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ON A TRIGONOMETRIC INEQUALITY IN ACUTE TRIANGLES

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Abstract. In this paper, we provide four new proofs of a trigonometric inequality in acute triangles, which is actually equivalent to an Oppenheim's inequality. We also prove several other inequalities which are equivalent to this inequality. Meanwhile, we establish an inequality chain which involves the results of Oppenheim's, the author's and others.

1. Introduction

Let ABC be an acute (non-obtuse) triangle. In a Chinese paper [6], the author established the following trigonometric inequality:

$$\sum \left(\frac{\cos B + \cos C}{\sin A}\right)^2 \leq 4, \tag{1.1}$$

where A, B, C are the angles of the acute triangle ABC and $\sum$ denotes the cyclic sum. Equality in (1.1) holds if and only if the triangle ABC is equilateral or right isosceles.

It is worth noticing that Walker's inequality (see [11], [13](p.248) and [15]) in acute (non-obtuse) triangles can be easily derived from (1.1). Walker's inequality says that for an acute triangle ABC we have

$$s^2 \geq 2R^2 + 8Rr + 3r^2, \tag{1.2}$$

where s, R and r are the semi-perimeter, circumradius and inradious of the triangle ABC, respectively. In fact, using the inequality (1.1) and the Cauchy-Schwarz inequality we immediately obtain

$$\sum \sin^2 A \geq \left(\sum \cos A\right)^2. \tag{1.3}$$

But using related identities in the triangle ABC (see [13]), we easily get the following identity:

$$\sum \sin^2 A - \left(\sum \cos A\right)^2 = \frac{s^2 - 2R^2 - 8Rr - 3r^2}{2R^2}. \tag{1.4}$$

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And, Walker's inequality (1.2) follows from (1.3) and (1.4).

On the other hand, it is easy to show that

$$\sum \sin^2 A - \left(\sum \cos A\right)^2 = 3 - \sum (\cos B + \cos C)^2. \tag{1.5}$$

Hence, by (1.3) we again obtain the following beautiful inequality:

$$\sum (\cos B + \cos C)^2 \le 3. \tag{1.6}$$

Recently, the author [12] obtained the following weighted generalization of this inequality:

$$\sum x^2 \ge \sum (z \cos B + y \cos C)^2, \tag{1.7}$$

where x, y, z are arbitrary real numbers. Equality in (1.7) holds if and only if $x : y : z = \sin A : \sin B : \sin C$.

In (1.7), taking $x = \cos A, y = \cos B, z = \cos C$, we get

$$4 \sum \cos^2 B \cos^2 C \le \sum \cos^2 A, \tag{1.8}$$

which was first proposed by A.Oppenheim in [14]. For a proof of (1.8), see the monograph [13](p.31-32). The equality condition of (1.8) is the same as that of (1.1). It is more interesting that inequality (1.1) is actually equivalent to Oppenheim's inequality (1.8). We will prove this statement in the following (see Remark 2.2 below). In addition, we will also show that inequality (1.1) is actually a direct consequence of the weighted inequality (1.7).

The paper is organized as follows. In the next section, we will present four new proofs of inequality (1.1). In Sect.3, we will prove several inequalities which are all equivalent to the inequality (1.1). In Sect.4, we will establish an inequality chain which contains inequalities (1.3), (1.6, (1.8), and others.

2. New proofs of inequality (1.1)

In [6], the author has proved the following identity:

$$\sum \left(\frac{\cos B + \cos C}{\sin A}\right)^2 = \frac{s^4 - (8R^2 + 4Rr - 2r^2)s^2 + 64R^4 + 96R^3 r + 52R^2 r^2 + 12Rr^3 + r^4}{4s^2R^2}. \tag{2.1}$$

Hence, we have

Lemma 2.1. *In any triangle ABC we have*

$$4 - \sum \left(\frac{\cos B + \cos C}{\sin A}\right)^2 = \frac{-s^4 + (24R^2 + 4Rr - 2r^2)s^2 - (2R + r)^2(4R + r)^2}{4s^2R^2}. \tag{2.2}$$

From the above identity, we see that inequality (1.1) is equivalent to the following inequality

$$Q_0 \equiv -s^4 + (24R^2 + 4Rr - 2r^2)s^2 - (2R + r)^2(4R + r)^2 \ge 0, \tag{2.3}$$

which involves geometric elements R, r and s.

In [6], the author proved that for the acute triangle ABC the following inequality holds:

$$s^2 \ge \frac{R(4R + r)^2}{5R - 4r}, \tag{2.4}$$

which is a result parallel to Walker's inequality (1.2). Then used (2.4) and Sondat's fundamental triangle inequality (see [4] (inequality 13.8), [8] and [13])

$$T_0 \equiv -s^4 + (4R^2 + 20Rr - 2r^2)s^2 - r(4R + r)^3 \geq 0 \tag{2.5}$$

(which is true for any triangle ABC) to complete the proof of the inequality (1.1).

In the following, we will give four new proofs of the inequality (1.1) based on Lemma 2.1.

2.1. **The first new proof of inequality (1.1).**

As usual, we denote the side lengths of the triangle ABC by a, b and c. Next, we will first use the weighted inequality (1.7) to prove inequality (1.1). In fact, the author finds that if we take $x = (b + c) \cos A, y = (c + a) \cos B$ and $z = (a + b) \cos C$ in (1.7), then the inequality (2.3) can be obtained. The detailed proof is as follows:

Proof. We first prove the following two identities:

$$E_1 \equiv \sum (b + c)^2 \cos^2 A = \frac{s}{2R}(8R^2 + 2Rr - r^2 - s^2), \tag{2.6}$$

$$E_2 \equiv \sum a \cos^2 B \cos^2 C = \frac{s}{4R^3}\left[(-2R + r)s^2 + (2R + r)(4R^2 + r^2)\right]. \tag{2.7}$$

Using $\cos^2 A = 1 - \frac{a^2}{4R^2}$, we easily get that

$$\sum (b + c)^2 \cos^2 A = 2\sum bc + 2\sum a^2 - \frac{1}{2R^2}\left(abc\sum a + \sum b^2c^2\right). \tag{2.8}$$

And then using the following known identities (cf.[13]):

$$\sum a = 2s, \tag{2.9}$$

$$abc = 4Rrs, \tag{2.10}$$

$$\sum bc = s^2 + 4Rr + r^2, \tag{2.11}$$

$$\sum a^2 = 2s^2 - 8Rr - 2r^2, \tag{2.12}$$

$$\sum b^2c^2 = s^4 - 2r(4R - r)s^2 + r^2(4R + r)^2, \tag{2.13}$$

we obtain identity (2.6) immediately.

Now, using $\cos^2 B = 1 - \frac{b^2}{4R^2}$ and $\cos^2 C = 1 - \frac{c^2}{4R^2}$, we easily get

$$E_2 = \sum a - \frac{1}{4R^2}\left(\sum a \sum a^2 - \sum a^3\right) + \frac{abc}{16R^4}\sum bc. \tag{2.14}$$

And, then using (2.9)-(2.11) and the known identity:

$$\sum a^3 = 2s(s^2 - 6Rr - 3r^2), \tag{2.15}$$

we further obtain (2.7).

