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NECESSARY AND SUFFICIENT CONDITIONS FOR A RATIO INVOLVING TRIGAMMA AND TETRAGAMMA FUNCTIONS TO BE MONOTONIC

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Dedicated to Professor Emeritus Christian Berg at University of Copenhagen in Denmark

Abstract. In the paper, by convolution theorem of the Laplace transforms, Bernstein's theorem for completely monotonic functions, and logarithmic concavity of a function involving exponential functions, the author

(a) finds necessary and sufficient conditions for a ratio involving trigamma and tetragamma functions to be monotonic on the right real semi-axis; and
(b) presents alternative proofs of necessary and sufficient conditions for a function and its negativity involving trigamma and tetragamma functions to be completely monotonic on the positive semi-axis.

These results generalizes known conclusions recently obtained by the author.

1. Motivations and main results

In the literature [1, Section 6.4], the function

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}\,t,\quad \Re(z)>0$$

and its logarithmic derivative $\psi(z)=[\ln\Gamma(z)]'=\frac{\Gamma'(z)}{\Gamma(z)}$ are called the classical Euler gamma function and digamma function respectively. Further, the functions $\psi'(z)$, $\psi''(z)$, $\psi'''(z)$, and $\psi^{(4)}(z)$ are known as trigamma, tetragamma, pentagamma, and hexagamma functions respectively. As a whole, all the derivatives $\psi^{(k)}(z)$ for $k\ge 0$ are known as polygamma functions.

Recall from Chapter XIII in [4], Chapter 1 in [17], and Chapter IV in [18] that, if a function $f(t)$ on an interval I has derivatives of all orders on I and satisfies inequalities

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$(-1)^n f^{(n)}(t) \ge 0$ for $t \in I$ and $n \in \{0\} \cup \mathbb{N}$, then we call $f(t)$ a completely monotonic function on I.

In [15, Theorem 4], the author turned out that,

(a) if and only if $\alpha \ge 2$, the function

$$\mathfrak{H}_\alpha(x) = \psi'(x) + x\psi''(x) + \alpha\big[x\psi'(x) - 1\big]^2 \tag{1.1}$$

is completely monotonic on $(0, \infty)$;

(b) if and only if $\alpha \le 1$, the function $-\mathfrak{H}_\alpha(x)$ is completely monotonic on $(0, \infty)$;

(c) the double inequality

$$-2 < \frac{\psi'(x) + x\psi''(x)}{[x\psi'(x) - 1]^2} < -1 \tag{1.2}$$

is valid and sharp in the sense that the constants -2 and -1 cannot be replaced by any bigger and smaller ones respectively.

In this paper, we mainly generalize the double inequality (1.2) by finding necessary and sufficient conditions in the following theorem.

Theorem 1.1. *Let*

$$H_\beta(x) = \frac{\psi'(x) + x\psi''(x)}{[x\psi'(x) - 1]^\beta} \tag{1.3}$$

on $(0, \infty)$ for $\beta \in \mathbb{R}$. Then the following conclusions are valid:

(a) *if and only if $\beta \ge 2$, the function $H_\beta(x)$ is decreasing on $(0, \infty)$, with the limits*

$$\lim_{x \to 0^+} H_\beta(x) = \begin{cases} -1, & \beta = 2 \\ 0, & \beta > 2 \end{cases} \quad \text{and} \quad \lim_{x \to \infty} H_\beta(x) = \begin{cases} -2, & \beta = 2 \\ -\infty, & \beta > 2; \end{cases} \tag{1.4}$$

(b) *if $\beta \le 1$, the function $H_\beta(x)$ is increasing on $(0, \infty)$, with the limits*

$$H_\beta(x) \to \begin{cases} -\infty, & x \to 0^+ \\ 0, & x \to \infty; \end{cases} \tag{1.5}$$

(c) *the double inequality (1.2) is true and sharp in the sense that the constants -2 and -1 cannot be replaced by any bigger and smaller ones respectively.*

The second aim of this paper is to supply alternative proofs of necessary and sufficient conditions on α for $\pm\mathfrak{H}_\alpha(x)$ in (1.1) to be completely monotonic on $(0, \infty)$.

