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CONVEX FUNCTIONS ON THE INTERVAL IN A REAL VECTOR SPACE

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Abstract. The article investigates convex functions defined on the interval in a real vector space. The first objective is to establish the most relevant inequalities relying on the support and secant lines. These are the Hermite-Hadamard and Jensen type inequalities. The second objective is to propose more appropriate notations related to the convex function subderivative and subdifferential.

1. Introduction

Throughout the article, the interval in a real vector space $\mathbb{X}$ is used. Let $a, b \in \mathbb{X}$ be a pair of different points. The interval between a and b is the set

$$[a, b] = \operatorname{conv}\{a, b\} = \{(1-t)a + tb : 0 \le t \le 1\}, \tag{1.1}$$

and its points are called convex combinations of a and b. It can also be said that the interval between a and b is the convex hull of the set $\{a, b\}$. Using $0 < t < 1$ in the above representation, we get the open interval (a, b) whose points are called interior points. The line through a and b is the set

$$\mathbb{L}_{\{a,b\}} = \operatorname{aff}\{a, b\} = \{(1-t)a + tb : -\infty < t < +\infty\},$$

and its points are called affine combinations of a and b. We also say that the line through a and b is the affine hull of the set $\{a, b\}$. As it is obvious that $[a, b] \subset \mathbb{L}_{\{a,b\}}$, a bounded interval is often called a line segment.

In the paper, the following two representations of points from the interval $[a, b]$ will also be employed. The point $b - a$ can be considered as the interval directional point. Let r_0 be a positive number, let $v = r_0(b - a)$, and let $c = (1 - t_0)a + t_0 b \in [a, b]$ be an interval point implying that $0 \le t_0 \le 1$. Then we have

$$[a, b] = \{c + tv = (1 - t_0 - r_0 t)a + (t_0 + r_0 t)b : t_a \le t \le t_b\}, \tag{1.2}$$

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where

$$t_a = -\frac{t_0}{r_0}, \quad t_b = \frac{1-t_0}{r_0}.$$

Here is $t_a = -t_0/r_0 \leq t \leq (1-t_0)/r_0 = t_b$ because it has to be $0 \leq t_0 + r_0 t \leq 1$. As regards the line, using the mark o for the origin in $\mathbb{X}$, we have

$$\mathbb{L}_{\{a,b\}} = c + \mathbb{L}_{\{o,v\}}.$$

2. Monotonicity of slopes

The slopes of convex functions defined on the interval $[a,b]$ are considered.

Lemma 2.1. *Let x_1, x_2, $x_3 \in [a,b]$ be points where $x_2 = (1-t)x_1 + tx_3$ for some number t satisfying $0 < t < 1$.*

Then each convex function $f : [a,b] \to \mathbb{R}$ satisfies the double inequality

$$\frac{f(x_2)-f(x_1)}{t} \leq \frac{f(x_3)-f(x_1)}{1} \leq \frac{f(x_3)-f(x_2)}{1-t}. \tag{2.1}$$

Proof. Applying the convexity of the function f to the given convex combination of the point x_2, we get

$$f(x_2) \leq (1-t)f(x_1) + tf(x_3),$$

and then moving $f(x_1)$ over to the left side, and dividing by t, we obtain

$$\frac{f(x_2)-f(x_1)}{t} \leq \frac{f(x_3)-f(x_1)}{1}.$$

Combining the above inequality with the convex combination

$$\frac{f(x_3)-f(x_1)}{1} = t\,\frac{f(x_2)-f(x_1)}{t} + (1-t)\,\frac{f(x_3)-f(x_2)}{1-t},$$

we reach the double inequality in (2.1). $\square$

Corollary 2.1. *Let $r_0 > 0$, let $v = r_0(b-a)$, let $c \in (a,b)$ be a point, and let $c+t_1v$, $c+t_2v$, $c+t_3v \in [a,b]$ be points where $t_1 < t_2 < t_3$.*

