho)+{}^\rho\mathcal{K}^\alpha_{b-}f(a^\rho)\right]\\=&\frac{\rho(b^\rho-a^\rho)}{2}\int_0^1[(1-\tau^\rho)^\alpha-\tau^{\rho\alpha}]\tau^{\rho-1}f'(\tau^\rho a^\rho+(1-\tau^\rho)b^\rho)\mathrm{d}\tau,\end{aligned}\tag{2.5}$$

which is Lemma 2.4 in [5].

Furthermore, taking $\chi^\rho=\frac{a^\rho+b^\rho}{2}$ in the identity(2.1), we obtain the Simpson type equality

$$\begin{aligned}&\frac{f(a^\rho)+4f(\frac{a^\rho+b^\rho}{2})+f(b^\rho)}{6}-\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho\mathcal{K}^\alpha_{\sqrt[\rho]{\frac{a^\rho+b^\rho}{2}}-}f(a^\rho)+{}^\rho\mathcal{K}^\alpha_{\sqrt[\rho]{\frac{a^\rho+b^\rho}{2}}+}f(b^\rho)\right]\\=&\frac{\rho(b^\rho-a^\rho)}{12}\int_0^1(1-3\tau^{\rho\alpha})\tau^{\rho-1}\left[f'\left(\frac{\tau^\rho}{2}a^\rho+\left(1-\frac{\tau^\rho}{2}\right)b^\rho\right)-f'\left(\left(1-\frac{\tau^\rho}{2}\right)a^\rho+\frac{\tau^\rho}{2}b^\rho\right)\right]\mathrm{d}\tau.\end{aligned}\tag{2.6}$$

Also, taking limits when $\rho\to1$ in the equality (2.6), we immediately get the Simpson type equality for Riemann- Liouville fractional integrals:

$$\begin{aligned}&\frac{f(a)+4f(\frac{a+b}{2})+f(b)}{6}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\mathcal{R}^\alpha_{\frac{a+b}{2}-}f(a)+\mathcal{R}^\alpha_{\frac{a+b}{2}+}f(b)\right]\\=&\frac{b-a}{12}\int_0^1(1-3\tau^\alpha)\left[f'\left(\frac{\tau}{2}a+\left(1-\frac{\tau}{2}\right)b\right)-f'\left(\left(1-\frac{\tau}{2}\right)a+\frac{\tau}{2}b\right)\right]\mathrm{d}\tau.\end{aligned}\tag{2.7}$$

In particular, taking limits when $\rho\to1$ with $\alpha=1$ in the equality (2.6), we immediately get the following Simpson type equality:

$$\begin{aligned}&\frac{f(a)+4f(\frac{a+b}{2})+f(b)}{6}-\frac{1}{b-a}\int_a^b f(\tau)\mathrm{d}\tau\\=&\frac{b-a}{12}\int_0^1(1-3\tau)\left[f'\left(\frac{\tau}{2}a+\left(1-\frac{\tau}{2}\right)b\right)-f'\left(\left(1-\frac{\tau}{2}\right)a+\frac{\tau}{2}b\right)\right]\mathrm{d}\tau.\end{aligned}\tag{2.8}$$

Next, we establish some Simpson type inequalities by the differentiability, the convexity and Lemma 2.1.

If the function f' is differentiable, then the following Simpson type inequality holds.

Theorem 2.1. *Suppose that $f:[a^\rho,b^\rho]\to\mathbb{R}$ is differentiable with $\rho>0$ and $0\le a<b$, and $f\in\mathfrak{X}^p_c(a^\rho,b^\rho)$. If f' is differentiable, then the inequality*

$$\begin{aligned}&\left|\frac{f(a^\rho)+4f(\frac{a^\rho+b^\rho}{2})+f(b^\rho)}{6}-\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho\mathcal{K}^\alpha_{\sqrt[\rho]{\frac{a^\rho+b^\rho}{2}}-}f(a^\rho)+{}^\rho\mathcal{K}^\alpha_{\sqrt[\rho]{\frac{a^\rho+b^\rho}{2}}+}f(b^\rho)\right]\right|\\\le&\frac{(b^\rho-a^\rho)^2}{24(\alpha+1)(\alpha+2)3^{2/\alpha}}\left[(4-3\alpha-\alpha^2)3^{2/\alpha}+4\alpha(\alpha+2)3^{1/\alpha}-2\alpha(\alpha+1)\right]\sup_{\zeta\in(a^\rho,b^\rho)}|f''(\zeta)|\end{aligned}\tag{2.9}$$

holds for any $\alpha > 0$.

Proof. Using the identity (2.6) and the differential mean value theorem for the function f', it can state that

$$\begin{aligned}&\frac{f(a^\rho)+4f(\frac{a^\rho+b^\rho}{2})+f(b^\rho)}{6}-\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho\mathcal{K}^\alpha_{\sqrt[\rho]{\frac{a^\rho+b^\rho}{2}}-}f(a^\rho)+{}^\rho\mathcal{K}^\alpha_{\sqrt[\rho]{\frac{a^\rho+b^\rho}{2}}+}f(b^\rho)\right]\\=&\frac{\rho(b^\rho-a^\rho)^2}{12}\int_0^1(1-3\tau^{\rho\alpha})\tau^{\rho-1}(1-\tau^\rho)f''(\zeta(\tau))\mathrm{d}\tau,\end{aligned}\tag{2.10}$$

where $\zeta(\tau)\in(a^\rho,b^\rho)$.

Namely,

$$\begin{aligned}&\left|\frac{f(a^\rho)+4f(\frac{a^\rho+b^\rho}{2})+f(b^\rho)}{6}-\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho\mathcal{K}^\alpha_{\sqrt[\rho]{\frac{a^\rho+b^\rho}{2}}-}f(a^\rho)+{}^\rho\mathcal{K}^\alpha_{\sqrt[\rho]{\frac{a^\rho+b^\rho}{2}}+}f(b^\rho)\right]\right|\\\leq&\frac{\rho(b^\rho-a^\rho)^2}{12}\int_0^1\tau^{\rho-1}(1-\tau^\rho)|1-3\tau^{\rho\alpha}||f''(\zeta(\tau))|\mathrm{d}\tau\\\leq&\frac{(b^\rho-a^\rho)^2}{24(\alpha+1)(\alpha+2)3^{2/\alpha}}\left[(4-3\alpha-\alpha^2)3^{2/\alpha}+4\alpha(\alpha+2)3^{1/\alpha}-2\alpha(\alpha+1)\right]\sup_{\zeta\in(a^\rho,b^\rho)}|f''(\zeta)|.\end{aligned}\tag{2.11}$$