2. Lemmas

The following lemmas are necessary in this paper.

Lemma 2.1 (Convolution theorem for the Laplace transforms [18, pp. 91–92])**.** *Let $f_k(t)$ for $k = 1, 2$ be piecewise continuous in arbitrary finite intervals included in $(0, \infty)$. If there exist some constants $M_k > 0$ and $c_k \ge 0$ such that $|f_k(t)| \le M_k e^{c_k t}$ for $k = 1, 2$, then*

$$\int_0^\infty \left[\int_0^t f_1(u) f_2(t - u) \,\mathrm{d}\, u\right] e^{-st} \,\mathrm{d}\, t = \int_0^\infty f_1(u) e^{-su} \,\mathrm{d}\, u \int_0^\infty f_2(v) e^{-sv} \,\mathrm{d}\, v.$$

Lemma 2.2 ([9, Theorem 6.1])**.** *If $f(x)$ is differentiable and logarithmically concave on $(-\infty,\infty)$, then the product $f(x)f(x_0-x)$ for any fixed number $x_0\in\mathbb{R}$ is increasing in $x\in\left(-\infty,\frac{x_0}{2}\right)$ and decreasing in $x\in\left(\frac{x_0}{2},\infty\right)$.*

Lemma 2.3. *Let*

$$h(t)=\begin{cases}\dfrac{e^t(e^t-1-t)}{(e^t-1)^2}, & t\ne0;\\[2mm] \dfrac12, & t=0.\end{cases}$$

Then the following conclusions are valid:

(a) *the function $h(t)$*

i *satisfies the identity*

$$h(t)+h(-t)=1 \tag{2.1}$$

on $(-\infty,\infty)$;

ii *is infinitely differentiable on $(-\infty,\infty)$, increasing from $(-\infty,\infty)$ onto $(0,1)$, convex on $(-\infty,0)$, concave on $(0,\infty)$, and logarithmically concave on $(-\infty,\infty)$;*

(b) *the function $\frac{h(2t)}{h^2(t)}$ is increasing from $(-\infty,0)$ onto $(0,2)$ and decreasing from $(0,\infty)$ onto $(1,2)$;*

(c) *the double inequality*

$$1<\frac{h(2t)}{h^2(t)}<2 \tag{2.2}$$

is valid on $(0,\infty)$ and sharp in the sense that the lower bound 1 and the upper bound 2 cannot be replaced by any larger scalar and any smaller scalar respectively;

(d) *for any fixed $t>0$, the function $h(st)h((1-s)t)$ is increasing in $s\in\left(0,\frac12\right)$.*

Proof. It is straightforward to prove the identity (2.1) on $(-\infty,\infty)$.

When $t\ne0$, we can rewrite $h(t)$ as

$$h(t)=\frac{e^t(e^t-1-t)/t^2}{[(e^t-1)/t]^2}=\frac{e^t\sum_{k=2}^\infty\frac{t^{k-2}}{k!}}{\left(\sum_{k=1}^\infty\frac{t^{k-1}}{k!}\right)^2}=\frac{e^t\sum_{k=0}^\infty\frac{t^k}{(k+2)!}}{\left[\sum_{k=0}^\infty\frac{t^k}{(k+1)!}\right]^2}$$

which implies that $t=0$ is a removable discontinuous point. Hence, the function $h(t)$ is infinitely differentiable on $(-\infty,\infty)$.

