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A NEW VERSION OF INTEGRAL INEQUALITIES FOR A LINEAR FUNCTION OF BOUNDED VARIATION

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Abstract. In this paper, some new integral inequalities are developed by using a 7-step linear kernel for the function of bounded variation. Applications of quadrature rule and probability density function are also provided. We also constructed some generalized trapezoid and midpoint inequalities for the linear functions of bounded variations.

1. Introduction

In 1938, the integral inequalities are established by Ukrainian Mathematician A. M. Ostrowski [1] named as Ostrowski's inequality. Dragomir gave the first generalization of integral inequalities for the function of bounded variation [12–17]. Several authors have worked on the Ostrowski's type inequalities for the function of bounded variation (see for example [9–11, 18–20]). Recently some researchers [2, 8] also worked on function of bounded variation. In this paper, we extend the work of Budak *et al.* [7, 8] by using 7-step kernel.

2. Main Results

With the help of 7-step kernel, we develop a new version of integral inequalities for a function of bounded variation.

Key words and phrases. Ostrowski's inequality, 7-step kernel, Bounded variation.

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Theorem 2.1. *Consider $\mathcal{L} : [\acute{a}, \acute{e}] \to \mathbb{R}$ be such that $\mathcal{L}$ is a continuous function of bounded variation on $[\acute{a}, \acute{e}]$, then we obtain*

$$\begin{aligned}
&\left|\frac{1}{8}\left[\mathcal{L}\left(\frac{3\acute{a}+\acute{u}}{4}\right)+\mathcal{L}\left(\frac{\acute{a}+\acute{u}}{2}\right)+2\mathcal{L}(\acute{u})+2\mathcal{L}(\acute{a}+\acute{e}-\acute{u})\right.\right.\\
&\left.\left.+\mathcal{L}\left(\frac{\acute{a}+2\acute{e}-\acute{u}}{2}\right)+\mathcal{L}\left(\frac{\acute{a}+4\acute{e}-\acute{u}}{4}\right)\right]-\frac{1}{\acute{e}-\acute{a}}\int_{\acute{a}}^{\acute{e}}\mathcal{L}(\check{z})d\check{z}\right|\\
&\leq \frac{1}{\acute{e}-\acute{a}}G(\acute{u})\bigvee_{\acute{a}}^{\acute{e}}(\mathcal{L}), \qquad (2.1)
\end{aligned}$$

for all $\acute{u} \in \left[\acute{a}, \frac{\acute{a}+\acute{e}}{2}\right]$, where

$$G(\acute{u}) = \max\left\{\left(\frac{\acute{u}-\acute{a}}{4}\right), \frac{3}{4}\left(\acute{u}-\frac{2\acute{a}+\acute{e}}{3}\right)+\frac{1}{4}|\acute{u}-\acute{a}|, \left(\acute{u}-\frac{\acute{a}+\acute{e}}{2}\right)\right\}$$

and $\bigvee_{\acute{a}}^{\acute{e}}(\mathcal{L})$ indicates the total variation of $\mathcal{L}$ on $[\acute{a}, \acute{e}]$.

Proof. To prove our required result, first we introduce a mapping

$$P(\acute{u},\check{z}) = \begin{cases}
\check{z}-\acute{a} \quad , & \check{z} \in \left[\acute{a}, \frac{3\acute{a}+\acute{u}}{4}\right] \\
\check{z}-\frac{7\acute{a}+\acute{e}}{8} \quad , & \check{z} \in \left(\frac{3\acute{a}+\acute{u}}{4}, \frac{\acute{a}+\acute{u}}{2}\right] \\
\check{z}-\frac{3\acute{a}+\acute{e}}{4} \quad , & \check{z} \in \left(\frac{\acute{a}+\acute{u}}{2}, \acute{u}\right] \\
\check{z}-\frac{\acute{a}+\acute{e}}{2} \quad , & \check{z} \in (\acute{u}, \acute{a}+\acute{e}-\acute{u}] \\
\check{z}-\frac{\acute{a}+3\acute{e}}{4} \quad , & \check{z} \in \left(\acute{a}+\acute{e}-\acute{u}, \frac{\acute{a}+2\acute{e}-\acute{u}}{2}\right] \\
\check{z}-\frac{\acute{a}+7\acute{e}}{8} \quad , & \check{z} \in \left(\frac{\acute{a}+2\acute{e}-\acute{u}}{2}, \frac{\acute{a}+4\acute{e}-\acute{u}}{4}\right] \\
\check{z}-\acute{e} \quad , & \check{z} \in \left(\frac{\acute{a}+4\acute{e}-\acute{u}}{4}, \acute{e}\right]
\end{cases} \qquad (2.2)$$