The proof of Theorem 2.1 is complete. □

Making limits when $\rho\to1$ in the inequality (2.9), we immediately get the Simpson type inequality for Riemann-Liouville fractional integrals:

$$\begin{aligned}&\left|\frac{f(a)+4f(\frac{a+b}{2})+f(b)}{6}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\mathcal{R}^\alpha_{\frac{a+b}{2}-}f(a)+\mathcal{R}^\alpha_{\frac{a+b}{2}+}f(b)\right]\right|\\\leq&\frac{(b-a)^2}{24(\alpha+1)(\alpha+2)3^{2/\alpha}}\left[(4-3\alpha-\alpha^2)3^{2/\alpha}+4\alpha(\alpha+2)3^{1/\alpha}-2\alpha(\alpha+1)\right]\sup_{\zeta\in(a,b)}|f''(\zeta)|.\end{aligned}\tag{2.12}$$

In particular, taking limits when $\rho\to1$ with $\alpha=1$ in the inequality (2.9), we get the Simpson type inequality as following

$$\left|\frac{f(a)+4f(\frac{a+b}{2})+f(b)}{6}-\frac{1}{b-a}\int_a^b f(\tau)\mathrm{d}\tau\right|\leq\frac{4(b-a)^2}{81}\sup_{\zeta\in(a,b)}|f''(\zeta)|.\tag{2.13}$$

If the function $|f'|$ is convex, then the below result holds.

Theorem 2.2. *Suppose that $f : [a^\rho, b^\rho] \to \mathbb{R}$ is differentiable with $\rho > 0$ and $0 \le a < b$, and $f \in \mathfrak{X}_c^p(a^\rho, b^\rho)$. If $|f'|$ is convex, then the inequality*

$$\begin{aligned}
&\left|\frac{f(a^\rho)+4f(\chi^\rho)+f(b^\rho)}{6} - \frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}^\alpha_{\chi-}f(a^\rho)}{(\chi^\rho-a^\rho)^\alpha} + \frac{{}^\rho\mathcal{K}^\alpha_{\chi+}f(b^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right]\right| \\
\le& \frac{1}{12(\alpha+1)(\alpha+2)3^{2/\alpha}}\Big\{(b^\rho-\chi^\rho)\Big[\Big((4+3\alpha-\alpha^2)3^{2/\alpha}+2\alpha(\alpha+1)\Big)|f'(\chi^\rho)| \\
&+\Big((4-3\alpha-\alpha^2)3^{2/\alpha}+4\alpha(\alpha+2)3^{1/\alpha}-2\alpha(\alpha+1)\Big)|f'(b^\rho)|\Big] \\
&+(\chi^\rho-a^\rho)\Big[\Big((4+3\alpha-\alpha^2)3^{2/\alpha}+2\alpha(\alpha+1)\Big)|f'(\chi^\rho)| \\
&+\Big((4-3\alpha-\alpha^2)3^{2/\alpha}+4\alpha(\alpha+2)3^{1/\alpha}-2\alpha(\alpha+1)\Big)|f'(a^\rho)|\Big]\Big\} \qquad (2.14)
\end{aligned}$$

holds for any $\alpha > 0$ and any $\chi \in [a, b]$.

Proof. Using Lemma 2.1 and the convexity of the function $|f'|$, we have

$$\begin{aligned}
&\left|\frac{f(a^\rho)+4f(\chi^\rho)+f(b^\rho)}{6} - \frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}^\alpha_{\chi-}f(a^\rho)}{(\chi^\rho-a^\rho)^\alpha} + \frac{{}^\rho\mathcal{K}^\alpha_{\chi+}f(b^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right]\right| \\
\le& \frac{\rho(b^\rho-\chi^\rho)}{6}\int_0^1 \tau^{\rho-1}|1-3\tau^{\rho\alpha}||f'((1-\tau^\rho)b^\rho+\tau^\rho\chi^\rho)|\mathtt{d}\tau \\
&+\frac{\rho(\chi^\rho-a^\rho)}{6}\int_0^1 \tau^{\rho-1}|1-3\tau^{\rho\alpha}||f'(\tau^\rho\chi^\rho+(1-\tau^\rho)a^\rho)|\mathtt{d}\tau \\
\le& \frac{\rho(b^\rho-\chi^\rho)}{6}\int_0^1 \tau^{\rho-1}|1-3\tau^{\rho\alpha}|[(1-\tau^\rho)|f'(b^\rho)|+\tau^\rho|f'(\chi^\rho)|]\mathtt{d}\tau \\
&+\frac{\rho(\chi^\rho-a^\rho)}{6}\int_0^1 \tau^{\rho-1}|1-3\tau^{\rho\alpha}|\Big[\tau^\rho|f'(\chi^\rho)|+(1-\tau^\rho)|f'(a^\rho)|\mathtt{d}\tau\Big]. \qquad (2.15)
\end{aligned}$$

By simple computation, the inequality (2.14) is obtained which completes the proof of Theorem 2.2. □

If we take $\chi^\rho = \frac{a^\rho+b^\rho}{2}$ in the inequality (2.14), then we get a Simpson type inequality.

$$\begin{aligned}
&\left|\frac{f(a^\rho)+4f\left(\frac{a^\rho+b^\rho}{2}\right)+f(b^\rho)}{6} - \frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho\mathcal{K}^\alpha_{\sqrt[\rho]{\frac{a^\rho+b^\rho}{2}}-}f(a^\rho) +{}^\rho\mathcal{K}^\alpha_{\sqrt[\rho]{\frac{a^\rho+b^\rho}{2}}+}f(b^\rho)\right]\right| \\
\le& \frac{(b^\rho-a^\rho)}{24(\alpha+1)(\alpha+2)3^{2/\alpha}}\Big\{\big[(4-3\alpha-\alpha^2)3^{2/\alpha}+4\alpha(\alpha+2)3^{1/\alpha}-2\alpha(\alpha+1)\big]\big[|f'(a^\rho)|+|f'(b^\rho)|\big] \\
&+2\big[(4+3\alpha-\alpha^2)3^{2/\alpha}+2\alpha(\alpha+1)\big]\left|f'\left(\frac{a^\rho+b^\rho}{2}\right)\right|\Big\}. \qquad (2.16)
\end{aligned}$$