Then each convex function $f : [a,b] \to \mathbb{R}$ satisfies the double inequality

$$\frac{f(c+t_2v)-f(c+t_1v)}{t_2-t_1} \leq \frac{f(c+t_3v)-f(c+t_1v)}{t_3-t_1} \leq \frac{f(c+t_3v)-f(c+t_2v)}{t_3-t_2}. \tag{2.2}$$

Proof. We put $x_1 = c+t_1v$, $x_2 = c+t_2v$ and $x_3 = c+t_3v$. Since $t_2 \in (t_1,t_3)$, it follows that $t_2 = (1-t)t_1 + tt_3$ for some number t satisfying $0 < t < 1$. Then we have the relations

$$t = \frac{t_2-t_1}{t_3-t_1}$$

and $x_2 = (1-t)x_1 + tx_3$. This leads us to apply formula (2.1) to the given points and the above coefficient, and so realize formula (2.2). $\square$

Corollary 2.2. *Let $r_0 > 0$, let $v = r_0(b-a)$, let $c = (1-t_0)a + t_0 b \in (a,b)$ be a point, and let $f : [a,b] \to \mathbb{R}$ be a convex function.*

Then the slopes function $f_{(v,c)} : [t_a, 0) \cup (0, t_b] \to \mathbb{R}$ defined by

$$f_{(v,c)}(t) = \frac{f(c+tv) - f(c)}{t} \tag{2.3}$$

is nondecreasing and bounded, and it satisfies the inequality

$$f_{(v,c)}(0-) \le f_{(v,c)}(0+). \tag{2.4}$$

Proof. As for the monotonicity of $f_{(v,c)}$, let $t_1, t_2 \in [t_a, 0) \cup (0, t_b]$ be numbers such that $t_1 < t_2$. Applying the double inequality in (2.2) to each of ordered triples $t_1 < t_2 < 0$, $t_1 < 0 < t_2$ and $0 < t_1 < t_2$, we always get $f_{(v,c)}(t_1) \le f_{(v,c)}(t_2)$.

Then the bordering inequality $f_{(v,c)}(t_a) \le f_{(v,c)}(t) \le f_{(v,c)}(t_b)$ holds for every $t \in [t_a, 0) \cup (0, t_b]$, and so the function $f_{(v,c)}$ is bounded.

Since the function $f_{(v,c)}$ is nondecreasing and bounded, its left and right limits exist at $t = 0$ satisfying

$$f_{(v,c)}(0-) = \lim_{t \to 0-} f_{(v,c)}(t) \le \lim_{t \to 0+} f_{(v,c)}(t) = f_{(v,c)}(0+),$$

which confirms the inequality in formula (2.4). ☐

The inequality in (2.4) implies that the slopes set $[f_{(v,c)}(0-), f_{(v,c)}(0+)]$ is nonempty, it contains at least one number. The slopes functions $f_{(v,c)}$ and $f_{(b-a,c)}$ satisfy the relation

$$\frac{1}{r_0} f_{(v,c)}\left(\frac{t}{r_0}\right) = f_{(b-a,c)}(t)$$

for every $t \in [-t_0, 0) \cup (0, 1-t_0]$. Relying on the above equality, we finish the section by presenting the lemma on slopes functions and sets.

Lemma 2.2. *Let $r_{1,2} > 0$, let $v_{1,2} = r_{1,2}(b-a)$, let $c = (1-t_0)a + t_0 b \in (a,b)$ be a point, and let $f : [a,b] \to \mathbb{R}$ be a convex function.*

Then the slopes functions relation

$$\frac{1}{r_1} f_{(v_1,c)}\left(\frac{t}{r_1}\right) = \frac{1}{r_2} f_{(v_2,c)}\left(\frac{t}{r_2}\right) \tag{2.5}$$

holds for every $t \in [-t_0, 0) \cup (0, 1-t_0]$. Letting t tend to zero from the negative and positive sides, it follows the slopes sets relation

$$\frac{1}{r_1}[f_{(v_1,c)}(0-), f_{(v_1,c)}(0+)] = \frac{1}{r_2}[f_{(v_2,c)}(0-), f_{(v_2,c)}(0+)]. \tag{2.6}$$

3. Main results - Inequalities

In this section, the Hermite-Hadamard and Jensen type inequalities referring to a convex function on the interval in a real vector space are obtained.