Standard computation shows that

$$h'(t)=\frac{e^t[e^t(t-2)+t+2]}{(e^t-1)^3}=\frac{e^t}{(e^t-1)^3}\sum_{k=3}^\infty(k-2)\frac{t^k}{k!},$$

which is positive on $(0,\infty)$, and

$$\begin{aligned}h''(t)&=-\frac{e^t\left[e^{2t}(t-3)+4e^tt+t+3\right]}{(e^t-1)^4}\\&=-\frac{e^t}{(e^t-1)^4}\left(\frac{t^5}{30}+\sum_{k=6}^\infty\left[(k-6)2^{k-1}+4k\right]\frac{t^k}{k!}\right),\end{aligned}$$

which is negative on $(0,\infty)$. Hence, we see that the function $h(t)$ is increasing and concave on $(0,\infty)$.

Differentiating with respect to t on both sides of (2.1) gives $h'(t) = h'(-t)$ and $h''(t) = -h''(-t)$ on $(-\infty, \infty)$. From this, we conclude that the function $h(t)$ is increasing and convex on $(-\infty, 0)$.

Taking the logarithm of $h(t)$, differentiating, and expanding yield

$$\begin{aligned}[\ln h(t)]'' &= -\frac{e^{3t}(t-3) + e^{2t}(2t+7) - e^t(2t^2+3t+5) + 1}{(e^t-1)^2(e^t-1-t)^2} \\ &= -\frac{\sum_{k=6}^{\infty}\left[3^{k-1}(k-9) + (2^k-1)(k+5) + 2(2^k-k^2)\right]\frac{t^k}{k!}}{(e^t-1)^2(e^t-1-t)^2} \\ &= -\frac{\frac{t^6}{36} + \frac{2t^7}{45} + \frac{3t^8}{80} + \frac{167t^9}{7560} + \frac{439t^{10}}{43200} + \frac{293t^{11}}{75600} + \cdots}{(e^t-1)^2(e^t-1-t)^2}\end{aligned}$$

and

$$\begin{aligned}[\ln h(-t)]'' &= -\frac{e^t\left[e^{3t} - e^{2t}(2t^2-3t+5) + e^t(7-2t) - t - 3\right]}{(e^t-1)^2[e^t(t-1)+1]^2} \\ &= -\frac{e^t\sum_{k=6}^{\infty}\left[3^k - 2^{k-1}(k^2-3k+10) + 2k(2^{k-2}-1) + 7\right]\frac{t^k}{k!}}{(e^t-1)^2[e^t(t-1)+1]^2} \\ &= -\frac{e^t\left(\frac{t^6}{36} + \frac{7t^7}{180} + \frac{7t^8}{240} + \frac{233t^9}{15120} + \frac{1933t^{10}}{302400} + \cdots\right)}{(e^t-1)^2[e^t(t-1)+1]^2}.\end{aligned}$$

By calculus, we can verify that $2^k - k^2 \ge 0$ for $k \ge 4$ and $3^k - 2^{k-1}(k^2-3k+10) > 0$ for $k \ge 8$. Accordingly, the second derivatives $[\ln h(t)]''$ and $[\ln h(-t)]''$ of the logarithms of $h(t)$ and $h(-t)$ are all negative on $(0, \infty)$. As a result, the function $h(t)$ is logarithmically concave on $(0, \infty)$.

Straightforward calculation gives

$$\begin{aligned}\frac{h(2t)}{h^2(t)} &= \frac{(e^t-1)^2(e^{2t}-1-2t)}{(e^t+1)^2(e^t-1-t)^2}, \\ \frac{h(-2t)}{h^2(-t)} &= \frac{(e^t-1)^2\left[e^{2t}(2t-1)+1\right]}{(e^t+1)^2[e^t(t-1)+1]^2}, \\ \left[\frac{h(2t)}{h^2(t)}\right]' &= -\frac{2(e^t-1)\left[e^{4t}(t-2) + 2e^{3t} + e^{2t}(4t+2) - 2e^t(2t^2+2t+1) - t\right]}{(e^t+1)^3(e^t-1-t)^3} \\ &= -\frac{4(e^t-1)\sum_{k=6}^{\infty}\left[2^{2k-3}(k-8) + 3^k + 2k(2^{k-1}-k) + (2^k-1)\right]\frac{t^k}{k!}}{(e^t+1)^3(e^t-1-t)^3} \\ &= -\frac{2(e^t-1)}{(e^t+1)^3(e^t-1-t)^3}\left(\frac{2t^6}{9} + \frac{19t^7}{45} + \frac{13t^8}{30} + \frac{299t^9}{945} + \cdots\right),\end{aligned}$$