Also, making limits when $\rho \to 1$ in the inequality (2.16), we immediately get the Simpson type inequalities for Riemann-Liouville fractional integrals:

$$\begin{aligned}
&\left|\frac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\mathcal{R}^\alpha_{\frac{a+b}{2}-}f(a)+\mathcal{R}^\alpha_{\frac{a+b}{2}+}f(b)\right]\right| \\
\le& \frac{b-a}{24(\alpha+1)(\alpha+2)3^{2/\alpha}}\Big\{\big[(4-3\alpha-\alpha^2)3^{2/\alpha}+4\alpha(\alpha+2)3^{1/\alpha}-2\alpha(\alpha+1)\big]\big[|f'(a)|+|f'(b)|\big] \\
&+2\big[(4+3\alpha-\alpha^2)3^{2/\alpha}+2\alpha(\alpha+1)\big]\left|f'\left(\frac{a+b}{2}\right)\right|\Big\}. \qquad (2.17)
\end{aligned}$$

In particular, taking limits when $\rho \to 1$ with $\alpha = 1$ in the inequality (2.16), we get the Simpson type inequality as following

$$\left|\frac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6}-\frac{1}{b-a}\int_a^b f(\tau)\mathrm{d}\tau\right| \leq \frac{b-a}{324}\left\{8\left[|f'(a)|+|f'(b)|\right]+11\left|f'\left(\frac{a+b}{2}\right)\right|\right\}. \tag{2.18}$$

If we consider the convexity of $|f'|$ in Equation (2.6), another Simpson type inequality also is derived.

Theorem 2.3. *Suppose that $f : [a^\rho, b^\rho] \to \mathbb{R}$ is differentiable with $\rho > 0$ and $0 \leq a < b$, and $f \in \mathfrak{X}_c^p(a^\rho, b^\rho)$. If $|f'|$ is convex, then the inequality*

$$\begin{aligned}&\left|\frac{f(a^\rho)+4f\left(\frac{a^\rho+b^\rho}{2}\right)+f(b^\rho)}{6}-\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho\mathcal{K}^\alpha_{\sqrt[\rho]{\frac{a^\rho+b^\rho}{2}}-}f(a^\rho)+{}^\rho\,\mathcal{K}^\alpha_{\sqrt[\rho]{\frac{a^\rho+b^\rho}{2}}+}f(b^\rho)\right]\right|\\ \leq&\frac{(b^\rho-a^\rho)\left(2-\alpha+2\alpha 3^{-1/\alpha}\right)}{12(\alpha+1)}\left[|f'(a^\rho)|+|f'(b^\rho)|\right]\end{aligned} \tag{2.19}$$

holds for any $\alpha > 0$.

Proof. Using the identity (2.6) and the convexity of the function $|f'|$, it note that

$$\begin{aligned}&\left|\frac{f(a^\rho)+4f\left(\frac{a^\rho+b^\rho}{2}\right)+f(b^\rho)}{6}-\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho\mathcal{K}^\alpha_{\sqrt[\rho]{\frac{a^\rho+b^\rho}{2}}-}f(a^\rho)+{}^\rho\,\mathcal{K}^\alpha_{\sqrt[\rho]{\frac{a^\rho+b^\rho}{2}}+}f(b^\rho)\right]\right|\\ \leq&\frac{\rho(b^\rho-a^\rho)}{12}\int_0^1\tau^{\rho-1}|1-3\tau^{\rho\alpha}|\left[\left|f'\left(\frac{\tau^\rho}{2}a^\rho+\left(1-\frac{\tau^\rho}{2}\right)b^\rho\right)\right|+\left|f'\left(\left(1-\frac{\tau^\rho}{2}\right)a^\rho+\frac{\tau^\rho}{2}b^\rho\right)\right|\right]\mathrm{d}\tau\\ \leq&\frac{\rho(b^\rho-a^\rho)}{12}\left[|f'(a^\rho)|+|f'(b^\rho)|\right]\int_0^1\tau^{\rho-1}|1-3\tau^{\rho\alpha}|\mathrm{d}\tau\\ =&\frac{(b^\rho-a^\rho)\left(2-\alpha+2\alpha 3^{-1/\alpha}\right)}{12(\alpha+1)}\left[|f'(a^\rho)|+|f'(b^\rho)|\right],\end{aligned} \tag{2.20}$$

which complete the proof of Theorem 2.3. □

Specially, making limits when $\rho \to 1$ in the inequality (2.19), we immediately get the Simpson type inequalities for Riemann-Liouville fractional integrals:

$$\begin{aligned}&\left|\frac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\mathcal{R}^\alpha_{\frac{a+b}{2}-}f(a)+\mathcal{R}^\alpha_{\frac{a+b}{2}+}f(b)\right]\right|\\ \leq&\frac{(b-a)\left(2-\alpha+2\alpha 3^{-1/\alpha}\right)}{12(\alpha+1)}\left[|f'(a)|+|f'(b)|\right].\end{aligned} \tag{2.21}$$

Also, taking limits when $\rho \to 1$ with $\alpha = 1$ in the inequality (2.19), we get the Simpson type inequality as following

$$\left|\frac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6}-\frac{1}{b-a}\int_a^b f(\tau)\mathrm{d}\tau\right| \leq \frac{5(b-a)}{72}\left[|f'(a)|+|f'(b)|\right]. \tag{2.22}$$

If the function $|f'|^q (q > 1)$ is convex, then the below Simpson type inequality holds.

Theorem 2.4. *Suppose that $f : [a^\rho, b^\rho] \to \mathbb{R}$ is differentiable with $\rho > 0$ and $0 \le a < b$, and $f \in \mathfrak{X}_c^p(a^\rho, b^\rho)$. If $|f'|^q (q > 1)$ is convex, then the inequality*