Two affine functions are usually attached to a convex function $f : [a,b] \to \mathbb{R}$. Let $r_0 > 0$, let $v = r_0(b-a)$, let $c \in (a,b)$ be an interval interior point, and let $k \in [f_{(v,c)}(0-), f_{(v,c)}(0+)]$ be a number.

The affine function $h_{(c,k)} : \mathbb{L}_{\{a,b\}} \to \mathbb{R}$ presented by the equation

$$h_{(c,k)}(c+tv) = kt + f(c) \tag{3.1}$$

is the support line of f at c with the slope k. Its graph passes through the point $(c, f(c))$ in $\mathbb{L}_{\{a,b\}} \times \mathbb{R}$. It can be visualized as the line

$$u(t) = kt + f(c)$$

passing through the point $(0, f(c))$ in $\mathbb{R} \times \mathbb{R}$ with the slope k.

The affine function $h_{(a,b)} : \mathbb{L}_{\{a,b\}} \to \mathbb{R}$ presented by the equation

$$h_{(a,b)}(c+tv) = (1 - t_0 - r_0 t)f(a) + (t_0 + r_0 t)f(b) \tag{3.2}$$

is the secant line of f at a and b. Its graph passes through the points $(a, f(a))$ and $(b, f(b))$ in $\mathbb{L}_{\{a,b\}} \times \mathbb{R}$. It can be visualized as the line

$$v(t) = \frac{t_b - t}{t_b - t_a} f(a) + \frac{t - t_a}{t_b - t_a} f(b) = (1 - t_0 - r_0 t)f(a) + (t_0 + r_0 t)f(b)$$

passing through the points $(t_a, f(a))$ and $(t_b, f(b))$ in $\mathbb{R} \times \mathbb{R}$.

Lemma 3.1. *Let $r_0 > 0$, let $v = r_0(b-a)$, let $c \in (a,b)$ be a point, let $f : [a,b] \to \mathbb{R}$ be a convex function, and let $k \in [f_{(v,c)}(0-), f_{(v,c)}(0+)]$ be a number.*

Then the double inequality

$$h_{(c,k)}(c+tv) \le f(c+tv) \le h_{(a,b)}(c+tv) \tag{3.3}$$

holds for every $t \in [t_a, t_b]$.

Proof. Respecting the equations of the support and secant lines in formula (3.1) and formula (3.2), we have to show that the double inequality

$$kt + f(c) \le f(c+tv) \le (1 - t_0 - r_0 t)f(a) + (t_0 + r_0 t)f(b) \tag{3.4}$$

is valid for every $t \in [t_a, t_b]$.

To prove the left-hand side of the inequality in formula (3.4), we will observe the cases $t = 0$, $t > 0$ and $t < 0$. If $t = 0$, then the trivial inequality $f(c) \le f(c)$ represents the left-hand side of (3.4). If $t > 0$, then the inequality

$$k \le \frac{f(c+tv) - f(c)}{t}$$

is valid because the function $f_{(v,c)}$ is nondecreasing. Multiplying this inequality by t, and then moving $f(c)$ over to the left side, we get the left-hand side of the double inequality in (3.4). If $t < 0$, then the reverse inequality

$$\frac{f(c+tv) - f(c)}{t} \le k$$

is valid, and doing as above, we obtain the left-hand side of the double inequality in (3.4).

To prove the right-hand side of the inequality in (3.4), we apply the convexity of f to the convex combination $c + tv = (1 - t_0 - r_0 t)a + (t_0 + r_0 t)b$. $\square$

In order to prepare the theorem below, the following is handled. If $f : [a, b] \to \mathbb{R}$ is a convex function, then the related function $\tilde{f} : [t_a, t_b] \to \mathbb{R}$ defined by

$$\tilde{f}(t) = f(c + tv) = f\big(c + r_0 t(b-a)\big) \tag{3.5}$$

is convex. Therefore, the function $\tilde{f}$ is integrable. In this context, we will utilize the above function as integrand with the variable t.