for all $\acute{u} \in \left[\acute{a}, \frac{\acute{a}+\acute{e}}{2}\right]$.
Integrating by parts, by using (2.2) we get the following identity

$$\begin{aligned}
&\frac{1}{\acute{e}-\acute{a}}\int_{\acute{a}}^{\acute{e}}\mathcal{L}(\check{z})d\check{z}\\
&=\frac{1}{8}\left[\mathcal{L}\left(\frac{3\acute{a}+\acute{u}}{4}\right)+\mathcal{L}\left(\frac{\acute{a}+\acute{u}}{2}\right)+2\mathcal{L}(\acute{u})+2\mathcal{L}(\acute{a}+\acute{e}-\acute{u})\right.\\
&\left.+\mathcal{L}\left(\frac{\acute{a}+2\acute{e}-\acute{u}}{2}\right)+\mathcal{L}\left(\frac{\acute{a}+4\acute{e}-\acute{u}}{4}\right)\right]-\int_{\acute{a}}^{\acute{e}}P(\acute{u},\check{z})d\mathcal{L}(\check{z}).
\end{aligned}$$

Again by using (2.2), we have

$$\begin{aligned}
&\left|\int_{\acute{a}}^{\acute{e}} P(\acute{u},\check{z})d\mathcal{L}(\check{z})\right| \\
&\leq \left|\int_{\acute{a}}^{\frac{3\acute{a}+\acute{u}}{4}} (\check{z}-\acute{a})\, d\mathcal{L}(\check{z})\right| + \left|\int_{\frac{3\acute{a}+\acute{u}}{4}}^{\frac{\acute{a}+\acute{u}}{2}} \left(\check{z}-\frac{7\acute{a}+\acute{e}}{8}\right) d\mathcal{L}(\check{z})\right| \\
&+ \left|\int_{\frac{\acute{a}+\acute{u}}{2}}^{\acute{u}} \left(\check{z}-\frac{3\acute{a}+\acute{e}}{4}\right) d\mathcal{L}(\check{z})\right| + \left|\int_{\acute{u}}^{\acute{a}+\acute{e}-\acute{u}} \left(\check{z}-\frac{\acute{a}+\acute{e}}{2}\right) d\mathcal{L}(\check{z})\right| \\
&+ \left|\int_{\acute{a}+\acute{e}-\acute{u}}^{\frac{\acute{a}+2\acute{e}-\acute{u}}{2}} \left(\check{z}-\frac{\acute{a}+3\acute{e}}{4}\right) d\mathcal{L}(\check{z})\right| + \left|\int_{\frac{\acute{a}+2\acute{e}-\acute{u}}{2}}^{\frac{\acute{a}+4\acute{e}-\acute{u}}{4}} \left(\check{z}-\frac{\acute{a}+7\acute{e}}{8}\right) d\mathcal{L}(\check{z})\right| \\
&+ \left|\int_{\frac{\acute{a}+4\acute{e}-\acute{u}}{4}}^{\acute{e}} (\check{z}-\acute{e})\, d\mathcal{L}(\check{z})\right|. \qquad (2.3)
\end{aligned}$$