$$\begin{aligned}
&\left|\frac{f(a^\rho)+4f\left(\frac{a^\rho+b^\rho}{2}\right)+f(b^\rho)}{6}-\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho\mathcal{K}^\alpha_{\sqrt[\rho]{\frac{a^\rho+b^\rho}{2}}-}f(a^\rho)+{}^\rho\mathcal{K}^\alpha_{\sqrt[\rho]{\frac{a^\rho+b^\rho}{2}}+}f(b^\rho)\right]\right| \\
\le&\frac{\rho(b^\rho-a^\rho)(q-1)^{1-1/q}}{2^{\frac{2q+1}{q}}3^{\frac{q(\alpha+1)+1}{\alpha q}}}\left(\frac{2\rho\alpha(q-1)+[2\rho(q-r)-\rho\alpha(q-1)+2(r-1)]3^{\frac{\rho(q-r)+r-1}{\rho\alpha(q-1)}}}{[\rho(q-r)+r-1][\rho(q-r)+\rho\alpha(q-1)+r-1]}\right)^{1-1/q} \\
&\times\left\{\left[\frac{2\rho\alpha+[2(\rho-1)r+\rho(2-\alpha)+2]3^{\frac{(\rho-1)r+\rho+1}{\rho\alpha}}}{[(\rho-1)r+\rho+1][(\rho-1)r+\rho(\alpha+1)+1]}|f'(a^\rho)|^q\right.\right. \\
&+\left(\frac{4\rho\alpha3^{\frac{1}{\alpha}}+2[2(\rho-1)r-\rho\alpha+2]3^{\frac{(\rho-1)r+\rho+1}{\rho\alpha}}}{[(\rho-1)r+1][(\rho-1)r+\rho\alpha+1]}\right. \\
&\left.\left.-\frac{2\rho\alpha+[2(\rho-1)r+\rho(2-\alpha)+2]3^{\frac{(\rho-1)r+\rho+1}{\rho\alpha}}}{[(\rho-1)r+\rho+1][(\rho-1)r+\rho(\alpha+1)+1]}\right)|f'(b^\rho)|^q\right]^{1/q} \\
&+\left[\left(\frac{4\rho\alpha3^{\frac{1}{\alpha}}+2[2(\rho-1)r-\rho\alpha+2]3^{\frac{(\rho-1)r+\rho+1}{\rho\alpha}}}{[(\rho-1)r+1][(\rho-1)r+\rho\alpha+1]}\right.\right. \\
&\left.-\frac{2\rho\alpha+[2(\rho-1)r+\rho(2-\alpha)+2]3^{\frac{(\rho-1)r+\rho+1}{\rho\alpha}}}{[(\rho-1)r+\rho+1][(\rho-1)r+\rho(\alpha+1)+1]}\right)|f'(a^\rho)|^q \\
&\left.\left.+\frac{2\rho\alpha+[2(\rho-1)r+\rho(2-\alpha)+2]3^{\frac{(\rho-1)r+\rho+1}{\rho\alpha}}}{[(\rho-1)r+\rho+1][(\rho-1)r+\rho(\alpha+1)+1]}|f'(b^\rho)|^q\right]^{1/q}\right\} \qquad (2.23)
\end{aligned}$$

holds for any $\alpha > 0$ and $0 \le r \le q$.

Proof. Using the identity (2.6), Hölder's inequality and the convexity of the function $|f'|^q (q > 1)$, we have

$$\begin{aligned}
&\left|\frac{f(a^\rho)+4f\left(\frac{a^\rho+b^\rho}{2}\right)+f(b^\rho)}{6}-\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho\mathcal{K}^\alpha_{\sqrt[\rho]{\frac{a^\rho+b^\rho}{2}}-}f(a^\rho)+{}^\rho\mathcal{K}^\alpha_{\sqrt[\rho]{\frac{a^\rho+b^\rho}{2}}+}f(b^\rho)\right]\right| \\
\le&\frac{\rho(b^\rho-a^\rho)}{12}\int_0^1\tau^{\rho-1}|1-3\tau^{\rho\alpha}| \\
&\times\left[\left|f'\left(\frac{\tau^\rho}{2}a^\rho+\left(1-\frac{\tau^\rho}{2}\right)b^\rho\right)\right|+\left|f'\left(\left(1-\frac{\tau^\rho}{2}\right)a^\rho+\frac{\tau^\rho}{2}b^\rho\right)\right|\right]\mathrm{d}\tau \\
\le&\frac{\rho(b^\rho-a^\rho)}{12}\left(\int_0^1\tau^{\frac{(\rho-1)(q-r)}{q-1}}|1-3\tau^{\rho\alpha}|\mathrm{d}\tau\right)^{1-1/q} \\
&\times\left\{\left[\int_0^1\tau^{(\rho-1)r}|1-3\tau^{\rho\alpha}|\left(\frac{\tau^\rho}{2}|f'(a^\rho)|^q+\left(1-\frac{\tau^\rho}{2}\right)|f'(b^\rho)|^q\right)\mathrm{d}\tau\right]^{1/q}\right. \\
&\left.+\left[\int_0^1\tau^{(\rho-1)r}|1-3\tau^{\rho\alpha}|\left(\left(1-\frac{\tau^\rho}{2}\right)|f'(a^\rho)|^q+\frac{\tau^\rho}{2}|f'(b^\rho)|^q\right)\mathrm{d}\tau\right]^{1/q}\right\}. \qquad (2.24)
\end{aligned}$$

By direct calculation, we obtain the inequality (2.23) which completes the proof of Theorem 2.4. ☐