Let us compute the following sum in terms of R, r and s:

$$E_3 \equiv \sum (2a + b + c)^2 \cos^2 B \cos^2 C.$$

Note that $2a + b + c = a + 2s$ and

$$\begin{aligned} E_3 =& \sum (a+2s)^2 \cos^2 B \cos^2 C \\ =& \sum a^2 \cos^2 B \cos^2 C + 4s \sum a \cos^2 B \cos^2 C + 4s^2 \sum \cos^2 B \cos^2 C, \\ =& 4R^2 \prod \cos^2 A \sum \tan^2 A + 4s \sum a \cos^2 B \cos^2 C + 4s^2 \prod \cos^2 A \sum \sec^2 A, \end{aligned}$$

where $\prod$ denote the cyclic products. Then using identity (2.7) and the following known identities (see [13]):

$$\prod \cos A = \frac{s^2 - (2R+r)^2}{4R^2}, \tag{2.16}$$

$$\sum \tan^2 A = \frac{4r^2s^2 - 2(s^2 - r^2 - 4Rr)\left[s^2 - (2R+r)^2\right]}{\left[s^2 - (2R+r)^2\right]^2}, \tag{2.17}$$

$$\sum \sec^2 A = \frac{(s^2 + r^2 - 4R^2)^2 - 8R(R+r)\left[s^2 - (2R+r)^2\right]}{\left[s^2 - (2R+r)^2\right]^2}, \tag{2.18}$$

we further obtain

$$E_3 = \frac{1}{4R^2} E_4, \tag{2.19}$$

where

$$\begin{aligned} E_4 =& s^6 - (26R^2 + 4Rr - 2r^2)s^4 + (88R^4 + 96R^3r + 48R^2r^2 \\ &+ 12Rr^3 + r^4)s^2 - 2(4R+r)(2R+r)^2R^2r. \end{aligned}$$

Finally, it follows from (2.6) and (2.19) that

$$E_1 - E_3 = \frac{-s^4 + (24R^2 + 4Rr - 2r^2)s^2 - (2R+r)^2(4R+r)^2}{4R^2}. \tag{2.20}$$

Now, taking $x = (b+c)\cos A, y = (c+a)\cos B$ and $z = (a+b)\cos C$ in (1.7), we get

$$\sum (b+c)^2 \cos^2 A \geq \sum (2a+b+c)^2 \cos^2 B \cos^2 C,$$

which shows that $E_1 \geq E_3$. Hence, by identity (2.20) we deduce that inequality (2.3) holds for the acute triangle ABC. Further, by Lemma 2.1 we deduce that inequality (1.1) holds. □

2.2. **The second new proof of inequality (1.1).**

In [11], the author remarked that Walker's inequality (1.2) can be obtained from the following identity:

$$\sum (c^2 + a^2 - b^2)(a^2 + b^2 - c^2)(b-c)^2 = 32r^2s^2(s^2 - 2R^2 - 8Rr - 3r^2). \tag{2.21}$$

Next, we will prove a similar identity which shows that the inequality (2.3) is valid for the acute triangle ABC.

Proof. We compute the sum

$$F_0 \equiv \sum b^2c^2(c^2 + a^2 - b^2)(a^2 + b^2 - c^2)(b-c)^2$$

in terms of R, r and s. First, it is easy to get that

$$\begin{aligned} F_0 =& 2\sum b^3c^3 \sum a^4 + 2(abc)^2 \sum b^2c^2 + \sum a^4 \sum a^6 \\ & - 4(abc)^3 \sum a - \sum a^2 \sum a^8 - 4\sum b^5c^5 \end{aligned} \tag{2.22}$$

Using the previous identities (2.9)-(2.13) and the following known identities (see [10]):

$$\sum a^4 = 2s^4 - 4(4R+3r)s^2r + 2(4R+r)^2r^2, \tag{2.23}$$

$$\begin{aligned} \sum a^6 =& 2s^6 - 6(4R+5r)rs^4 + 6(24R^2+24Rr+5r^2)r^2s^2 \\ & - 2(4R+r)^3r^3, \end{aligned} \tag{2.24}$$

$$\begin{aligned} \sum a^8 =& 2s^8 - 8(4R+7r)rs^6 + 20(16R^2+24Rr+7r^2)r^2s^4 \\ & - 8(4R+r)(32R^2+32Rr+7r^2)r^3s^2 + 2(4R+r)^4r^4, \end{aligned} \tag{2.25}$$

$$\sum b^3c^3 = s^6 - 3(4R-r)rs^4 + 3r^4s^2 + (4R+r)^3r^3, \tag{2.26}$$

$$\begin{aligned} \sum b^5c^5 =& s^{10} - 5(4R-r)s^8r + 10(8R^2-4Rr+r^2)s^6r^2 \\ & + 10s^4r^6 + 5(4R+r)^2s^2r^6 + (4R+r)^5r^5, \end{aligned} \tag{2.27}$$

we obtain that

$$F_0 = 64r^4s^2\left[-s^4 + (24R^2+4Rr-2r^2)s^2 - (2R+r)^2(4R+r)^2\right], \tag{2.28}$$

For the acute triangle ABC, it is clear that we have inequality:

$$F_0 \equiv \sum b^2c^2(c^2+a^2-b^2)(a^2+b^2-c^2) \geq 0.$$

Hence, inequality (2.3) follows from identity (2.28). And we further deduce by Lemma 2.1 that inequality (1.1) holds for the acute triangle ABC. □

2.3. The third new proof of inequality (1.1).

In the proof of (2.4) given in [6], the author used the following well-known inequality (see [1]-[4]):

$$s^2 \geq 2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R^2-2Rr}, \tag{2.29}$$

which is actually equivalent to Sondat's inequality (2.5).

Next, we will give a simpler proof of inequality (2.4), and then use Sondat's inequality (2.5) to finish the proof of inequality (1.1).

Proof. We first prove the acute triangle inequality (2.4), namely

$$(5R-4r)s^2 - R(4R+r)^2 \geq 0, \tag{2.30}$$

which can be rewritten as

$$2(R+r)(s^2-2R^2-8Rr-3r^2) + 3(R-2r)\left[s^2-(2R+r)^2\right] \geq 0. \tag{2.31}$$

This inequality can be obtained from Walker's inequality (1.2), Euler's inequality

$$R \geq 2r \tag{2.32}$$

(which holds for any triangle), and Ciamberlini's acute triangle inequality (see [5]):

$$s \geq 2R + r \tag{2.33}$$

(which follows from identity (2.16)). Hence inequality (2.30) is proved.

To prove the inequality (2.3), i.e., $Q_0 \geq 0$, we can rewrite Q_0 as follows:

$$Q_0 = T_0 + 4R\left[(5R-4r)s^2 - R(4R+r)^3\right]. \tag{2.34}$$

Thus, by Sondat's inequality (2.5) and inequality (2.30), we conclude that the inequality $Q_0 \geq 0$ holds for the acute triangle ABC. Further, by Lemma 2.1 we deduce that the inequality (1.1) holds for the acute triangle ABC. □

2.4. **The fourth new proof of inequality (1.1).**

Next, we will use Oppenheim's inequality (1.8) to prove inequality (1.1).