$$
\begin{aligned}
\left[\frac{h(-2t)}{h^2(-t)}\right]' &= -\frac{2e^t(e^t-1)\left[e^{4t}t-2e^{3t}(2t^2-2t+1)-2e^{2t}(2t-1)+2e^t-t-2\right]}{(e^t+1)^3[e^t(t-1)+1]^3}\\
&= -\frac{4e^t(e^t-1)}{(e^t+1)^3[e^t(t-1)+1]^3}\sum_{k=6}^{\infty}\begin{pmatrix}3^{k-2}\left(k\left[\left(\frac{4}{3}\right)^{k-2}-2k\right]+(8k-9)\right)\\ +\left[(2^{k-4}-1)k+1\right]2^k+1\end{pmatrix}\frac{t^k}{k!}\\
&= -\frac{2e^t(e^t-1)}{(e^t+1)^3[e^t(t-1)+1]^3}\biggl(\frac{2t^6}{9}+\frac{7t^7}{15}+\frac{47t^8}{90}+\frac{43t^9}{105}+\frac{4741t^{10}}{18900}\\
&\quad+\frac{641t^{11}}{5040}+\frac{15059t^{12}}{272160}+\frac{5281t^{13}}{249480}+\frac{1519051t^{14}}{209563200}+\frac{2921293t^{15}}{1297296000}+\dotsb\biggr).
\end{aligned}
$$

By calculus, we can verify that the sequence

$$
\left(\frac{4}{3}\right)^{k-2}-2k=\frac{9}{16}\left[\left(\frac{4}{3}\right)^{k}-\frac{32}{9}k\right]
$$

is increasing for

$$
k\ge\frac{\ln\frac{32}{9}-\ln\ln\frac{4}{3}}{\ln\frac{4}{3}}=8.74024101586\dots
$$

and is positive for all $k\ge14$. Consequently, both of the derivatives $\bigl[\frac{h(2t)}{h^2(t)}\bigr]'$ and $\bigl[\frac{h(-2t)}{h^2(-t)}\bigr]'$ are all negative on $(0,\infty)$. In a word, the function $\frac{h(2t)}{h^2(t)}$ is increasing on $(-\infty,0)$ and decreasing on $(0,\infty)$.

By the L'Hospital rule, we obtain

$$
\lim_{t\to0}\frac{h(2t)}{h^2(t)}=2,\quad \lim_{t\to-\infty}\frac{h(2t)}{h^2(t)}=0,\quad \lim_{t\to\infty}\frac{h(2t)}{h^2(t)}=1.
$$

Direct differentiation gives

$$
\begin{aligned}
\frac{\mathrm{d}[h(st)h((1-s)t)]}{\mathrm{d}\,s} &= th'(st)h((1-s)t)-th(st)h'((1-s)t)\\
&= th(st)h((1-s)t)\left[\frac{h'(st)}{h(st)}-\frac{h'((1-s)t)}{h((1-s)t)}\right]\\
&= th(st)h((1-s)t)\left[\frac{\mathrm{d}\ln h(s)}{\mathrm{d}\,s}\bigg|_{s=st}-\frac{\mathrm{d}\ln h(s)}{\mathrm{d}\,s}\bigg|_{s=(1-s)t}\right]\\
&> 0
\end{aligned}
$$

for $0<s<\frac{1}{2}$, where we used in the last step the facts that $st<(1-s)t$ for $0<s<\frac{1}{2}$ and that $h(t)$ is logarithmically concave on $(-\infty,\infty)$. Accordingly, for any fixed $t>0$, the function $h(st)h((1-s)t)$ is increasing in $s\in(0,\frac{1}{2})$.