In particular, making $r=0$, $r=1$ and $r=q$, respectively, then

$$\begin{aligned}
&\left|\frac{f(a^\rho)+4f\left(\frac{a^\rho+b^\rho}{2}\right)+f(b^\rho)}{6}-\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho\mathcal{K}^\alpha_{\sqrt[\rho]{\frac{a^\rho+b^\rho}{2}}-}f\left(a^\rho\right)+{}^\rho\mathcal{K}^\alpha_{\sqrt[\rho]{\frac{a^\rho+b^\rho}{2}}+}f\left(b^\rho\right)\right]\right|\\
\leq&\frac{\rho(b^\rho-a^\rho)(q-1)^{1-1/q}}{2^{\frac{2q+1}{q}}3^{\frac{q(\alpha+1)+1}{\alpha q}}}\left(\frac{2\rho\alpha(q-1)+[2\rho q-\rho\alpha(q-1)-2]3^{\frac{\rho q-1}{\rho\alpha(q-1)}}}{(\rho q-1)[\rho q+\rho\alpha(q-1)-1]}\right)^{1-1/q}\\
&\times\left\{\left[\frac{2\rho\alpha+[2+\rho(2-\alpha)]3^{\frac{\rho+1}{\rho\alpha}}}{(\rho+1)[\rho(\alpha+1)+1]}|f'(a^\rho)|^q\right.\right.\\
&\left.+\left(\frac{4\rho\alpha3^{\frac{1}{\alpha}}+2(2-\rho\alpha)3^{\frac{\rho+1}{\rho\alpha}}}{\rho\alpha+1}-\frac{2\rho\alpha+[\rho(2-\alpha)+2]3^{\frac{\rho+1}{\rho\alpha}}}{(\rho+1)[\rho(\alpha+1)+1]}\right)|f'(b^\rho)|^q\right]^{1/q}\\
&+\left[\left(\frac{4\rho\alpha3^{\frac{1}{\alpha}}+2(2-\rho\alpha)3^{\frac{\rho+1}{\rho\alpha}}}{\rho\alpha+1}-\frac{2\rho\alpha+[\rho(2-\alpha)+2]3^{\frac{\rho+1}{\rho\alpha}}}{(\rho+1)[\rho(\alpha+1)+1]}\right)|f'(a^\rho)|^q\right.\\
&\left.\left.+\frac{2\rho\alpha+[2+\rho(2-\alpha)]3^{\frac{\rho+1}{\rho\alpha}}}{(\rho+1)[\rho(\alpha+1)+1]}|f'(b^\rho)|^q\right]^{1/q}\right\} \qquad (2.25)
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{f(a^\rho)+4f\left(\frac{a^\rho+b^\rho}{2}\right)+f(b^\rho)}{6}-\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho\mathcal{K}^\alpha_{\sqrt[\rho]{\frac{a^\rho+b^\rho}{2}}-}f\left(a^\rho\right)+{}^\rho\mathcal{K}^\alpha_{\sqrt[\rho]{\frac{a^\rho+b^\rho}{2}}+}f\left(b^\rho\right)\right]\right|\\
\leq&\frac{b^\rho-a^\rho}{2^{\frac{2q+1}{q}}3^{\frac{q(\alpha+1)+1}{\alpha q}}}\left(\frac{2\alpha+(2-\alpha)3^{1/\alpha}}{1+\alpha}\right)^{1-1/q}\left\{\left[\frac{2\alpha+(4-\alpha)3^{2/\alpha}}{2(\alpha+2)}|f'(a^\rho)|^q\right.\right.\\
&\left.+\left(\frac{8\alpha(\alpha+2)3^{1/\alpha}-2\alpha(\alpha+1)-3(4-\alpha-\alpha^2)3^{2/\alpha}}{2(\alpha+1)(\alpha+2)}\right)|f'(b^\rho)|^q\right]^{1/q}\\
&+\left[\frac{2\alpha+(4-\alpha)3^{2/\alpha}}{\alpha+2}|f'(b^\rho)|^q\right.\\
&\left.\left.+\left(\frac{8\alpha(\alpha+2)3^{1/\alpha}-2\alpha(\alpha+1)-3(4-\alpha-\alpha^2)3^{2/\alpha}}{2(\alpha+1)(\alpha+2)}\right)|f'(a^\rho)|^q\right]^{1/q}\right\} \qquad (2.26)
\end{aligned}$$

and

$$\left|\frac{f(a^\rho)+4f\left(\frac{a^\rho+b^\rho}{2}\right)+f(b^\rho)}{6}-\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho\mathcal{K}^\alpha_{\sqrt[\rho]{\frac{a^\rho+b^\rho}{2}}-}f\left(a^\rho\right)+{}^\rho\mathcal{K}^\alpha_{\sqrt[\rho]{\frac{a^\rho+b^\rho}{2}}+}f\left(b^\rho\right)\right]\right|$$

$$\leq \frac{\rho(b^\rho - a^\rho)}{2^{\frac{2q+1}{q}} 3^{\frac{q(\alpha+1)+1}{\alpha q}}} \left(\frac{2\rho\alpha + (2-\rho\alpha)3^{\frac{1}{\rho\alpha}}}{\rho\alpha+1} \right)^{1-1/q}$$
$$\times \left\{ \left[\left(\frac{2\rho\alpha + [2(\rho-1)q + \rho(2-\alpha)+2]3^{\frac{(\rho-1)q+\rho+1}{\rho\alpha}}}{[(\rho-1)q+\rho+1][(\rho-1)q+\rho(\alpha+1)+1]} |f'(a^\rho)|^q \right.\right.\right.$$
$$+ \left(\frac{4\rho\alpha 3^{\frac{1}{\alpha}} + 2[2(\rho-1)q - \rho\alpha + 2]3^{\frac{(\rho-1)q+\rho+1}{\rho\alpha}}}{[(\rho-1)q+1][(\rho-1)q+\rho\alpha+1]} \right.$$
$$\left.\left. - \frac{2\rho\alpha + [2(\rho-1)q + \rho(2-\alpha)+2]3^{\frac{(\rho-1)q+\rho+1}{\rho\alpha}}}{[(\rho-1)q+\rho+1][(\rho-1)q+\rho(\alpha+1)+1]} \right) |f'(b^\rho)|^q \right]^{1/q}$$
$$+ \left[\left(\frac{2\rho\alpha + [2(\rho-1)q + \rho(2-\alpha)+2]3^{\frac{(\rho-1)q+\rho+1}{\rho\alpha}}}{[(\rho-1)q+\rho+1][(\rho-1)q+\rho(\alpha+1)+1]} |f'(b^\rho)|^q \right.\right.$$
$$+ \left(\frac{4\rho\alpha 3^{\frac{1}{\alpha}} + 2[2(\rho-1)q - \rho\alpha + 2]3^{\frac{(\rho-1)q+\rho+1}{\rho\alpha}}}{[(\rho-1)q+1][(\rho-1)q+\rho\alpha+1]} \right.$$
$$\left.\left.\left. - \frac{2\rho\alpha + [2(\rho-1)q + \rho(2-\alpha)+2]3^{\frac{(\rho-1)q+\rho+1}{\rho\alpha}}}{[(\rho-1)q+\rho+1][(\rho-1)q+\rho(\alpha+1)+1]} \right) |f'(a^\rho)|^q \right]^{1/q} \right\}. \tag{2.27}$$

Also, making limitss when $\rho \to 1$ in the inequality (2.23), we immediately get the Simpson type inequality for Riemann-Liouville fractional integrals:

$$\left| \frac{f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)}{6} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[\mathcal{R}^\alpha_{\frac{a+b}{2}-} f(a) + \mathcal{R}^\alpha_{\frac{a+b}{2}+} f(b) \right] \right|$$
$$\leq \frac{b-a}{2^{\frac{2q+1}{q}} 3^{\frac{q(\alpha+1)+1}{\alpha q}}} \left(\frac{2\alpha + (2-\alpha)3^{\frac{1}{\alpha}}}{1+\alpha} \right)^{1-1/q} \left\{ \left[\left(\frac{4\alpha 3^{\frac{1}{\alpha}} + 2(2-\alpha)3^{\frac{2}{\alpha}}}{\alpha+1} - \frac{2\alpha + (4-\alpha)3^{\frac{2}{\alpha}}}{2(\alpha+2)} \right) |f'(b)|^q \right.\right.$$
$$\left. + \frac{2\alpha + (4-\alpha)3^{\frac{2}{\alpha}}}{2(\alpha+2)} |f'(a)|^q \right]^{1/q} + \left[\left(\frac{4\alpha 3^{\frac{1}{\alpha}} + 2(2-\alpha)3^{\frac{2}{\alpha}}}{\alpha+1} - \frac{2\alpha + (4-\alpha)3^{\frac{2}{\alpha}}}{2(\alpha+2)} \right) |f'(a)|^q \right.$$
$$\left.\left. + \frac{2\alpha + (4-\alpha)3^{\frac{2}{\alpha}}}{2(\alpha+2)} |f'(b)|^q \right]^{1/q} \right\}. \tag{2.28}$$