Proof. Obviously, Oppenheim's inequality (1.8) is equivalent to

$$K_0 \equiv \sum \cos^2 A + 8\sum \cos A \prod \cos A - 4\left(\sum \cos B \cos C\right)^2 \geq 0.$$

Using the previous identity (2.16) and the following known identities (see [13]):

$$\sum \cos A = 1 + \frac{r}{R}, \tag{2.35}$$

$$\sum \cos^2 A = \frac{6R^2 + 4Rr + r^2 - s^2}{2R^2}, \tag{2.36}$$

$$\sum \cos B \cos C = \frac{s^2 - 4R^2 + r^2}{4R^2} \tag{2.37}$$

we get

$$K_0 = \frac{-s^4 + (14R^2 + 8Rr - 2r^2)s^2 - 36R^4 - 56R^3 r - 30R^2 r^2 - 8Rr^3 - r^4}{4R^4}. \tag{2.38}$$

Hence, we obtain the following inequality:

$$Q_1 \equiv -s^4 + (14R^2 + 8Rr - 2r^2)s^2 - 36R^4 - 56R^3 r - 30R^2 r^2 - 8Rr^3 - r^4 \tag{2.39}$$

We now use $Q_1 \geq 0$ to prove $Q_0 \geq 0$. It is easy to check the following identity:

$$Q_0 - Q_1 = 2RM_1, \tag{2.40}$$

where

$$M_1 = (5R-2r)s^2 - 14R^3 - 20R^2 r - 11Rr^2 - 2r^3.$$

Since $Q_1 \geq 0$, to prove $Q_0 \geq 0$ we need to show that $M_1 \geq 0$. But we can rewrite M_1 in two ways as follows:

$$M_1 = (5R-2r)\left[s^2 - (2R+r)^2\right] + 2(3R+2r)(R^2 - 2Rr - r^2), \tag{2.41}$$

$$M_1 = (5R-2r)(s^2 - 2R^2 - 8Rr - 3r^2) - 4(R-2r)(R^2 - 2Rr - r^2). \tag{2.42}$$

If $R^2 - 2Rr - r^2 \geq 0$, then by (2.41), Ciamberlini's inequality (2.33), and Euler's inequality $R \geq 2r$ one sees that $M_1 \geq 0$ holds. If $R^2 - 2Rr - r^2 < 0$, then by (2.42), Euler's inequality, Walker's inequality (1.3) one also sees that $M_1 \geq 0$ holds. Therefore, we conclude that inequality $M_1 \geq 0$ holds for all acute triangles. This completes the proof of $Q_0 \geq 0$. Finally, by Lemma 2.1 we know that the inequality (1.1) holds. □

Remark 2.1. Indeed, inequality $Q_0 \geq Q_1$ can be improved to $Q_0 \geq 2Q_1$ (we omit the proof here).

Remark 2.2. Inequality (1.1) is equivalent to Oppenheim's inequality (1.8). To show this conclusion, it remains to prove that (1.8) can be proved by using (1.1) (since we have used (1.1) to prove (1.8). It is easy to check the following identity

$$4Q_1 - Q_0 = 3T_0 + 4RM_2, \tag{2.43}$$

where T_0 is the same as in (2.5) and

$$M_2 = (5R - 8r)s^2 - 20R^3 + 16R^2r + 19Rr^2 + 4r^3.$$

In the same way used to prove $M_1 \geq 0$, it is easily shown that $M_2 \geq 0$ for all acute triangles. Hence, by identity (2.43) and Sondat's inequality (2.5) one deduces that $4Q_1 \geq Q_0$. And then by $Q_0 \geq 0$ we have $Q_1 \geq 0$. This shows that Oppenheim's inequality (1.8) can be derived from inequality (1.1). Therefore, inequality (1.1) is actually equivalent to Oppenheim's inequality (1.8).

3. Several new equivalent forms of the inequalities (1.1)

In this section, we will prove several new inequalities which are equivalent to inequality (1.1).

Theorem 3.1. *Let ABC be an acute triangle, then*

$$\sum \sin^2 2A \leq \left(\sum \cos A\right)^2, \tag{3.1}$$

$$\sum \frac{\cos B \cos C}{1 + \cos A} \leq \frac{1}{2}. \tag{3.2}$$

Both inequalities are equivalent to inequality (1.1). Equalities in (3.1) and (3.2) hold if and only if the triangle ABC is equilateral or right isosceles.

Proof. (i) In order to prove the inequality (3.1), we first prove that for any triangle ABC the following inequality holds:

$$\sum \sin^2 A \leq \left(\sum \sin \frac{A}{2}\right)^2, \tag{3.3}$$

that is

$$\sum \sin^2 A \leq \sum \sin^2 \frac{A}{2} + 2\prod \sin \frac{A}{2} \sum \frac{1}{\sin \frac{A}{2}}.$$

In view of the well-known inequality

$$\sin \frac{A}{2} \leq \frac{a}{b+c}, \tag{3.4}$$

we only need to prove that

$$\sum \sin^2 A \leq \sum \sin^2 \frac{A}{2} + 2\prod \sin \frac{A}{2} \sum \frac{b+c}{a}. \tag{3.5}$$

Using the following known identities(see [13]):

$$\sum \sin^2 A = \frac{s^2 - 4Rr - r^2}{2R^2}, \tag{3.6}$$

$$\sum \sin^2 \frac{A}{2} = \frac{2R - r}{2R}, \tag{3.7}$$

$$\prod \sin \frac{A}{2} = \frac{r}{4R}, \tag{3.8}$$

$$\sum \frac{b+c}{a} = \frac{s^2 - 2Rr + r^2}{2Rr}, \tag{3.9}$$

we easily know that the above inequality (3.5) is equivalent to

$$4R^2 + 4Rr + 3r^2 \geq s^2, \tag{3.10}$$

which is the well-known Gerretsen inequality (see [4] and [1]). Thus, inequality (3.3) is proved. When the triangle ABC is an acute triangle, we may make the substitution $A \to \pi - 2A, B \to \pi - 2B, C \to \pi - 2C$ in (3.3) and then inequality (3.1) follows immediately.