Theorem 3.1. *Let $r_0 > 0$, let $v = r_0(b-a)$, let $c = (1-t_0)a + t_0 b \in (a, b)$ be a point, let $f : [a, b] \to \mathbb{R}$ be a convex function, and let $k \in [f_{(v,c)}(0-), f_{(v,c)}(0+)]$ be a number.*

Then we have the double inequality

$$\frac{1-2t_0}{2r_0} k + f(c) \le r_0 \int_{t_a}^{t_b} f\big(c + r_0 t(b-a)\big)\, dt \le \frac{f(a) + f(b)}{2}. \tag{3.6}$$

Proof. Integrating the inequality in formula (3.4) over the interval $[t_a, t_b]$ by the variable t, using the calculations

$$\int_{t_a}^{t_b} dt = \frac{1}{r_0}, \quad \int_{t_a}^{t_b} t\, dt = \frac{1-2t_0}{2r_0^2}$$

and

$$\int_{t_a}^{t_b} (t_0 + r_0 t)\, dt = \int_{t_a}^{t_b} (1 - t_0 - r_0 t)\, dt = \frac{1}{2r_0},$$

and multiplying by the number r_0, we obtain the inequality in (3.6). □

If $t_0 = 1/2$, then k disappears from the double inequality in (3.6). Using the interval midpoint $c = (a+b)/2$, in which case $t_0 = 1/2$, the previous theorem is reduced as follows.

Corollary 3.1. *Each convex function $f : [a, b] \to \mathbb{R}$ satisfies the double inequality*

$$f\left(\frac{a+b}{2}\right) \le r_0 \int_{-\frac{1}{2r_0}}^{\frac{1}{2r_0}} f\left(\left(\frac{1}{2} - r_0 t\right)a + \left(\frac{1}{2} + r_0 t\right)b\right) dt \le \frac{f(a) + f(b)}{2} \tag{3.7}$$

for every positive number r_0.

Taking $\mathbb{X} = \mathbb{R}$, and so utilizing the real numbers interval $[a, b]$, and applying the substitution

$$x = \left(\frac{1}{2} - r_0 t\right)a + \left(\frac{1}{2} + r_0 t\right)b = \frac{a+b}{2} + r_0(b-a)t$$

to the middle member in formula (3.7), we gain the well known Hermite-Hadamard inequality (see [3] and [2])

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x)\, dx \le \frac{f(a) + f(b)}{2}. \tag{3.8}$$

As a preparation for the next theorem, we consider a bounded function $\tilde{g} : [t_a, t_b] \to \mathbb{R}$. If the image of $\tilde{g}$ is contained in the interval $[a_1, b_1] \subset \mathbb{R}$, and if $\tilde{g}$ is integrable, then its integral arithmetic mean

$$c_1 = \frac{1}{t_b - t_a} \int_{t_a}^{t_b} \tilde{g}(t)\, dt = r_0 \int_{t_a}^{t_b} \tilde{g}(t)\, dt$$

is contained in $[a_1, b_1]$. Namely, integrating the inequality $a_1 \le \tilde{g}(t) \le b_1$ over the interval $[t_a, t_b]$ by the variable t, it follows that $a_1 \le c_1 \le b_1$.

Attaching an affine function $h : \mathbb{R} \to \mathbb{R}$ to the above integral arithmetic mean is done by the rule

$$h\left(r_0 \int_{t_a}^{t_b} \tilde{g}(t)\,dt\right) = r_0 \int_{t_a}^{t_b} h(\tilde{g}(t))\,dt.$$

Theorem 3.2. *Let $r_0 > 0$, let $c = (1-t_0)a + t_0 b \in [a,b]$ be a point, let $g : [a,b] \to \mathbb{R}$ be a bounded function such that the related function $\tilde{g} : [t_a, t_b] \to \mathbb{R}$ defined by $\tilde{g}(t) = g(c + r_0 t(b-a))$ is integrable, let $c_1 = r_0 \int_{t_a}^{t_b} \tilde{g}(t)\,dt$, and let $[a_1, b_1] \subset \mathbb{R}$ be an interval containing the image of g.*