Since $h(st)h((1-s)t)=h(st)h(t-st)=h(x)h(t-x)$ for $x=st$, it is ready from Lemma 2.2 that, for any fixed $t>0$, the function $h(st)h((1-s)t)$ is increasing in $x=st\in(0,\frac{t}{2})$, equivalently, in $s\in(0,\frac{1}{2})$. The proof of Lemma 2.3 is complete. □

Lemma 2.4 (Bernstein's theorem [18, p. 161, Theorem 12b])**.** *A function $f(x)$ is completely monotonic on $(0,\infty)$ if and only if*

$$
f(x)=\int_0^\infty e^{-xt}\,\mathrm{d}\,\sigma(t),\quad x\in(0,\infty), \tag{2.3}
$$

where $\sigma(s)$ is non-decreasing and the integral in (2.3) converges for $x \in (0,\infty)$.

The integral representation (2.3) means that a function $f(t)$ is completely monotonic on $(0,\infty)$ if and only if it is a Laplace transform of a non-decreasing measure $\sigma(s)$ on $(0,\infty)$.

3. Proofs of main results

In this section, we are in a position to give a proof of Theorem 1.1.

Proof of Theorem 1.1. In the proof of [15, Theorem 4], the author established that

$$x\psi'(x) - 1 = \int_0^\infty h(t)e^{-xt}\,\mathrm{d}\,t > 0 \tag{3.1}$$

and

$$\psi'(x) + x\psi''(x) = -\int_0^\infty th(t)e^{-xt}\,\mathrm{d}\,t < 0. \tag{3.2}$$

If the function $H_\beta(x)$ is decreasing, then its first derivative

$$H'_\beta(x) = \frac{[2\psi''(x) + x\psi'''(x)][x\psi'(x)-1] - \beta[\psi'(x)+x\psi''(x)]^2}{[x\psi'(x)-1]^{\beta+1}} \le 0,$$

that is,

$$\begin{aligned}\beta &\ge \frac{[2\psi''(x)+x\psi'''(x)][x\psi'(x)-1]}{[\psi'(x)+x\psi''(x)]^2} = \frac{x^3[2\psi''(x)+x\psi'''(x)]x[x\psi'(x)-1]}{x^4[\psi'(x)+x\psi''(x)]^2}\\ &\to \frac{\lim_{x\to0^+}(x^3[2\psi''(x)+x\psi'''(x)])\lim_{x\to0^+}(x[x\psi'(x)-1])}{(\lim_{x\to0^+}(x^2[\psi'(x)+x\psi''(x)]))^2} = \frac{2\times1}{(-1)^2} = 2\end{aligned}$$

as $x \to 0^+$, where we used (3.1), (3.2), and the limit

$$\lim_{x\to0^+}[x^k\psi^{(k-1)}(x)] = (-1)^k(k-1)!, \quad k \ge 1$$

in [8, p. 260, (2.2)] and [19, p. 769]. Hence, the necessary condition for $H_\beta(x)$ to be decreasing on $(0,\infty)$ is $\beta \ge 2$.

By virtue of (3.1) and (3.2), the function $H_\beta(x)$ defined in (1.3) can be rewritten as

$$H_\beta(x) = -\frac{\int_0^\infty th(t)e^{-xt}\,\mathrm{d}\,t}{[\int_0^\infty h(t)e^{-xt}\,\mathrm{d}\,t]^\beta}.$$

Since

$$\frac{\mathrm{d}\,H_\beta(x)}{\mathrm{d}\,x} = \frac{\int_0^\infty t^2h(t)e^{-xt}\,\mathrm{d}\,t\int_0^\infty h(t)e^{-xt}\,\mathrm{d}\,t - \beta[\int_0^\infty th(t)e^{-xt}\,\mathrm{d}\,t]^2}{[\int_0^\infty h(t)e^{-xt}\,\mathrm{d}\,t]^{\beta+1}},$$

in order to prove that the function $H_\beta(x)$ is decreasing on $(0,\infty)$, it is sufficient to show the inequality

$$\beta\left[\int_0^\infty th(t)e^{-xt}\,\mathrm{d}\,t\right]^2 \ge \int_0^\infty t^2h(t)e^{-xt}\,\mathrm{d}\,t\int_0^\infty h(t)e^{-xt}\,\mathrm{d}\,t. \tag{3.3}$$