It is familiar that if $h, g : [\acute{a}, \acute{e}] \to \mathbb{R}$ are such that h is continuous on $[\acute{a}, \acute{e}]$ and g is of bounded variation on $[\acute{a}, \acute{e}]$, then $\int_{\acute{a}}^{\acute{e}} h(\check{z})dg(\check{z})$ exists and

$$\left|\int_{\acute{a}}^{\acute{e}} h(\check{z})dg(\check{z})\right| \leq \sup_{\check{z}\in[\acute{a},\acute{e}]} |h(\check{z})| \bigvee_{\acute{a}}^{\acute{e}}(g). \qquad (2.4)$$

By using (2.4) for each term in (2.3), we get

$$\begin{aligned}
&\left|\frac{1}{8}\left[\mathcal{L}\left(\frac{3\acute{a}+\acute{u}}{4}\right) + \mathcal{L}\left(\frac{\acute{a}+\acute{u}}{2}\right) + 2\mathcal{L}(\acute{u}) + 2\mathcal{L}(\acute{a}+\acute{e}-\acute{u})\right.\right. \\
&\left.\left. +\mathcal{L}\left(\frac{\acute{a}+2\acute{e}-\acute{u}}{2}\right) + \mathcal{L}\left(\frac{\acute{a}+4\acute{e}-\acute{u}}{4}\right)\right] - \frac{1}{\acute{e}-\acute{a}}\int_{\acute{a}}^{\acute{e}} \mathcal{L}(\check{z})d\check{z}\right| \\
&\leq \frac{1}{\acute{e}-\acute{a}} G(\acute{u}) \bigvee_{\acute{a}}^{\acute{e}}(\mathcal{L})
\end{aligned}$$

for all $\acute{u} \in \left[\acute{a}, \frac{\acute{a}+\acute{e}}{2}\right]$. Hence proved our result (2.1). □

Corollary 2.1. *By putting $\acute{u} = \acute{a}$ in Theorem 2.1, we get the result of Dragomir [14].*

Corollary 2.2. *By putting $\acute{u} = \frac{\acute{a}+\acute{e}}{2}$ in Theorem 2.1, we get*

$$\begin{aligned}
&\left|\frac{1}{8}\left[\mathcal{L}\left(\frac{7\acute{a}+\acute{e}}{8}\right) + \mathcal{L}\left(\frac{3\acute{a}+\acute{e}}{4}\right) + 4\mathcal{L}\left(\frac{\acute{a}+\acute{e}}{2}\right)\right.\right. \\
&\left.\left. +\mathcal{L}\left(\frac{\acute{a}+3\acute{e}}{4}\right) + \mathcal{L}\left(\frac{\acute{a}+7\acute{e}}{8}\right)\right] - \frac{1}{\acute{e}-\acute{a}}\int_{\acute{a}}^{\acute{e}} \mathcal{L}(\check{z})d\check{z}\right| \\
&\leq \frac{1}{4}\bigvee_{\acute{a}}^{\acute{e}}(\mathcal{L}).
\end{aligned}$$

Corollary 2.3. *By putting* $ú = \frac{3á+é}{4}$ *in Theorem* 2.1*, we get*

$$\left|\frac{1}{8}\mathcal{L}\left(\frac{15á+é}{16}\right)+\frac{1}{8}\mathcal{L}\left(\frac{7á+é}{8}\right)+\frac{1}{4}\mathcal{L}\left(\frac{3á+é}{4}\right)+\frac{1}{4}\mathcal{L}\left(\frac{7á+5é}{4}\right)\right.$$
$$\left.+\frac{1}{8}\mathcal{L}\left(\frac{á+7é}{8}\right)+\frac{1}{8}\mathcal{L}\left(\frac{á+15é}{16}\right)-\frac{1}{é-á}\int_{á}^{é}\mathcal{L}(ž)dž\right|$$
$$\leq \frac{1}{8}\bigvee_{á}^{é}(\mathcal{L}).$$