Then each convex function $f : [a_1, b_1] \to \mathbb{R}$ satisfies the double inequality

$$\begin{aligned} f\left(r_0 \int_{t_a}^{t_b} g(c + r_0 t(b-a))\,dt\right) &\le r_0 \int_{t_a}^{t_b} f\left(g(c + r_0 t(b-a))\right) dt \\ &\le \frac{b_1 - c_1}{b_1 - a_1} f(a_1) + \frac{c_1 - a_1}{b_1 - a_1} f(b_1). \end{aligned} \tag{3.9}$$

Proof. If $c_1 \in (a_1, b_1)$, then any support line $h_{(c_1,k_1)}(x_1) = k_1(x_1 - c_1) + f(c_1)$ of f at c_1, and the secant line

$$h_{(a_1,b_1)}(x_1) = \frac{b_1 - x_1}{b_1 - a_1} f(a_1) + \frac{x_1 - a_1}{b_1 - a_1} f(b_1) \tag{3.10}$$

of f at a_1 and b_1 are recommended to be used.

By plugging the equality $f(c_1) = h_{(c_1,k_1)}(c_1)$, the affinity of $h_{(c_1,k_1)}$, and the inequality $h_{(c_1,k_1)}(\tilde{g}(t)) \le f(\tilde{g}(t))$ into the calculation

$$\begin{aligned} f\left(r_0 \int_{t_a}^{t_b} \tilde{g}(t)\,dt\right) &= h_{(c_1,k_1)}\left(r_0 \int_{t_a}^{t_b} \tilde{g}(t)\,dt\right) \\ &= r_0 \int_{t_a}^{t_b} h_{(c_1,k_1)}(\tilde{g}(t))\,dt \\ &\le r_0 \int_{t_a}^{t_b} f(\tilde{g}(t))\,dt, \end{aligned}$$

one gets the left-hand side of the inequality in (3.9). By plugging the inequality $f(\tilde{g}(t)) \le h_{(a_1,b_1)}(\tilde{g}(t))$, the affinity of $h_{(a_1,b_1)}$, and the secant line equation in (3.10) for $x_1 = c_1$ into the calculation

$$\begin{aligned} r_0 \int_{t_a}^{t_b} f(\tilde{g}(t))\,dt &\le r_0 \int_{t_a}^{t_b} h_{(a_1,b_1)}(\tilde{g}(t))\,dt \\ &= h_{(a_1,b_1)}\left(r_0 \int_{t_a}^{t_b} \tilde{g}(t)\,dt\right) \\ &= \frac{b_1 - c_1}{b_1 - a_1} f(a_1) + \frac{c_1 - a_1}{b_1 - a_1} f(b_1), \end{aligned}$$

one gets the right-hand side of the inequality in (3.9). By connecting the above inequalities, one reaches the double inequality in (3.9).

If $c_1 = a_1$ or $c_1 = b_1$, then the function $\tilde{g}$ is almost everywhere equal to c_1, and the trivial double inequality $f(c_1) \le f(c_1) \le f(c_1)$ represents the inequality in (3.9). $\square$

If $\mathbb{X} = \mathbb{R}$, and so $[a, b] \subset \mathbb{R}$, then using the interval midpoint $c = (a + b)/2$, and applying the substitution $x = (a + b)/2 + r_0(b - a)t$ to the first and second member in formula (3.9), we get the inequality

$$\begin{aligned} f\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right) &\leq \frac{1}{b-a}\int_a^b f(g(x))\,dx \\ &\leq \frac{b_1 - c_1}{b_1 - a_1} f(a_1) + \frac{c_1 - a_1}{b_1 - a_1} f(b_1), \end{aligned} \tag{3.11}$$

where $c_1 = \int_a^b g(x)\,dx/(b - a)$. The left-hand side of this inequality presents a version of the integral form of very significant Jensen's inequality (see [5]).

The discrete form of Jensen's inequality was introduced in [4]. The connections of the classical forms of the Jensen and Hermite-Hadamard inequalities were presented in [8]. The functional forms of these inequalities concerning convex functions on the triangle were considered in [9].