By Lemma 2.1, the inequality (3.3) can be reformulated as

$$\beta\int_0^\infty\left[\int_0^t u(t-u)h(u)h(t-u)\,\mathrm{d}\,u\right]e^{-xt}\,\mathrm{d}\,t \ge \int_0^\infty\left[\int_0^t u^2h(u)h(t-u)\,\mathrm{d}\,u\right]e^{-xt}\,\mathrm{d}\,t. \tag{3.4}$$

Let

$$P(t)=\int_0^t u(t-u)h(u)h(t-u)\,\mathrm{d}\,u \quad \text{and} \quad Q(t)=\int_0^t u^2h(u)h(t-u)\,\mathrm{d}\,u.$$

The inequality (3.4) can be rewritten as

$$\int_0^\infty Q(t)\left[\frac{P(t)}{Q(t)}-\frac{1}{\beta}\right]e^{-xt}\,\mathrm{d}\,t\ge 0. \tag{3.5}$$

Changing the variable $u=\frac{(1+v)t}{2}$ results in

$$\frac{P(t)}{Q(t)}=\frac{\int_0^1(1-v^2)h\big(\frac{1+v}{2}t\big)h\big(\frac{1-v}{2}t\big)\,\mathrm{d}\,v}{\int_0^1(1+v^2)h\big(\frac{1+v}{2}t\big)h\big(\frac{1-v}{2}t\big)\,\mathrm{d}\,v}\to\frac{\int_0^1(1-v^2)\,\mathrm{d}\,v}{\int_0^1(1+v^2)\,\mathrm{d}\,v}=\frac{1}{2} \tag{3.6}$$

as $t\to0^+$ or $t\to\infty$, where we used the property in Lemma 2.3 that the function $h(t)$ is increasing from $(0,\infty)$ onto $(\frac{1}{2},1)$.

Let

$$S(t)=\int_0^1(1-v^2)h\Big(\frac{1+v}{2}t\Big)h\Big(\frac{1-v}{2}t\Big)\,\mathrm{d}\,v-\frac{1}{2}\int_0^1(1+v^2)h\Big(\frac{1+v}{2}t\Big)h\Big(\frac{1-v}{2}t\Big)\,\mathrm{d}\,v.$$

Then

$$\begin{aligned}S(t)&=\frac{3}{2}\int_0^1\Big(\frac{1}{3}-v^2\Big)h\Big(\frac{1+v}{2}t\Big)h\Big(\frac{1-v}{2}t\Big)\,\mathrm{d}\,v\\&=\frac{3}{2}\bigg[\int_0^{1/\sqrt3}\Big(\frac{1}{3}-v^2\Big)h\Big(\frac{1+v}{2}t\Big)h\Big(\frac{1-v}{2}t\Big)\,\mathrm{d}\,v\\&\quad+\int_{1/\sqrt3}^1\Big(\frac{1}{3}-v^2\Big)h\Big(\frac{1+v}{2}t\Big)h\Big(\frac{1-v}{2}t\Big)\,\mathrm{d}\,v\bigg].\end{aligned}$$

Employing the fourth conclusion in Lemma 2.3 leads to

$$\begin{aligned}S(t)&>\frac{3}{2}h\Big(\frac{1+1/\sqrt3}{2}t\Big)h\Big(\frac{1-1/\sqrt3}{2}t\Big)\bigg[\int_0^{1/\sqrt3}\Big(\frac{1}{3}-v^2\Big)\,\mathrm{d}\,v+\int_{1/\sqrt3}^1\Big(\frac{1}{3}-v^2\Big)\,\mathrm{d}\,v\bigg]\\&=0.\end{aligned}$$

Consequently, considering the limit in (3.6), we find that the inequality

$$\frac{P(t)}{Q(t)}>\frac{1}{2} \tag{3.7}$$

is valid for $t>0$ and is sharp in the sense that the scalar $\frac{1}{2}$ cannot be replaced by any larger number. This sharp inequality means that the inequality (3.5) is valid for all $\beta\ge2$. As a result, the condition $\beta\ge2$ is sufficient for $H_\beta(x)$ to be decreasing on $(0,\infty)$.