Corollary 2.4. *By putting* $ú = \frac{7á+é}{8}$ *in Theorem* 2.1*, we get*

$$\left|\frac{1}{8}\mathcal{L}\left(\frac{3á+ú}{4}\right)+\frac{1}{8}\mathcal{L}\left(\frac{á+ú}{2}\right)+\frac{1}{4}\mathcal{L}(ú)+\frac{1}{4}\mathcal{L}(á+é-ú)\right.$$
$$\left.+\frac{1}{8}\mathcal{L}\left(\frac{á+2é-ú}{2}\right)+\frac{1}{8}\mathcal{L}\left(\frac{á+4é-ú}{4}\right)-\frac{1}{é-á}\int_{á}^{é}\mathcal{L}(ž)dž\right|$$
$$\leq \frac{1}{32}\bigvee_{á}^{é}(\mathcal{L}).$$

Corollary 2.5. *Let* $\mathcal{L} \in C^1[á, é]$*, then we have the inequality*

$$\left|\frac{1}{8}\mathcal{L}\left(\frac{3á+ú}{4}\right)+\frac{1}{8}\mathcal{L}\left(\frac{á+ú}{2}\right)+\frac{1}{4}\mathcal{L}(ú)+\frac{1}{4}\mathcal{L}(á+é-ú)\right.$$
$$\left.+\frac{1}{8}\mathcal{L}\left(\frac{á+2é-ú}{2}\right)+\frac{1}{8}\mathcal{L}\left(\frac{á+4é-ú}{4}\right)-\frac{1}{é-á}\int_{á}^{é}\mathcal{L}(ž)dž\right|$$
$$\leq \frac{1}{é-á}\max\left\{\frac{ú-á}{4}, \frac{3}{4}\left(ú-\frac{2á+é}{3}\right)+\frac{1}{4}\left|ú-á\right|, ú-\frac{á+é}{2}\right\}\left\|\mathcal{L}'\right\|_1,$$

for all $ú \in \left[á, \frac{á+é}{2}\right]$, where $\|.\|_1$ is the L_1 norm.

3. An Application to Quadrature Rule

Suppose that the random division $I_n : á = ú_0 < ú_1 < ú_2 < .... < ú_n = é$ with $h_i = ú_{i+1} - ú_i$ and $v(h) = \max\{h_i| \ i = 0, 1, 2......n-1\}$. Then the following result holds:

Theorem 3.1. *Suppose* $\mathcal{L} : [á, é] \to \mathbb{R}$ *be as* $\mathcal{L}'$ *is continuous function of bounded variation on* $[á, é]$ *and* $\xi_i \in \left[ú_i\ , \frac{ú_i+ú_{i+1}}{2}\right]$ $(i = 0, 1, ...n-1)$*, then*

$$\int_{á}^{é}\mathcal{L}(ž)dž$$
$$= \frac{1}{8}\sum_{i=0}^{n-1}\left[\mathcal{L}\left(\frac{3ú_i+\xi_i}{4}\right)+\mathcal{L}\left(\frac{ú_i+\xi_i}{2}\right)+2\mathcal{L}(\xi_i)+2\mathcal{L}(ú_i+ú_{i+1}-\xi_i)\right.$$
$$\left.+\mathcal{L}\left(\frac{ú_i+2ú_{i+1}-\xi_i}{2}\right)+\mathcal{L}\left(\frac{ú_i+4ú_{i+1}-\xi_i}{4}\right)\right]h_i+\acute{R}\left(I_n, \mathcal{L}, \xi\right)$$

and the remainder $\acute{R}\left(I_n, \mathcal{L}, \xi\right)$ *satisfies*

$$\begin{aligned}
&\left|\acute{R}\left(I_n, \mathcal{L}, \xi\right)\right| \\
&\leq \max_{i=0,\ldots,n-1}\left\{\max\left\{\left(\frac{\xi_i-\acute{u}_i}{4}\right), \frac{3}{4}\left(\xi_i-\frac{2\acute{u}_i+\acute{u}_{i+1}}{3}\right)\right.\right. \\
&\left.\left.+\frac{1}{4}\left|\xi_i-\acute{u}_i\right|, \left(\xi_i-\frac{2\acute{u}_i+\acute{u}_{i+1}}{2}\right)\right\}\right\}\bigvee_{\acute{a}}^{\acute{e}}(\mathcal{L}). \qquad (3.1)
\end{aligned}$$