Now, we show that inequality (3.1) is equivalent to (1.1). Firstly, we use (3.1) to prove inequality (1.1). Noting that

$$\sum \sin^2 2A = 4 \sum \sin^2 A - 4 \sum \sin^4 A,$$

and using the Low of Sine and the previous identities (2.12) and (2.23), we get

$$\sum \sin^2 2A = \frac{-s^4 + (4R^2 + 8Rr + 6r^2)s^2 - r(4R + r)(2R + r)^2}{2R^4}. \tag{3.11}$$

Substituting (3.11) and the previous identity (2.35) into inequality (3.1), we can obtain the following inequality

$$Q_2 \equiv s^4 - (4R^2 + 8Rr + 6r^2)s^2 + 2R^4 + 20R^3 r + 22R^2 r^2 + 8Rr^3 + r^4 \geq 0. \tag{3.12}$$

which is equivalent to (3.1). We next use the above inequality to prove inequality (2.3), i.e., $Q_0 \geq 0$. It is easy to check that

$$Q_0 = 3Q_2 + 4T_0 + 2eM_4, \tag{3.13}$$

where $e = R - 2r, M_4 = (10R - 6r)s^2 - R(35R^2 + 20Rr + 3r^2)$. Thus, by Sondat's inequality (2.5), Euler's inequality $R \geq 2r$, and identity (3.13), to show $Q_0 \geq 0$ we need to prove that $M_4 \geq 0$. We can rewrite M_4 in two ways as follows:

$$M_4 = (10R - 6r)\left[s^2 - (2R + r)^2\right] + (5R + 6r)(R^2 - 2Rr - r^2), \tag{3.14}$$

$$M_4 = (10R - 6r)(s^2 - 2R^2 - 8Rr - 3r^2) - 3(5R - 6r)(R^2 - 2Rr - r^2). \tag{3.15}$$

Then, by Walker's inequality (1.2), Euler's inequality (2.32), and Ciamberlini's inequality (2.33), we immediately conclude that $M_4 \geq$ holds for all acute triangles. Hence, inequality $Q_0 \geq 0$ is proved.

Secondly, we use $Q_0 \geq 0$ to prove $Q_2 \geq 0$. It is easy to check that

$$8Q_2 = Q_0 + M_5, \tag{3.16}$$

where

$$M_5 = 9s^4 - (56R^2 + 68Rr + 46r^2)s^2 + 80R^4 + 256R^3r + 228R^2r^2 + 76Rr^3 + 9r^4.$$

Since $Q_0 \geq 0$, we need to prove that $M_5 \geq 0$. We consider the following two cases to finish the proof of $M_5 \geq 0$.

Case 1. Suppose that R and r satisfy $R^2 - 2Rr - r^2 \geq 0$.

It is easy to check that

$$\begin{aligned} M_5 =& 9\left[s^2 - (2R+r)^2\right]^2 + \left[s^2 - (2R+r)^2\right](16R^2 + 4Rr - 28r^2) \\ &+ 4r(12R+7r)(R^2 - 2Rr - r^2). \end{aligned} \tag{3.17}$$

Thus, by the hypothesis, Ciamberlini's inequality $s > 2R + r$, and $R \geq 2r$ one sees that $M_5 \geq 0$ holds.

Case 2. Suppose that R and r satisfy $R^2 - 2Rr - r^2 < 0$.

It is easy to check that

$$\begin{aligned} M_5 =& 9(s^2 - 2R^2 - 8Rr - 3r^2)^2 + (s^2 - 2R^2 - 8Rr - 3r^2)(-20R^2 + 76Rr \\ &+ 8r^2) + 4(R - 6r)(R - 2r)(R^2 - 2Rr - r^2). \end{aligned} \tag{3.18}$$

By the hypothesis $R^2 - 2Rr - r^2 < 0$, we have that $R < (\sqrt{2} + 1)r$, which yields $6r > R$ and $76r > 20R$. Hence, by Walker's inequality (1.2), Euler's inequality, and the hypothesis one sees that $M_5 \geq 0$ is true.

By combining the discussions of the above two cases, we conclude that $M_5 \geq 0$ holds for all acute triangles. Thus, we have proved $Q_2 \geq 0$ by using $Q_0 \geq 0$. And, we therefore complete the proof of the equivalence between inequalities $Q_0 \geq 0$ and $Q_2 \geq 0$. Hence, We have proved that inequality (3.1) is equivalent to inequality (1.1).

(ii) We now prove inequality (3.2). If $\triangle ABC$ is a right triangle, without loss of generality we may assume that $A = \pi/2$. In this case, (3.2) becomes

$$\cos B \cos C \leq \frac{1}{2},$$

which follows from the fact that $\cos B \cos C \leq \sin^2 \dfrac{A}{2}$ for $A = \pi/2$. And the equality occurs only when $B = C = \pi/4$. If $\triangle ABC$ is an acute triangle ($A, B, C \neq \pi/2$), then we have

$$\begin{aligned} &\sum \frac{\cos B \cos C}{1 + \cos A} \\ &= \sum \frac{\cos B \cos C}{1 - \cos B \cos C + \sin B \sin C} \\ &= \sum \frac{1}{\sqrt{(1 + \tan^2 B)(1 + \tan^2 C)} - 1 + \tan B \tan C} \\ &\leq \sum \frac{1}{(1 + \tan B \tan C) - 1 + \tan B \tan C} \\ &= \frac{1}{2} \sum \cot B \cot C = \frac{1}{2}, \end{aligned}$$

where we used the Cauchy-Schwarz inequality and the following identity

$$\sum \cot B \cot C = 1. \tag{3.19}$$

From the above discussions, we conclude that inequality (3.2) holds for any non-obtuse triangle ABC. Also, its equality condition is as mentioned in Theorem 3.1.

Next, we prove that inequality (3.2) is equivalent to inequality (1.1). In fact, the author finds the following identity relation:

$$\sum \left(\frac{\cos B + \cos C}{\sin A}\right)^2 - 2\sum \frac{\cos B \cos C}{1 + \cos A} = 3, \tag{3.20}$$

which clearly shows the claimed conclusion. This identity can be proved as follows: Putting $\cos A = u, \cos B = v, \cos C = w$, then (3.20) is equivalent to

$$\sum \frac{(v+w)^2}{1-u^2} - 2\sum \frac{vw}{1+u} = 3,$$

i.e.,

$$\sum (1-v^2)(1-w^2)(v+w)^2 - 2\prod (1-u)\sum vw - 3\prod (1-u^2) = 0.$$

Expanding and factorizing gives

$$\left(\sum u^2 + 2\prod u - 1\right)\left(2\sum u^2 - \sum v^2w^2 - 3\right) = 0, \tag{3.21}$$

which is required to prove. We recall that in any triangle ABC the following identity holds:

$$\sum \cos^2 A + 2\prod \cos A = 1, \tag{3.22}$$

so that

$$\sum u^2 + 2\prod u - 1 = 0. \tag{3.23}$$

Hence, identity (3.21) holds and identity (3.20) is proved. This completes the proof of the equivalence of inequalities (3.2) and (1.1). And we complete the proof of Theorem 3.1. ☐

Remark 3.1. By using the previous identities (2.16), (2.35)-(2.37), we can prove that

$$\sum \frac{\cos B \cos C}{1 + \cos A} - \frac{1}{2} = \frac{s^4 - (24R^2 + 4Rr - 2r^2)s^2 + (2R+r)^2(4R+r)^2}{8s^2R^2}, \tag{3.24}$$

which directly shows that inequality (3.2) is equivalent to (2.3).