4. Introducing a norm

It is supposed that the space $\mathbb{X}$ is endowed with some norm $\|\ \|$. We use this norm to determine the unit vector and directional derivative, and we certainly apply it to inequalities.

Let $x = (1 - t)a + tb \in [a, b]$ be an interval point. Then taking the norm of the equation $x - a = t(b - a)$, it follows that $t = \|x - a\|/\|b - a\|$, and so we have the representation

$$x = \frac{\|b - x\|}{\|b - a\|}a + \frac{\|x - a\|}{\|b - a\|}b. \tag{4.1}$$

Let $v = (b - a)/\|b - a\|$ be the directional unit point of the interval $[a, b]$, and let $c \in [a, b]$ be an interval point. Then using

$$r_0 = \frac{1}{\|b - a\|},\ \ t_0 = \frac{\|c - a\|}{\|b - a\|},\ \ t_a = -\|c - a\|,\ \ t_b = \|b - c\|,$$

the representation in formula (1.2) takes the form

$$[a, b] = \left\{c + tv = \frac{\|b - c\| - t}{\|b - a\|}a + \frac{\|c - a\| + t}{\|b - a\|}b : -\|c - a\| \leq t \leq \|b - c\|\right\}. \tag{4.2}$$

Let $v = (b - a)/\|b - a\|$, let $c \in (a, b)$ be an interval interior point, and let $f : [a, b] \to \mathbb{R}$ be a function. The directional derivative of f over v at c is the number

$$\frac{\partial f}{\partial v}(c) = f'_v(c) = \lim_{t\to 0} f_{(v,c)}(t) = \lim_{t\to 0} \frac{f(c + tv) - f(c)}{t}, \tag{4.3}$$

provided that the limit exists. If the function f is convex, then it has the left and right directional derivatives over v at c, the slopes set $[f'_v(c-), f'_v(c+)]$ is nonempty, and so f has a support line at c.

The equation of the support line $h_{(c,k)}$ of f at c with a slope $k \in [f'_v(c-), f'_v(c+)]$ remains in the same form as in equation (3.1),

$$h_{(c,k)}(c + tv) = kt + f(c).$$

The equation of the secant line $h_{(a,b)}$ of f at a and b can be expressed in the form

$$h_{(a,b)}(c+tv) = \frac{\|b-c\|-t}{\|b-a\|}f(a) + \frac{\|c-a\|+t}{\|b-a\|}f(b).$$

If $c+tv = x \in [a,b]$, then taking the norm of $tv = x-c$, it follows that

$$t = \mp\|x-c\| = \|x-a\| - \|c-a\|,$$

and thus the equations of the support and secant line segments are

$$h_{(c,k)}(x) = k(\|x-a\| - \|c-a\|) + f(c) \tag{4.4}$$

and

$$h_{(a,b)}(x) = \frac{\|b-x\|}{\|b-a\|}f(a) + \frac{\|x-a\|}{\|b-a\|}f(b). \tag{4.5}$$

Including the norm, Lemma 3.1 and Theorem 3.1 can be stated as follows.

Lemma 4.1. *Let $v = (b-a)/\|b-a\|$, let $c \in (a,b)$ be a point, let $f : [a,b] \to \mathbb{R}$ be a convex function, and let $k \in [f'_v(c-), f'_v(c+)]$ be a number.*

Then the double inequality

$$k(\|x-a\| - \|c-a\|) + f(c) \le f(x) \le \frac{\|b-x\|}{\|b-a\|}f(a) + \frac{\|x-a\|}{\|b-a\|}f(b) \tag{4.6}$$

holds for every $x \in [a,b]$.