Proof. Applying Theorem 2.1 for the interval $(\acute{u}_i, \acute{u}_{i+1})$, we get

$$\begin{aligned}
&\left|\int_{\acute{u}_i}^{\acute{u}_{i+1}} \mathcal{L}(\check{z})d\check{z}-\frac{\acute{u}_i\ -\ \acute{u}_{i+1}}{8}\left[\mathcal{L}\left(\frac{3\acute{u}_i+\xi_i}{4}\right)+\mathcal{L}\left(\frac{\acute{u}_i+\xi_i}{2}\right)+2\mathcal{L}(\xi_i)\right.\right. \\
&\left.\left.+2\mathcal{L}(\acute{u}_i+\acute{u}_{i+1}-\xi_i)+\mathcal{L}\left(\frac{\acute{u}_i+2\acute{u}_{i+1}-\xi_i}{2}\right)+\mathcal{L}\left(\frac{\acute{u}_i+4\acute{u}_{i+1}-\xi_i}{4}\right)\right]\right| \\
&\leq \max\left\{\left(\frac{\xi_i-\acute{u}_i}{4}\right), \frac{3}{4}\left(\xi_i-\frac{2\acute{u}_i+\acute{u}_{i+1}}{3}\right)+\frac{1}{4}\left|\xi_i-\acute{u}_i\right|,\right. \\
&\left.\left(\xi_i-\frac{2\acute{u}_i+\acute{u}_{i+1}}{2}\right)\right\}\bigvee_{\acute{a}}^{\acute{e}}(\mathcal{L}). \qquad (3.2)
\end{aligned}$$

Now

$$\begin{aligned}
&\left|\acute{R}\left(I_n, \mathcal{L}, \xi\right)\right| \\
&\leq \sum_{i=0}^{n-1}\max\left\{\left(\frac{\xi_i-\acute{u}_i}{4}\right), \frac{3}{4}\left(\xi_i-\frac{2\acute{u}_i+\acute{u}_{i+1}}{3}\right)\right. \\
&\left.+\frac{1}{4}\left|\xi_i-\acute{u}_i\right|, \left(\xi_i-\frac{2\acute{u}_i+\acute{u}_{i+1}}{2}\right)\right\}\bigvee_{\acute{u}_i}^{\acute{u}_{i+1}}(\mathcal{L}).
\end{aligned}$$

$$\begin{aligned}
&\leq \max_{i=0,\ldots,n-1}\left\{\max\left\{\left(\frac{\xi_i-\acute{u}_i}{4}\right), \frac{3}{4}\left(\xi_i-\frac{2\acute{u}_i+\acute{u}_{i+1}}{3}\right)\right.\right. \\
&\left.\left.+\frac{1}{4}\left|\xi_i-\acute{u}_i\right|+\left(\xi_i-\frac{2\acute{u}_i+\acute{u}_{i+1}}{2}\right)\right\}\right\}\sum_{i=0}^{n-1}\bigvee_{\acute{u}_i}^{\acute{u}_{i+1}}(\mathcal{L}).
\end{aligned}$$

$$\begin{aligned}
&\leq \max_{i=0,\ldots,n-1}\left\{\max\left\{\left(\frac{\xi_i-\acute{u}_i}{4}\right), \frac{3}{4}\left(\xi_i-\frac{2\acute{u}_i+\acute{u}_{i+1}}{3}\right)\right.\right. \\
&\left.\left.+\frac{1}{4}\left|\xi_i-\acute{u}_i\right|, \left(\xi_i-\frac{2\acute{u}_i+\acute{u}_{i+1}}{2}\right)\right\}\right\}\bigvee_{\acute{a}}^{\acute{e}}(\mathcal{L}).
\end{aligned}$$

Hence proved. □

Corollary 3.1. *By using Theorem* 3.1 *for* $\xi_i = \acute{u}_i$*, we have*

$$\int_{\acute{a}}^{\acute{e}} \mathcal{L}(\check{z})d\check{z} = \frac{1}{2}\sum_{i=0}^{n-1}\left[\mathcal{L}\left(\acute{u}_i\right)+\mathcal{L}(\acute{u}_{i+1})\right]h_i + R\left(I_n, \mathcal{L}\right),$$

where

$$|R(I_n,\mathcal{L})| \leq \frac{v(h)}{2}\bigvee_{\acute{a}}^{\acute{e}}(\mathcal{L}).$$