Specially, taking limits when $\rho \to 1$ with $\alpha = 1$ in the inequality (2.23), we get the Simpson type inequality as following

$$\left| \frac{f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)}{6} - \frac{1}{b-a} \int_a^b f(\tau) \mathrm{d}\tau \right|$$
$$\leq \frac{b-a}{36^{\frac{q+1}{q}}} \left(\frac{5}{2} \right)^{1-1/q} \left\{ \left[61|f'(b)|^q + 29|f'(a)|^q \right]^{1/q} + \left[29|f'(a)|^q + 61|f'(b)|^q \right]^{1/q} \right\}. \tag{2.29}$$

Furthermore, utilizing Lemma 2.1 reduces to the below inequality.

Theorem 2.5. *Suppose that* $f:[a^\rho,b^\rho]\to\mathbb{R}$ *is differentiable with* $\rho>0$ *and* $0\le a<b$, *and* $f\in\mathfrak{X}_c^p(a^\rho,b^\rho)$. *If* $|f'|^q(q>1)$ *is convex, then the inequality*

$$\begin{aligned}
&\left|\frac{f(a^\rho)+4f(\chi^\rho)+f(b^\rho)}{6}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}_{\chi-}^\alpha f(a^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}_{\chi+}^\alpha f(b^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right]\right|\\
\le&\frac{\rho(q-1)^{1-1/q}}{2\times 3^{\frac{q(\alpha+1)+1}{q\alpha}}}\left\{\frac{[2\rho(q-r)-\rho\alpha(q-1)+2(r-1)]3^{\frac{\rho(q-r)+r-1}{\rho\alpha(q-1)}}+2\rho\alpha(q-1)}{[\rho(q-r)+r-1][\rho(q-r)+\rho\alpha(q-1)+r-1]}\right\}^{1-1/q}\\
&\times\Bigg\{(b^\rho-\chi^\rho)\Bigg[\frac{[2(\rho-1)r+\rho(2-\alpha)+2]3^{\frac{(\rho-1)r+\rho+1}{\rho\alpha}}+2\rho\alpha}{[(\rho-1)r+\rho+1][(\rho-1)r+\rho(\alpha+1)+1]}|f'(\chi^\rho)|^q\\
&+\Bigg(\frac{[2(\rho-1)r-\rho\alpha+2]3^{\frac{(\rho-1)r+\rho+1}{\rho\alpha}}+2\rho\alpha 3^{1/\alpha}}{[(\rho-1)r+1][(\rho-1)r+\rho\alpha+1]}\\
&-\frac{[2(\rho-1)r+\rho(2-\alpha)+2]3^{\frac{(\rho-1)r+\rho+1}{\rho\alpha}}+2\rho\alpha}{[(\rho-1)r+\rho+1][(\rho-1)r+\rho(\alpha+1)+1]}\Bigg)|f'(b^\rho)|^q\Bigg]^{1/q}\\
&+(\chi^\rho-a^\rho)\Bigg[\frac{[2(\rho-1)r+\rho(2-\alpha)+2]3^{\frac{(\rho-1)r+\rho+1}{\rho\alpha}}+2\rho\alpha}{[(\rho-1)r+\rho+1][(\rho-1)r+\rho(\alpha+1)+1]}|f'(\chi^\rho)|^q\\
&+\Bigg(\frac{[2(\rho-1)r-\rho\alpha+2]3^{\frac{(\rho-1)r+\rho+1}{\rho\alpha}}+2\rho\alpha 3^{1/\alpha}}{[(\rho-1)r+1][(\rho-1)r+\rho\alpha+1]}\\
&-\frac{[2(\rho-1)r+\rho(2-\alpha)+2]3^{\frac{(\rho-1)r+\rho+1}{\rho\alpha}}+2\rho\alpha}{[(\rho-1)r+\rho+1][(\rho-1)r+\rho(\alpha+1)+1]}\Bigg)|f'(a^\rho)|^q\Bigg]^{1/q}\Bigg\}
\end{aligned}\tag{2.30}$$

holds for any $\alpha>0$, $0\le r\le q$ *and* $\chi\in[a,b]$.

Proof. Employing Lemma 2.1, Hölder's inequality and the convexity of the function $|f'|^q(q>1)$, we have

$$\begin{aligned}
&\left|\frac{f(a^\rho)+4f(\chi^\rho)+f(b^\rho)}{6}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}_{\chi-}^\alpha f(a^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}_{\chi+}^\alpha f(b^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right]\right|\\
\le&\frac{\rho(b^\rho-\chi^\rho)}{6}\int_0^1\tau^{\rho-1}|1-3\tau^{\rho\alpha}||f'((1-\tau^\rho)b^\rho+\tau^\rho\chi^\rho)|\mathtt{d}\tau\\
&+\frac{\rho(\chi^\rho-a^\rho)}{6}\int_0^1\tau^{\rho-1}|1-3\tau^{\rho\alpha}||f'(\tau^\rho\chi^\rho+(1-\tau^\rho)a^\rho)|\mathtt{d}\tau\\
\le&\frac{\rho(b^\rho-\chi^\rho)}{6}\left[\int_0^1\tau^{\frac{(\rho-1)(q-r)}{q-1}}|1-3\tau^{\rho\alpha}|\mathtt{d}\tau\right]^{1-1/q}\\
&\times\left[\int_0^1\tau^{(\rho-1)r}|1-3\tau^{\rho\alpha}|\left[(1-\tau^\rho)\left|f'(b^\rho)\right|^q+\tau^\rho\left|f'(\chi^\rho)\right|^q\right]\mathtt{d}\tau\right]^{1/q}\\
&+\frac{\rho(\chi^\rho-a^\rho)}{6}\left[\int_0^1\tau^{\frac{(\rho-1)(q-r)}{q-1}}|1-3\tau^{\rho\alpha}|\mathtt{d}\tau\right]^{1-1/q}\\
&\times\left[\int_0^1\tau^{(\rho-1)r}|1-3\tau^{\rho\alpha}|\left(\tau^\rho\left|f'(\chi^\rho)\right|^q+(1-\tau^\rho)\left|f'(a^\rho)\right|^q\right)\mathtt{d}\tau\right]^{1/q}.
\end{aligned}\tag{2.31}$$