Considering the characterization expressed by the integral representation (2.3) of completely monotonic functions on $(0,\infty)$, the inequality (3.5), and the sharpness of the inequality (3.7), we can conclude the necessary and sufficient condition $\beta\ge2$ alternatively.

From $0<1-v^2<1+v^2$ for $v\in(0,1)$ and the positivity of $h(t)$ on $(0,\infty)$, we derive $0<\frac{P(t)}{Q(t)}<1$ on $(0,\infty)$. This means that, for all $\beta\le1$, the function $H_\beta(x)$ is increasing on $(0,\infty)$.

The limits in (1.4) and (1.5) follow from the integral representation (3.1) and corresponding limits in [2, Proposition 4], [3, Proposition 14], or [15, Theorem 2]. The proof of Theorem 1.1 is complete. ☐

Corollary 3.1. *The function*

$$\beta\big[\psi'(x)+x\psi''(x)\big]^2-\big[2\psi''(x)+x\psi'''(x)\big]\big[x\psi'(x)-1\big]$$

is completely monotonic on $(0,\infty)$ *if and only if* $\beta\ge2$*, while its negativity is completely monotonic on* $(0,\infty)$ *for* $\beta\le1$.

Proof. This follows from the proof of Theorem 1.1. ☐

4. Alternative proofs of necessary and sufficient conditions

In this section, we supply alternative proofs of necessary and sufficient conditions on α for the functions $\pm\mathfrak{H}_\alpha(x)$ in (1.1) to be completely monotonic on $(0,\infty)$.

Theorem 4.1 ([15, Theorem 4])**.** *If and only if* $\alpha\ge2$*, the function* $\mathfrak{H}_\alpha(x)$ *defined in* (1.1) *is completely monotonic on* $(0,\infty)$*; if and only if* $\alpha\le1$*, the function* $-\mathfrak{H}_\alpha(x)$ *is completely monotonic on* $(0,\infty)$.

Alternative proof of sufficient conditions. By integral representations (3.1) and (3.2), we arrive at

$$\mathfrak{H}_\alpha(x)=\alpha\bigg[\int_0^\infty h(t)e^{-xt}\,\mathrm{d}\,t\bigg]^2-\int_0^\infty th(t)e^{-xt}\,\mathrm{d}\,t.$$

By Lemma 2.1, we obtain

$$\begin{aligned}\mathfrak{H}_\alpha(x)&=\alpha\int_0^\infty\bigg[\int_0^t h(u)h(t-u)\,\mathrm{d}\,u\bigg]e^{-xt}\,\mathrm{d}\,t-\int_0^\infty th(t)e^{-xt}\,\mathrm{d}\,t\\&=\int_0^\infty\bigg[\frac{\alpha}{t}\int_0^t h(u)h(t-u)\,\mathrm{d}\,u-h(t)\bigg]te^{-xt}\,\mathrm{d}\,t.\end{aligned}\tag{4.1}$$