Theorem 4.1. *Let $v = (b-a)/\|b-a\|$, let $c \in (a,b)$ be a point, let $f : [a,b] \to \mathbb{R}$ be a convex function, and let $k \in [f'_v(c-), f'_v(c+)]$ be a number.*

Then we have the double inequality

$$\begin{aligned} \frac{\|b-c\|-\|c-a\|}{2}k + f(c) \ &\le\ \frac{1}{\|b-a\|}\int_{-\|c-a\|}^{\|b-c\|} f\left(\frac{\|b-c\|-t}{\|b-a\|}a + \frac{\|c-a\|+t}{\|b-a\|}b\right)dt \\ &\le\ \frac{f(a)+f(b)}{2}. \end{aligned} \tag{4.7}$$

Putting $c = (a+b)/2$ in the double inequality in (4.7), the following is obtained.

Corollary 4.1. *Each convex function $f : [a,b] \to \mathbb{R}$ satisfies the double inequality*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) \ &\le\ \frac{1}{\|b-a\|}\int_{-\|b-a\|/2}^{\|b-a\|/2} f\left(\Big(\frac{1}{2} - \frac{t}{\|b-a\|}\Big)a + \Big(\frac{1}{2} + \frac{t}{\|b-a\|}\Big)b\right) dt \\ &\le\ \frac{f(a)+f(b)}{2}. \end{aligned} \tag{4.8}$$

If $[a,b] \subset \mathbb{R}$ is the interval of real numbers, then applying the substitution

$$x = \Big(\frac{1}{2} - \frac{t}{b-a}\Big)a + \Big(\frac{1}{2} + \frac{t}{b-a}\Big)b = \frac{a+b}{2} + t$$

to the middle member in formula (4.8), we gain the Hermite-Hadamard inequality in (3.8).

Using the norm, Theorem 3.2 can be formulated as follows.

Theorem 4.2. *Let $v=(b-a)/\|b-a\|$, let $c\in[a,b]$ be a point, let $g:[a,b]\to\mathbb{R}$ be a bounded function such that the related function $\tilde{g}:[-\|c-a\|,\|b-c\|]\to\mathbb{R}$ defined by*

$$\tilde{g}(t)=g\Big(\frac{\|b-c\|-t}{\|b-a\|}a+\frac{\|c-a\|+t}{\|b-a\|}b\Big)$$

is integrable, let

$$c_1=\frac{1}{\|b-a\|}\int_{-\|c-a\|}^{\|b-c\|}\tilde{g}(t)\,dt,$$

and let $[a_1,b_1]\subset\mathbb{R}$ be an interval containing the image of g.

Then each convex function $f:[a_1,b_1]\to\mathbb{R}$ satisfies the double inequality

$$\begin{aligned} f\Big(\frac{1}{\|b-a\|}\int_{-\|c-a\|}^{\|b-c\|}\tilde{g}(t)\,dt\Big) &\leq \frac{1}{\|b-a\|}\int_{-\|c-a\|}^{\|b-c\|}f\big(\tilde{g}(t)\big)\,dt \\ &\leq \frac{b_1-c_1}{b_1-a_1}f(a_1)+\frac{c_1-a_1}{b_1-a_1}f(b_1). \end{aligned} \tag{4.9}$$

If $\mathbb{X}=\mathbb{R}$, and thus $[a,b]\subset\mathbb{R}$, then using the interval midpoint $c=(a+b)/2$, and applying the substitution $x=(a+b)/2+t$ to the first and second member in (3.9), we get the inequality in (3.11).

More details on convex functions on the real line and normed linear space can be found in frequently quoted books [11] and [10]. Some inequalities of the Hermite-Hadamard type for h-convex functions defined on convex subsets in vector spaces, as well as the norm inequalities, were obtained in [1].

5. Subderivative, subdifferential and subtangent

Let $\mathbb{X}$ be a real normed space, let $\mathbb{I}\subseteq\mathbb{X}$ be an interval containing a pair of different points a and b, let $v=(b-a)/\|b-a\|$ be the related unit point, and let $\mathbb{L}_{\{o,v\}}$ be the line space in $\mathbb{X}$ spanned by the origin o and unit point v. Let $f:\mathbb{I}\to\mathbb{R}$ be a convex function, and let $c\in\mathbb{I}$ be an interval interior point. The notions that follow want to be recommended.