Remark 3.2. From the previous inequalities (1.3) and (3.1), we deduce that

$$\sum \sin^2 A \geq \sum \sin^2 2A \tag{3.25}$$

holds for the acute triangle ABC. Similar to the proof of the equivalence of (1.1) and (3.1), we can also prove that inequality (3.25) is actually equivalent to (1.1). On the other hand, by the angle transforms it can be seen that (3.25) is equivalent to the following inequality:

$$\sum \cos^2 \frac{A}{2} \geq \sum \sin^2 A, \tag{3.26}$$

which holds for any triangle ABC. Incidentally, inequality (3.26) can be generalized to the following ternary quadratic inequality (see [9]):

$$\sum x^2 \cos^2 \frac{A}{2} \geq \sum yz \sin^2 A \tag{3.27}$$

where x, y, z are arbitrary real numbers. And, by (3.27) we easily know again that the acute triangle inequality (3.25) can be generalized to

$$\sum x^2 \sin^2 A \geq \sum yz \sin^2 2A, \tag{3.28}$$

with equality if and only if $x = y = z$ and the triangle ABC is equilateral or $x = y = z$ and the triangle ABC is right isosceles.

In the following Theorem 3.2, we will point out that the trigonometric inequality (1.1) is equivalent to the two geometric inequalities. In what follows, we denote the altitudes of the triangle ABC by h_a, h_b, h_c and denote the radiff of excircles of the triangle ABC by r_a, r_b, r_c. In addition, denote the area of the triangle ABC by S.

Theorem 3.2. *Let ABC be an acute triangle, then*

$$\sum (h_a + r_a)^2 \leq 4s^2, \tag{3.29}$$

$$\sum \frac{(b+c)^2}{a^2}(s-b)^2(s-c)^2 \leq 4S^2. \tag{3.30}$$

Both inequalities of (3.29) and (3.30) are equivalent to inequality (1.1). Equalities in (3.29) and (3.30) hold if and only if the triangle ABC is equilateral or right isosceles.

Proof. We first prove that in any triangle ABC the following identity holds:

$$\frac{\cos B + \cos C}{\sin A} = \frac{h_a + r_a}{s}, \tag{3.31}$$

Let w_a be the lengths of the bisector of $\angle BAC$, then it is easy to obtain

$$h_a = w_a \cos \frac{B-C}{2} \tag{3.32}$$

and

$$S = \frac{1}{2}(b+c) w_a \sin \frac{A}{2}. \tag{3.33}$$

So we have

$$\frac{\cos B + \cos C}{\sin A} = \frac{\cos \dfrac{B-C}{2}}{\cos \dfrac{A}{2}} = \frac{h_a}{w_a \cos \dfrac{A}{2}} = \frac{h_a (b+c) \sin \dfrac{A}{2}}{2S \cos \dfrac{A}{2}} = \frac{b+c}{a} \tan \frac{A}{2}.$$

On the other hand, by the known formula $h_a = 2S/a$, $r_a = S/(s-a)$, and $S = rs$, we have

$$\frac{h_a + r_a}{s} = \frac{1}{s}\left(\frac{2S}{a} + \frac{S}{s-a}\right) = \frac{(b+c)S}{a(s-a)s} = \frac{(b+c)r}{a(s-a)} = \frac{b+c}{a} \tan \frac{A}{2}.$$

Hence, identity (3.31) is proved and it shows that inequality (1.1) is equivalent to inequality (3.29).

Since

$$h_a + r_a = \frac{(b+c)S}{a(s-a)}, \tag{3.34}$$

one sees that inequality (3.29) is equivalent to

$$\sum \frac{(b+c)^2}{a^2(s-a)^2} \leq \frac{4s^2}{S^2}.$$

Multiplying both sides by $\prod(s-a)^2$ and using Heron's formula

$$S=\sqrt{s(s-a)(s-b)(s-c)}, \tag{3.35}$$

we obtain inequality (3.30). Thus, we have proved that both inequalities (3.29) and (3.30) are equivalent to inequality (1.1).

Next, we will prove inequality (3.30). By Heron's formula and $s=(a+b+c)/2$, it is easy to see that (3.30) is equivalent to

$$M_6\equiv 4\sum a\prod(b+c-a)a^2-\sum b^2c^2(b+c)^2(c+a-b)^2(a+b-c)^2\geq 0. \tag{3.36}$$

Without loss of generality we may assume that $a=\max\{a,b,c\}$, i.e., $a\geq b$ and $a\geq c$. After analyzing, we obtain the identity:

$$\begin{aligned} M_6 =&2b^2c^2(b-c)^2(b+c-a)\left[b(a^2-b^2)+c(a^2-c^2)+abc\right]\\ &+(b^2+c^2-a^2)(a-b)(a-c)M_7, \end{aligned} \tag{3.37}$$

where

$$\begin{aligned} M_7 =&(b^2+c^2)a^4+(-b^3-c^3+b^2c+c^2b)a^3\\ &-(b^4+c^4+b^3c+bc^3-4b^2c^2)a^2\\ &+(b^5+c^5-b^4c-c^4b)a+b^5c+bc^5-2b^3c^3. \end{aligned}$$

Therefore, to prove $M_6\geq 0$ for the acute triangle ABC we only need to prove that the strict inequality $M_7>0$ holds for any triangle ABC. Putting $b+c-a=2x, c+a-b=2y, a+b-c=2z$, then we have $a=y+z, b=z+x, c=x+y (x,y,z>0)$. Also, it is easy to get

$$\begin{aligned} M_7 =&4(y-z)^2x^4+12(y+z)(y-z)^2x^3+(8y^4+12zy^3-8y^2z^2\\ &+12z^3y+8z^4)x^2+8yz(y+z)^3x+8y^2z^2(y+z)^2. \end{aligned}$$

Noting that $y^4+z^4-y^2z^2>0$, we see that $M_7>0$ holds. Thus, the inequalities (3.36) and (3.30) are proved. Moreover, from identity (3.37) we easily obtain the equality conditions of (3.30) as stated in Theorem 3.2. This completes the proof of Theorem 3.2. □

Remark 3.3. By identity (3.31), we see that inequalities (1.6) is equivalent to

$$\sum(h_a+r_a)^2\sin^2 A\leq 3s^2. \tag{3.38}$$

Remark 3.4. We can prove that inequality (3.30) is better than the following known inequality (see [13]):

$$(4S)^6\geq 27\prod(b^2+c^2-a^2)^2, \tag{3.39}$$

which is true for the acute triangle ABC.

4. An inequality chain

Considering relations between inequality (1.1) and Oppenheim's inequality (1.8), the author first finds that

$$\frac{16\sum\cos^2 B\cos^2 C}{\sum\cos^2 A}\leq\sum\left(\frac{\cos B+\cos C}{\sin A}\right)^2. \tag{4.1}$$

Further, we find that the above inequality can be refined and extended to the following inequality chain.