By logarithmic concavity of $h(t)$ in Lemma 2.3 and by Lemma 2.2, we acquire

$$\begin{aligned}\frac{\alpha}{t}\int_0^t h(u)h(t-u)\,\mathrm{d}\,u-h(t)&\le\frac{\alpha}{t}\int_0^t h\Big(\frac{t}{2}\Big)h\Big(t-\frac{t}{2}\Big)\,\mathrm{d}\,u-h(t)\\&=\alpha\bigg[h\Big(\frac{t}{2}\Big)\bigg]^2-h(t)\\&=\bigg[h\Big(\frac{t}{2}\Big)\bigg]^2\bigg(\alpha-\frac{h(t)}{[h(\frac{t}{2})]^2}\bigg)\end{aligned}$$

and

$$\begin{aligned}\frac{\alpha}{t}\int_0^t h(u)h(t-u)\,\mathrm{d}\,u-h(t)&\ge\frac{\alpha}{t}\int_0^t h(0)h(t)\,\mathrm{d}\,u-h(t)\\&=[\alpha h(0)-1]h(t)\\&=\Big(\frac{\alpha}{2}-1\Big)h(t).\end{aligned}$$

By the double inequality (2.2) in Lemma 2.3, when $\alpha \le 1$, we deduce

$$\frac{\alpha}{t}\int_0^t h(u)h(t-u)\,\mathrm{d}\,u - h(t) < 0, \quad t \in (0,\infty);$$

when $\alpha \ge 2$, we have

$$\frac{\alpha}{t}\int_0^t h(u)h(t-u)\,\mathrm{d}\,u - h(t) > 0, \quad t \in (0,\infty).$$

Consequently, when $\alpha \ge 2$, the function $\mathfrak{H}_\alpha(x)$ is completely monotonic on $(0,\infty)$; when $\alpha \le 1$, the function $-\mathfrak{H}_\alpha(x)$ is completely monotonic on $(0,\infty)$.

By the way, the proof of necessary conditions on α for $\pm\mathfrak{H}_\alpha(x)$ to be completely monotonic on $(0,\infty)$ is the same as in the proof of [15, Theorem 4]. The required proof is complete. □

Alternative proof of necessary and sufficient conditions. We can rewrite the integral representation (4.1) alternatively as

$$\begin{aligned}\mathfrak{H}_\alpha(x) &= \int_0^\infty \biggl[\alpha\int_0^1 h(st)h((1-s)t)\,\mathrm{d}\,s - h(t)\biggr] te^{-xt}\,\mathrm{d}\,t\\ &= \int_0^\infty \biggl[2\alpha\int_{1/2}^1 h(st)h((1-s)t)\,\mathrm{d}\,s - h(t)\biggr] te^{-xt}\,\mathrm{d}\,t\\ &= \int_0^\infty \biggl[2\alpha\frac{\int_{1/2}^1 h(st)h((1-s)t)\,\mathrm{d}\,s}{h(t)} - 1\biggr] h(t)te^{-xt}\,\mathrm{d}\,t.\end{aligned}$$

From the last property in Lemma 2.3, we see that, for any fixed $t > 0$, the function $h(st)h((1-s)t)$ is decreasing in $s \in \bigl(\frac{1}{2},1\bigr)$. Accordingly, the inequalities

$$\frac{1}{4} = \frac{h(0)}{2} < \frac{\int_{1/2}^1 h(st)h((1-s)t)\,\mathrm{d}\,s}{h(t)} < \frac{1}{2}\frac{\bigl[h\bigl(\frac{t}{2}\bigr)\bigr]^2}{h(t)} < \frac{1}{2}$$

are valid and sharp for $t \in (0,\infty)$, where we used the upper bound in the double inequality (2.2) and its sharpness. Therefore, due to these sharpness, by virtue of Lemma 2.4, we conclude that, if and only if $\alpha \ge 2$, the function $\mathfrak{H}_\alpha(x)$ is completely monotonic on $(0,\infty)$; if and only if $\alpha \le 1$, the function $-\mathfrak{H}_\alpha(x)$ is completely monotonic on $(0,\infty)$. The required proof is complete. □

Remark 4.1. This paper is a revised version of the electronic preprint [13], was reported between 11:10–11:25 on 20 November 2020, the 3rd International Conference on Mathematical and Related Sciences: Current Trends and Developments (ICMRS 2020) in Turkey, and is the second one in a series of articles including [5–7, 10–12, 14–16].

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