The subderivative set of f over v at c contains numbers between the left and right directional derivatives. It is the set

$$I=\mathrm{Der}_{(v,c)}f=[f'_v(c-),f'_v(c+)] \tag{5.1}$$

whose numbers are called subderivatives or slopes of f over v at c.

The subdifferential collection of f over v at c contains linear functionals on the line space $\mathbb{L}_{\{o,v\}}$ attached to subderivatives. It is the collection

$$\mathcal{D}=\mathrm{Dif}_{(v,c)}f=\big\{h:\mathbb{L}_{\{o,v\}}\to\mathbb{R}\,|\,h(tv)=kt\text{ with }k\in I\big\} \tag{5.2}$$

whose functionals are called subdifferentials of f over v at c. This means that a linear functional $h:\mathbb{L}_{\{o,v\}}\to\mathbb{R}$ belongs to $\mathcal{D}$ if and only if it satisfies the inequality

$$f'_v(c-)\,t\leq h(tv)\leq f'_v(c+)\,t \tag{5.3}$$

for every $t \geq 0$. The collection $\mathcal{D}$ is convex set in the vector space of all linear functionals on $\mathbb{L}_{\{o,v\}}$. Namely, if $h_1(tv) = k_1 t$ and $h_2(tv) = k_2 t$ with $k_1, k_2 \in I$, then a convex combination $h = \alpha_1 h_1 + \alpha_2 h_2$ satisfies $h(tv) = (\alpha_1 k_1 + \alpha_2 k_2)t$ with $k = \alpha_1 k_1 + \alpha_2 k_2 \in I$.

The subtangent collection of f over v at c contains affine functions on the line $c + \mathbb{L}_{\{o,v\}}$ attached to subderivatives. It is the collection

$$\mathcal{T} = \mathrm{Tan}_{(v,c)} f = \{h : c + \mathbb{L}_{\{o,v\}} \to \mathbb{R} \,|\, h(c+tv) = kt + f(c) \text{ with } k \in I\} \tag{5.4}$$

whose functions are called subtangents or support lines of f over v at c. This implies that an affine function $h : c + \mathbb{L}_{\{o,v\}} \to \mathbb{R}$ belongs to $\mathcal{T}$ if and only if it satisfies the inequality

$$f'_v(c-)\, t + f(c) \leq h(c+tv) \leq f'_v(c+)\, t + f(c) \tag{5.5}$$

for every $t \geq 0$. According to Lemma 3.1, each function $h \in \mathcal{T}$ satisfies the inequality $h(c+tv) \leq f(c+tv)$ for every number t such that $c + tv \in \mathbb{I}$. The collection $\mathcal{T}$ is convex set in the vector space of all affine functions on $c + \mathbb{L}_{\{o,v\}}$.

If $\mathbb{X} = \mathbb{R}$, then $\mathbb{I}$ is the interval of real numbers, $v = 1$ is the unit, and so the directional derivative of f over v at c is reduced to the usual derivative of f at c. In this case, the subderivative set I, subderivative collection $\mathcal{D}$, and subtangent collection $\mathcal{T}$ of f at c are as follows:

$$I = \mathrm{Der}_c f = [f'(c-), f'(c+)] \tag{5.6}$$

$$\mathcal{D} = \mathrm{Dif}_c f = \{h : \mathbb{R} \to \mathbb{R} \,|\, h(x) = kx \text{ with } k \in I\} \tag{5.7}$$

$$\mathcal{T} = \mathrm{Tan}_c f = \{h : \mathbb{R} \to \mathbb{R} \,|\, h(x) = k(x-c) + f(c) \text{ with } k \in I\} \tag{5.8}$$

Formula (5.7) follows from formula (5.2) by using $v = 1$ and $t = x$, as well as formula (5.8) follows from formula (5.4) by using $v = 1$ and $c + t = x$.

The notion of the subdifferential was introduced by Moreau and Rockafellar in the early 1960s (see [6] and [12]) as follows. Given a convex function f on the open interval of real numbers and an interval point c, the subdifferential of f at c is the set $\partial f(c) = [f'(c-), f'(c+)]$. The concept of the subdifferential of a convex function on the open set in a real normed space can be found in [7].

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