Theorem 4.1. *Let ABC be an acute triangle ABC, then*

$$\begin{aligned}
&\frac{16\sum\cos^2 B\cos^2 C}{\sum\cos^2 A}\leq\frac{4\sum\sin^2 2A}{\sum\sin^2 A}\leq\frac{4\sum\sin^2 2A}{\left(\sum\cos A\right)^2}\leq 8\sum\frac{\cos B\cos C}{1+\cos A}\\
&\leq-\frac{8\sum\cos B\cos C}{\sum\cos 2A}\leq\frac{4\left(\sum\cos A\right)^2}{\sum\sin^2 A}\leq\sum\left(\frac{\cos B+\cos C}{\sin A}\right)^2\\
&\leq 1+\sum(\cos B+\cos C)^2\leq 3-\frac{2\sum\cos B\cos C}{\sum\cos 2A}\\
&\leq 3+\sum\left(\frac{\cos B+\cos C}{\sin A}\right)^2\cos^2 A\leq 4.
\end{aligned}\tag{4.2}$$

All the equalities in (4.2) hold if and only if the acute triangle ABC is equilateral or right isosceles.

Obviously, the above inequality chain (4.2) gives a refinement of the equivalent form of Oppenheim's inequality (1.8). It contains the previous inequalities (1.1), (1.3), (1.6), (1.8), (3.1), and (3.2) etc. The last inequality of (4.2) is equivalent to

$$\sum\left(\frac{\cos B+\cos C}{\sin A}\right)^2\cos^2 A\leq 1. \tag{4.3}$$

Also, by identity (3.31) one sees that inequality (4.3) is equivalent to

$$\sum(h_a+r_a)^2\cos^2 A\leq s^2. \tag{4.4}$$

which adding inequality (3.38) gives the previous inequality (3.29).

In addition, the following inequality

$$\sum\left(\frac{\cos B+\cos C}{\sin A}\right)^2\cos 2A\geq -2 \tag{4.5}$$

can be easily obtained from (4.2).

In order to prove Theorem 4.1, we first give some identities.

Putting

$$k_1 = \frac{4\sum\cos^2 B\cos^2 C}{\sum\cos^2 A}, \quad k_2 = \frac{\sum\sin^2 2A}{\sum\sin^2 A},$$
$$k_3 = \frac{\sum\sin^2 2A}{\left(\sum\cos A\right)^2}, \quad k_4 = 2\sum\frac{\cos B\cos C}{1+\cos A},$$
$$k_5 = -\frac{2\sum\cos B\cos C}{\sum\cos 2A}, \quad k_6 = \frac{\left(\sum\cos A\right)^2}{\sum\sin^2 A},$$
$$k_7 = \frac{1}{4}\sum\left(\frac{\cos B+\cos C}{\sin A}\right)^2, \quad k_8 = \frac{1}{4}+\frac{1}{4}\sum\left(\cos B+\cos C\right)^2,$$
$$k_9 = \frac{3}{4}-\frac{\sum\cos B\cos C}{2\sum\cos 2A}, \quad k_{10} = \frac{3}{4}+\frac{1}{4}\sum\left(\frac{\cos B+\cos C}{\sin A}\right)^2\cos^2 A,$$

then we have

Lemma 4.1. *In any triangle ABC, the following identities hold:*

$$k_1 = \frac{s^4+(-16R^2-8Rr+2r^2)s^2+(12R^2+4Rr+r^2)(2R+r)^2}{2R^2(6R^2+4Rr+r^2-s^2)}, \tag{4.6}$$
$$k_2 = \frac{-s^4+(4R^2+8Rr+6r^2)s^2-r(4R+r)(2R+r)^2}{R^2(s^2-4Rr-r^2)}, \tag{4.7}$$
$$k_3 = \frac{-s^4+(4R^2+8Rr+6r^2)s^2-r(4R+r)(2R+r)^2}{2R^2(R+r)^2}, \tag{4.8}$$
$$k_4 = \frac{s^4+(-20R^2-4Rr+2r^2)s^2+(2R+r)^2(4R+r)^2}{4s^2R^2}, \tag{4.9}$$
$$k_5 = \frac{4R^2-r^2-s^2}{6R^2+8Rr+2r^2-2s^2}, \tag{4.10}$$
$$k_6 = \frac{2(R+r)^2}{s^2-4Rr-r^2}. \tag{4.11}$$
$$k_7 = \frac{s^4-(8R^2+4Rr-2r^2)s^2+64R^4+96R^3r+52R^2r^2+12Rr^3+r^4}{16s^2R^2}, \tag{4.12}$$
$$k_8 = \frac{10R^2+8Rr+3r^2-s^2}{8R^2}, \tag{4.13}$$
$$k_9 = \frac{22R^2+24Rr+5r^2-7s^2}{8(3R^2+4Rr+r^2-s^2)}, \tag{4.14}$$
$$k_{10} = \frac{3s^4-(12R^2+20Rr+4r^2)s^2+(2R+r)^2(4R+r)^2}{16s^2R^2}. \tag{4.15}$$

The above identities can be obtained by using the previous identities, we omit the details. The proof of the third inequality of (4.2) will be used the following lemma.

Lemma 4.2. *In an acute triangle ABC, we have*

$$s^2 \geq 16Rr - 3r^2 - \frac{4r^3}{R}, \tag{4.16}$$

with equality if and only if the acute triangle ABC is equilateral or right isosceles.

Inequality (4.16) was first established by the author in a Chinese paper [7]. In the recent paper [11], the author gave a new direct proof.

We are now ready to prove Theorem 4.1.

Proof. Clearly, inequality chain (4.2) is equivalent to

$$k_1 \leq k_2 \leq k_3 \leq k_4 \leq k_5 \leq k_6 \leq k_7 \leq k_8 \leq k_9 \leq k_{10} \leq 1. \tag{4.17}$$

Next, we will prove each of these inequalities in turn.

(I) By identities (4.6) and (4.7) given in Lemma 4.1, it is easy to obtain

$$k_2 - k_1 = \frac{[s^2-(2R+r)^2][s^4-2r(4R+7r)s^2+r^2(4R+r)^2]}{2R^2(s^2-4Rr-r^2)(6R^2+4Rr+r^2-s^2)}. \tag{4.18}$$

In view of Ciamberlini's inequality (2.33) and Gerretsen's inequality (3.10), to prove $k_2 \geq k_1$ we only need to prove that

$$X_1 \equiv s^4 - 2r(4R+7r)s^2 + r^2(4R+r)^2 \geq 0, \tag{4.19}$$

which can be rewritten as follows:

$$\begin{aligned} X_1 =& (s^2+8Rr)(s^2-16Rr+5r^2) + 19r^2(4R^2+4Rr+3r^2-s^2) \\ &+ 4r^2(17R+7r)(R-2r). \end{aligned}$$

Thus, by Euler's inequality $R \geq 2r$ and Gerretsen's inequality (see [13]):

$$s^2 \geq 16Rr - 5r^2, \tag{4.20}$$

which holds for all triangles, one sees that $X_1 \geq 0$ is true for any triangle ABC. Hence inequality $k_2 \geq k_1$ is proved.

(II) The second inequality of (4.2) is equivalent to the previous inequality (1.3). Thus, it holds for the acute triangle ABC.