By integration, we obtain the inequality (2.30) which prove Theorem 2.5. □

In particular, making $r=0$, $r=1$ and $r=q$ in the inequality (2.30), respectively, we have

$$\begin{aligned}
&\left|\frac{f(a^\rho)+4f(\chi^\rho)+f(b^\rho)}{6}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}^\alpha_{\chi-}f(a^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}^\alpha_{\chi+}f(b^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right]\right|\\
\leq&\frac{\rho(q-1)^{1-1/q}}{2\times 3^{\frac{q(\alpha+1)+1}{q\alpha}}}\left\{\frac{[2\rho q-\rho\alpha(q-1)-2]3^{\frac{\rho q-1}{\rho\alpha(q-1)}}+2\rho\alpha(q-1)}{(\rho q-1)[\rho q+\rho\alpha(q-1)-1]}\right\}^{1-1/q}\\
&\times\left\{(b^\rho-\chi^\rho)\left[\left(\frac{(2-\rho\alpha)3^{\frac{1+\rho}{\rho\alpha}}+2\rho\alpha 3^{\frac{1}{\alpha}}}{\rho\alpha+1}-\frac{[\rho(2-\alpha)+2]3^{\frac{\rho+1}{\rho\alpha}}+2\rho\alpha}{(\rho+1)[\rho(\alpha+1)+1]}\right)\left|f'(b^\rho)\right|^q\right.\right.\\
&\left.+\frac{[\rho(2-\alpha)+2]3^{\frac{\rho+1}{\rho\alpha}}+2\rho\alpha}{(\rho+1)[\rho(\alpha+1)+1]}\left|f'(\chi^\rho)\right|^q\right]^{1/q}\\
&+(\chi^\rho-a^\rho)\left[\left(\frac{(2-\rho\alpha)3^{\frac{1+\rho}{\rho\alpha}}+2\rho\alpha 3^{\frac{1}{\alpha}}}{\rho\alpha+1}-\frac{[\rho(2-\alpha)+2]3^{\frac{\rho+1}{\rho\alpha}}+2\rho\alpha}{(\rho+1)[\rho(\alpha+1)+1]}\right)\left|f'(a^\rho)\right|^q\right.\\
&\left.\left.+\frac{[\rho(2-\alpha)+2]3^{\frac{\rho+1}{\rho\alpha}}+2\rho\alpha}{(\rho+1)[\rho(\alpha+1)+1]}\left|f'(\chi^\rho)\right|^q\right]^{1/q}\right\}
\end{aligned}\tag{2.32}$$

and

$$\begin{aligned}
&\left|\frac{f(a^\rho)+4f(\chi^\rho)+f(b^\rho)}{6}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}^\alpha_{\chi-}f(a^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}^\alpha_{\chi+}f(b^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right]\right|\\
\leq&\frac{1}{2^{\frac{q+1}{q}}3^{\frac{q(\alpha+1)+1}{q\alpha}}}\left\{\frac{(2-\alpha)3^{\frac{1}{\alpha}}+2\alpha}{\alpha+1}\right\}^{1-1/q}\\
&\times\left\{(b^\rho-\chi^\rho)\left[\frac{(4-3\alpha-\alpha^2)3^{\frac{2}{\alpha}}+4\alpha(\alpha+2)3^{\frac{1}{\alpha}}-2\alpha(\alpha+1)}{(\alpha+1)(\alpha+2)}\left|f'(b^\rho)\right|^q\right.\right.\\
&\left.+\frac{(4-\alpha)3^{\frac{2}{\alpha}}+2\alpha}{\alpha+2}\left|f'(\chi^\rho)\right|^q\right]^{1/q}+(\chi^\rho-a^\rho)\left[\frac{(4-\alpha)3^{\frac{2}{\alpha}}+2\alpha}{\alpha+2}\left|f'(\chi^\rho)\right|^q\right.\\
&\left.\left.+\frac{(4-3\alpha-\alpha^2)3^{\frac{2}{\alpha}}+4\alpha(\alpha+2)3^{\frac{1}{\alpha}}-2\alpha(\alpha+1)}{(\alpha+1)(\alpha+2)}\left|f'(a^\rho)\right|^q\right]^{1/q}\right\}
\end{aligned}\tag{2.33}$$

and

$$\left|\frac{f(a^\rho)+4f(\chi^\rho)+f(b^\rho)}{6}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}^\alpha_{\chi-}f(a^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}^\alpha_{\chi+}f(b^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right]\right|$$

$$\le \frac{\rho}{2\times 3^{\frac{q(\alpha+1)+1}{q\alpha}}}\left\{\frac{(2-\rho\alpha)3^{\frac{1}{\rho\alpha}}+2\rho\alpha}{\rho\alpha+1}\right\}^{1-1/q}$$
$$\times\Bigg\{(b^{\rho}-\chi^{\rho})\Bigg[\Bigg(\frac{[2(\rho-1)q-\rho\alpha+2]3^{\frac{(\rho-1)q+\rho+1}{\rho\alpha}}+2\rho\alpha 3^{\frac{1}{\alpha}}}{[(\rho-1)q+1][(\rho-1)q+\rho\alpha+1]}$$
$$-\frac{[2(\rho-1)q+\rho(2-\alpha)+2]3^{\frac{(\rho-1)q+\rho+1}{\rho\alpha}}+2\rho\alpha}{[(\rho-1)q+\rho+1][(\rho-1)q+\rho(\alpha+1)+1]}\Bigg)|f'(b^{\rho})|^{q}$$
$$+\frac{[2(\rho-1)q+\rho(2-\alpha)+2]3^{\frac{(\rho-1)q+\rho+1}{\rho\alpha}}+2\rho\alpha}{[(\rho-1)q+\rho+1][(\rho-1)q+\rho(\alpha+1)+1]}|f'(\chi^{\rho})|^{q}\Bigg]^{1/q}$$
$$+(\chi^{\rho}-a^{\rho})\Bigg[\Bigg(\frac{[2(\rho-1)q-\rho\alpha+2]3^{\frac{(\rho-1)q+\rho+1}{\rho\alpha}}+2\rho\alpha 3^{\frac{1}{\alpha}}}{[(\rho-1)q+1][(\rho-1)q+\rho\alpha+1]}$$
$$-\frac{[2(\rho-1)q+\rho(2-\alpha)+2]3^{\frac{(\rho-1)q+\rho+1}{\rho\alpha}}+2\rho\alpha}{[(\rho-1)q+\rho+1][(\rho-1)q+\rho(\alpha+1)+1]}\Bigg)|f'(a^{\rho})|^{q}$$
$$+\frac{[2(\rho-1)q+\rho(2-\alpha)+2]3^{\frac{(\rho-1)q+\rho+1}{\rho\alpha}}+2\rho\alpha}{[(\rho-1)q+\rho+1][(\rho-1)q+\rho(\alpha+1)+1]}|f'(\chi^{\rho})|^{q}\Bigg]^{1/q}\Bigg\}. \tag{2.34}$$