(III) By identities (4.8) and (4.9), we easily obtain

$$4s^2R^2(R+r)^2(k_4-k_3) = X_2, \tag{4.21}$$

where

$$\begin{aligned} X_2 =& 2s^6 - (7R^2+14Rr+11r^2)s^4 + (-20R^4-12R^3r+14R^2r^2 \\ &+ 16Rr^3+4r^4)s^2 + (4R+r)^2(2R+r)^2(R+r)^2. \end{aligned}$$

In order to prove $X_2 \geq 0$, we let

$$\begin{aligned} s_0 &= s^2 - (2R+r)^2, \\ s_1 &= s^2 - 2R^2 - 8Rr - 3r^2, \\ s_2 &= (5R-4r)s^2 - R(4R+r)^2. \end{aligned}$$

Then by Ciamberlini's inequality (2.33), Walker's inequality (1.2), and the previous inequality (2.30) we have $s_0 \geq 0, s_1 \geq 0$, and $s_2 \geq 0$ respectively. After analyzing, we obtain the following identity

$$(5R-4r)X_2 = s_0\left[2(5R-4r)s_1^2 + (9R^2+26Rr+3r^2)s_2\right] + X_3, \tag{4.22}$$

where

$$\begin{aligned} X_3 =& (24R^5+104R^4r-336R^3r^2+228R^2r^3+376Rr^4+92r^5)s^2 \\ &- 4(6R^5+16R^4r-61R^3r^2+65R^2r^3+83Rr^4+19r^5)(2R+r)^2. \end{aligned}$$

From (4.22), it can be seen that to prove $X_2 \geq 0$ it remains to show that $X_3 \geq 0$. We next consider the following two cases to complete the proof of $X_3 \geq 0$.

Case 1. R and r satisfy $R^2 - 2Rr - r^2 > 0$.

Let $e = R - 2r$, then by Euler's inequality $R \geq 2r$ we have

$$\begin{aligned} &24R^5+104R^4r-336R^3r^2+228R^2r^3+376Rr^4+92r^5 \\ &= 24e^5+344e^4r+1456e^3r^2+2628e^2r^3+2504er^4+1500r^5 > 0. \end{aligned} \tag{4.23}$$

Thus, by Ciamberlini's inequality $s \geq 2R+r$ and the hypothesis, we know

$$\begin{aligned} X_3 \geq& (24R^5+104R^4r-336R^3r^2+228R^2r^3+376Rr^4+92r^5)(2R+r)^2 \\ &- 4(6R^5+16R^4r-61R^3r^2+65R^2r^3+83Rr^4+19r^5)(2R+r)^2 \\ =& 4r(5R-4r)(R^2-r^2-2Rr)(2R+r)^3 > 0. \end{aligned}$$

Case 2. R and r satisfy $R^2 - 2Rr - r^2 \leq 0$.

In this case, by Lemma 4.2 and inequality (4.23), to prove $X_3 \geq 0$ we need to prove that

$$\begin{aligned} &(24R^5+104R^4r-336R^3r^2+228R^2r^3+376Rr^4+92r^5)(16R^2r-3Rr^2-4r^3) \\ &-4R(6R^5+16R^4r-61R^3r^2+65R^2r^3+83Rr^4+19r^5)(2R+r)^2 \geq 0. \end{aligned}$$

After arranging, we know the above inequality is equivalent to

$$-4(R-2r)(R^2-2Rr-r^2)(24R^5+88R^4r-292R^3r^2-2R^2r^3+163Rr^4+46r^5) \geq 0,$$

which can be rewritten as follows

$$-4e(R^2-2Rr-r^2)(24e^5+328e^4r+1372e^3r^2+2278e^2r^3+1387er^4+204r^5) \geq 0.$$

By Euler's inequality $e \geq 0$ and the hypothesis, we know that the above inequality holds.

Combining the discussions of the above two cases, we conclude that inequality $X_3 \geq 0$ holds for all acute triangles. This completes the proof of $X_2 \geq 0$, and inequality $k_4 \geq k_3$ is proved.

(IV) By Lemma 4.1, it is easy to obtain the following identity

$$k_5 - k_4 = \frac{(s+2R+r)(s-2R-r)X_4}{4s^2R^2(s^2-3R^2-4Rr-r^2)}, \tag{4.24}$$

where

$$X_4 = -s^4 + (21R^2+4Rr-2r^2)s^2 - (R+r)(3R+r)(4R+r)^2.$$

By Ciamberlini's inequality $s \geq 2R+r$, to prove $k_5 \geq k_4$ we need to prove $X_4 \geq 0$. But, X_4 can be rewritten as

$$X_4 = T_0 + R\left[(17R-16r)s^2 - 3R(4R+r)^2\right], \tag{4.25}$$

where T_0 is the same as in (2.5). Thus, it remains to prove that

$$s^2 \geq \frac{3R(4R+r)^2}{17R-16r}. \tag{4.26}$$

Since

$$\frac{R(4R+r)^2}{5R-4r} - \frac{3R(4R+r)^2}{17R-16r} = \frac{2R(4R+r)^2(R-2r)}{(5R-4r)(17R-16r)} \geq 0,$$

thus by the previous inequality (2.4) we deduce that inequality (4.26) holds. Hence, inequality $k_5 \geq k_4$ is proved.

(V) From identities (4.10) and (4.11), we obtain

$$k_6 - k_5 = \frac{(6R^2+4Rr+r^2-s^2)(s^2-2R^2-8Rr-3r^2)}{2(s^2-4Rr-r^2)(s^2-3R^2-4Rr-r^2)}. \tag{4.27}$$

Thus, by Gerretsen's inequality (3.10), Walker's inequality (1.2), and Ciamberlini's inequality (2.33), we see that $k_6 \geq k_5$ holds for the acute triangle ABC.

(VI) The seventh inequality of (4.2) is actually valid for any triangle ABC and can be easily obtained by using the Cauchy-Schwarz inequality.

(VII) By Lemma 4.1, we have

$$k_8 - k_7 = \frac{\left[s^2-(2R+r)^2\right]\left[3(4R^2+4Rr+3r^2-s^2)+4(R+r)(R-2r)\right]}{16s^2R^2}. \tag{4.28}$$

Then by (2.33), (3.10), and Euler's inequality $R \geq 2r$, one sees that $k_8 \geq k_7$ holds for the acute triangle ABC.

(VIII) By Lemma 4.1, we get

$$k_9 - k_8 = \frac{(s+2R+r)(s-2R-r)(s^2-2R^2-8Rr-3r^2)}{8(s^2-3R^2-4Rr-r^2)}. \tag{4.29}$$

Thus by (1.2) and (2.33) we see that $k_9 \geq k_8$ holds.