Meantime, taking $\chi^{\rho}=\frac{a^{\rho}+b^{\rho}}{2}$ in the inequality (2.30), we obtain the following Simpson type inequality:

$$\left|\frac{f(a^{\rho})+4f\left(\frac{a^{\rho}+b^{\rho}}{2}\right)+f(b^{\rho})}{6}-\frac{2^{\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(b^{\rho}-a^{\rho})^{\alpha}}\left[{}^{\rho}\mathcal{K}^{\alpha}_{\sqrt[\rho]{\frac{a^{\rho}+b^{\rho}}{2}}-}f(a^{\rho})+{}^{\rho}\mathcal{K}^{\alpha}_{\sqrt[\rho]{\frac{a^{\rho}+b^{\rho}}{2}}+}f(b^{\rho})\right]\right|$$
$$\le\frac{\rho(q-1)^{1-1/q}(b^{\rho}-a^{\rho})}{4\times 3^{\frac{q(\alpha+1)+1}{q\alpha}}}\left\{\frac{[2\rho(q-r)-\rho\alpha(q-1)+2(r-1)]3^{\frac{\rho(q-r)+r-1}{\rho\alpha(q-1)}}+2\rho\alpha(q-1)}{[\rho(q-r)+r-1][\rho(q-r)+\rho\alpha(q-1)+r-1]}\right\}^{1-1/q}$$
$$\times\Bigg\{\Bigg[\Bigg(\frac{[2(\rho-1)r-\rho\alpha+2]3^{\frac{(\rho-1)r+\rho+1}{\rho\alpha}}+2\rho\alpha 3^{1/\alpha}}{[(\rho-1)r+1][(\rho-1)r+\rho\alpha+1]}$$
$$-\frac{[2(\rho-1)r+\rho(2-\alpha)+2]3^{\frac{(\rho-1)r+\rho+1}{\rho\alpha}}+2\rho\alpha}{[(\rho-1)r+\rho+1][(\rho-1)r+\rho(\alpha+1)+1]}\Bigg)|f'(b^{\rho})|^{q}$$
$$+\frac{[2(\rho-1)r+\rho(2-\alpha)+2]3^{\frac{(\rho-1)r+\rho+1}{\rho\alpha}}+2\rho\alpha}{[(\rho-1)r+\rho+1][(\rho-1)r+\rho(\alpha+1)+1]}\left|f'\left(\frac{a^{\rho}+b^{\rho}}{2}\right)\right|^{q}\Bigg]^{1/q}$$
$$+\Bigg[\Bigg(\frac{[2(\rho-1)r-\rho\alpha+2]3^{\frac{(\rho-1)r+\rho+1}{\rho\alpha}}+2\rho\alpha 3^{1/\alpha}}{[(\rho-1)r+1][(\rho-1)r+\rho\alpha+1]}$$
$$-\frac{[2(\rho-1)r+\rho(2-\alpha)+2]3^{\frac{(\rho-1)r+\rho+1}{\rho\alpha}}+2\rho\alpha}{[(\rho-1)r+\rho+1][(\rho-1)r+\rho(\alpha+1)+1]}\Bigg)|f'(a^{\rho})|^{q}$$
$$+\frac{[2(\rho-1)r+\rho(2-\alpha)+2]3^{\frac{(\rho-1)r+\rho+1}{\rho\alpha}}+2\rho\alpha}{[(\rho-1)r+\rho+1][(\rho-1)r+\rho(\alpha+1)+1]}\left|f'\left(\frac{a^{\rho}+b^{\rho}}{2}\right)\right|^{q}\Bigg]^{1/q}\Bigg\}. \tag{2.35}$$

Also, making limits when $\rho \to 1$ in the inequality (2.35), we immediately get the Simpson type inequality for Riemann-Liouville fractional integrals:

$$\begin{aligned}
&\left|\frac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathcal{R}^{\alpha}_{\frac{a+b}{2}-}f(a)+\mathcal{R}^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\right|\\
\leq&\frac{b-a}{4\times 3^{\frac{q(\alpha+1)+1}{q\alpha}}}\left\{\frac{(2-\alpha)3^{\frac{1}{\alpha}}+2\alpha}{1+\alpha}\right\}^{1-1/q}\left\{\left[\left(\frac{(2-\alpha)3^{\frac{2}{\alpha}}+2\alpha 3^{1/\alpha}}{\alpha+1}-\frac{(4-\alpha)3^{\frac{2}{\alpha}}+2\alpha}{2(\alpha+2)}\right)\left|f'(b)\right|^{q}\right.\right.\\
&\left.+\frac{(4-\alpha)3^{\frac{2}{\alpha}}+2\alpha}{2(\alpha+2)}\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\right]^{1/q}+\left[\left(\frac{(2-\alpha)3^{\frac{2}{\alpha}}+2\alpha 3^{1/\alpha}}{\alpha+1}-\frac{(4-\alpha)3^{\frac{2}{\alpha}}+2\alpha}{2(\alpha+2)}\right)\left|f'(a)\right|^{q}\right.\\
&\left.\left.+\frac{(4-\alpha)3^{\frac{2}{\alpha}}+2\alpha}{2(\alpha+2)}\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\right]^{1/q}\right\}. \qquad (2.36)
\end{aligned}$$

Specially, taking limits when $\rho \to 1$ with $\alpha = 1$ in the inequality (2.35), we get the Simpson type inequality as following

$$\begin{aligned}
&\left|\frac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6}-\frac{1}{b-a}\int_a^b f(\tau)\mathrm{d}\tau\right|\\
\leq&\frac{b-a}{2^{\frac{2q+1}{q}}6^{\frac{q+1}{q}}}\left(\frac{5}{2}\right)^{1-1/q}\left\{\left[16\left|f'(b)\right|^{q}+29\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\right]^{1/q}+\left[16\left|f'(a)\right|^{q}+29\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\right]^{1/q}\right\}. \qquad (2.37)
\end{aligned}$$

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[1]College of Mathematics and Physics,
Inner Mongolia University for Nationalities,
Tongliao Inner Mongolia Autonomous Region 028043, China
E-mail address: 2651721297@qq.com
E-mail address: shuhong7682@163.com (Corresponding author)