(IX) It follows from identities (4.14) and (4.15) that

$$k_{10} - k_9 = \frac{X_5}{16s^2R^2(s^2-3R^2-4Rr-r^2)}, \tag{4.30}$$

where

$$\begin{aligned} X_5 =& 3s^6 - (35R^2+32Rr+7r^2)s^4 + (4R+r)(2R+r)(18R^2+18Rr+5r^2)s^2 \\ & - (R+r)(3R+r)(4R+r)^2(2R+r)^2. \end{aligned}$$

We can rewrite X_5 as follows:

$$X_5 = 11R^2T_0 + 3s_0s_1^2 + 2X_6, \tag{4.31}$$

where s_0, s_1, T_0 are the same as above and

$$\begin{aligned} X_6 =& 7r(2R+r)s^4 + (20R^4-152R^3r-158R^2r^2-108Rr^3-20r^4)s^2 \\ & - 72R^6 + 296R^5r + 616R^4r^2 + 688R^3r^3 + 410R^2r^4 + 118Rr^5 + 13r^6. \end{aligned}$$

Since $s_0 \geq 0$ and $T_0 \geq 0$, to prove $X_5 \geq 0$ it remains to show that $X_6 \geq 0$. Similar to the proof of inequality $X_3 \geq 0$, we next consider two cases to finish the proof of $X_6 \geq 0$.

We let $s_0 = s^2 - (2R + r)^2$ and then it is easy to check the following identity:

$$\begin{aligned} X_6 =& 7r(2R + r)s_0^2 + 4(2R^2 + 2Rr - 5r^2)(R^2 - 2Rr - r^2)R^2 \\ &+ s_0 \left[10(R^2 - 2Rr - r^2)(2R^2 + 3r^2) + 36Rr^3 + 24r^4\right]. \end{aligned} \tag{4.32}$$

In view of Euler's inequality $R \geq 2r$ and Ciamberlini's inequality (2.34), we see that if $R^2 - 2Rr - r^2 > 0$ then $X_6 \geq 0$ holds. Thus, it remains to prove that if $R^2 - 2Rr - r^2 \leq 0$ then $X_6 \geq 0$.

We let $s_1 = s^2 - 2R^2 - 8Rr - 3r^2$, then by Walker's inequality we have $s_1 \geq 0$. Now, it is easy to check that

$$\begin{aligned} X_6 =& 7r(2R + r)s_1^2 + X_7 s_1 - 4(R - 2r)(R^2 - 2Rr - r^2) \\ &\cdot \left[4(2R - r)(R^2 - 2Rr - r^2) - 9Rr^2 - 6r^3\right], \end{aligned} \tag{4.33}$$

where $X_7 = 20R^4 - 96R^3 r + 94R^2 r^2 + 88Rr^3 + 22r^4$. We first show that $X_7 > 0$ holds for any triangle ABC. In fact, it easy to check that

$$X_7 = 2(e^2 - r^2)^2 + 64e^3 r + 2e^2 r^2 + 2(9e^4 - 24er^3 + 62r^4), \tag{4.34}$$

where $e = R - 2r$. When $e \geq 2r$, it is obvious that $9e^4 - 24er^3 > 0$. When $e < 2r$, it is obvious that $-24er^3 + 62r^4 > 0$. We therefore conclude that inequality $9e^4 - 24er^3 + 62r^4 > 0$ for any triangle ABC. Hence, inequality $X_7 > 0$ is proved.

Note that $s_1 \geq 0, 2R - r > 0$ and $X_7 > 0$, from (4.33) one sees that if $R^2 - 2Rr - r^2 \leq 0$ then $X_6 \geq 0$. So we have proved that $X_6 \geq 0$ holds for all acute triangles. Hence, $X_5 \geq 0$ and $X_{10} \geq X_9$ are proved.

(X) Finally, we prove inequality $k_{10} \leq 1$. It is easy to obtain the following identity;

$$1 - k_{10} = \frac{\left[s^2 - (2R + r)^2\right]\left[3(4R^2 + 4Rr + 3r^2 - s^2) + 4(R - 2r)(R + r)\right]}{16s^2R^2}. \tag{4.35}$$

Then by Euler's inequality$R \geq 2r$, Ciamberlini's inequality (2.33), and Gerretsen's inequality (3.10) we see that $k_{10} \leq 1$ holds.

From the above deduction of inequality chain (4.2), it is easy to determine the equality condition for every inequality in (4.2). The proof of Theorem 4.1 is completed. □

Remark 4.1. The first inequality of (4.2) can also be proved by using the following simple method: Letting $x = \cos^2 A, y = \cos^2 B, z = \cos^2 C$, then the first inequality of (4.2) becomes

$$\frac{16\sum yz}{\sum x} - \frac{16\sum x(1-x)}{\sum(1-x)} \leq 0,$$

which is equivalent to

$$\frac{\left(1 - \sum x\right)\sum(y - z)^2}{\left(3 - \sum x\right)\sum x} \geq 0. \tag{4.36}$$

Recall that in the acute triangle ABC we have the following known inequality (see [4]):

$$\sum \sin^2 A > 2, \tag{4.37}$$

and then it follows that $\sum x < 1$. Again, note that $x, y, z > 0$, we see that inequality (4.36) holds and the first inequality of (4.2) is proved.

Remark 4.2. Inequality (4.5) is equivalent to

$$\sum \frac{\cos^2 \dfrac{B-C}{2}}{\cos^2 \dfrac{A}{2}} \cos 2A \geq -2. \tag{4.38}$$

Motivated by this inequality, the author finds that the following inequality

$$\sum \frac{\cos 2A}{\cos^2 \dfrac{A}{2}} \leq -2 \tag{4.39}$$

holds for the acute triangle ABC. In fact, it is easy to prove the following identity:

$$\sum \frac{\cos 2A}{\cos^2 \dfrac{A}{2}} + 2 = \frac{R(4R+r)^2 - (5R-4r)s^2}{Rs^2}, \tag{4.40}$$

which shows that inequality (4.39) is equivalent to the previous inequality (2.4).

Remark 4.3. From the previous identity (2.34) and (4.40), we conclude that Sondat's fundamental inequality (2.5) is equivalent to the following trigonometric inequality:

$$\sum \left(\frac{\cos B + \cos C}{\sin A}\right)^2 - \sum \frac{\cos 2A}{\cos^2 \dfrac{A}{2}} \leq 6, \tag{4.41}$$

or equivalently

$$\sum \frac{\cos^2 \dfrac{B-C}{2} - \cos 2A}{\cos^2 \dfrac{A}{2}} \leq 6. \tag{4.42}$$

In fact, if real numbers A, B, C satisfy $A + B + C = \pi$, then we can prove the following identity (we omit the proof here):

$$6 - \sum \frac{\cos^2 \dfrac{B-C}{2} - \cos 2A}{\cos^2 \dfrac{A}{2}} = \frac{\prod (\sin B - \sin C)^2}{\prod \sin^2 A}, \tag{4.43}$$

Therefore, both inequalities (4.41) and (4.42) are actually valid for real numbers A, B, C satisfying $A + B + C = \pi$. Therefore, replacing A, B, C by $\pi - 2A, \pi - 2B, \pi - 2C$ in (4.42) respectively, we deduce that that the following inequality

$$\sum \frac{\cos^2(B-C) - \cos 4A}{\sin^2 A} \leq 6 \tag{4.44}$$

holds for any triangle ABC. In fact, inequality (4.44) is also equivalent to Sondat's fundamental triangle inequality (2.5).

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