4. An Application to Probability Density Function

Let X be an arbitrary variable with finite interval $[\acute{a},\acute{e}]$, with the probability density function $\mathcal{L}:[\acute{a},\acute{e}]\longrightarrow[0,1]$ and with the cumulative distribution function

$$F(\acute{u}) = \Pr(X\leq\acute{u}) = \int_{\acute{a}}^{\acute{u}}\mathcal{L}(\check{z})\,d\check{z},$$

then the following result holds:

Theorem 4.1. *With the assumption of Theorem* 2.1*, we have*

$$\begin{aligned}&\left|\frac{1}{8}\left[F\left(\frac{3\acute{a}+\acute{u}}{4}\right)+F\left(\frac{\acute{a}+\acute{u}}{2}\right)+2F(\acute{u})+2F(\acute{a}+\acute{e}-\acute{u})\right.\right.\\&\left.+F\left(\frac{\acute{a}+2\acute{e}-\acute{u}}{2}\right)+F\left(\frac{\acute{a}+4\acute{e}-\acute{u}}{4}\right)\right]-\frac{\acute{e}-\ddot{E}(X)}{\acute{e}-\acute{a}}\\&\leq\frac{1}{\acute{e}-\acute{a}}G(\acute{u})\bigvee_{\acute{a}}^{\acute{e}}(\mathcal{L})\end{aligned}\tag{4.1}$$

for all $\acute{u}\in\left[\acute{a},\frac{\acute{a}+\acute{e}}{2}\right]$*, where* $\ddot{E}(X)$ *is the assumption of* X*.*

Proof. If we apply Theorem 2.1 for which is monotonic non-decreasing, we get

$$\begin{aligned}&\left|\frac{1}{8}\left[F\left(\frac{3\acute{a}+\acute{u}}{4}\right)+F\left(\frac{\acute{a}+\acute{u}}{2}\right)+2F(\acute{u})+2F(\acute{a}+\acute{e}-\acute{u})\right.\right.\\&\left.+F\left(\frac{\acute{a}+2\acute{e}-\acute{u}}{2}\right)+F\left(\frac{\acute{a}+4\acute{e}-\acute{u}}{4}\right)\right]-\frac{1}{\acute{e}-\acute{a}}\int_{\acute{a}}^{\acute{e}}F(\check{z})d\check{z}\\&\leq\frac{1}{\acute{e}-\acute{a}}G(\acute{u})\bigvee_{\acute{a}}^{\acute{e}}(\mathcal{L}).\end{aligned}$$

Since

$$\ddot{E}(X)=\int_{\acute{a}}^{\acute{e}}\check{z}F(\check{z})d\check{z}=\acute{e}-\int_{\acute{a}}^{\acute{e}}F(\check{z})d\check{z}.$$

Thus the proof is completed. □

Corollary 4.1. *By putting* $\acute{u}=\frac{\acute{a}+\acute{e}}{2}$ *in Theorem* 4.1*, we have*

$$\begin{aligned}&\left|\frac{1}{8}\left[F\left(\frac{7\acute{a}+\acute{e}}{8}\right)+F\left(\frac{3\acute{a}+\acute{e}}{4}\right)+2F\left(\frac{\acute{a}+\acute{e}}{2}\right)+2F\left(\frac{\acute{a}+\acute{e}}{2}\right)\right.\right.\\&\left.\left.+F\left(\frac{\acute{a}+3\acute{e}}{4}\right)+F\left(\frac{\acute{a}+7\acute{e}}{8}\right)\right]-\frac{\acute{e}-\ddot{E}(X)}{\acute{e}-\acute{a}}\right|\\&\leq\frac{1}{4}.\end{aligned}$$

Corollary 4.2. *By putting $\acute{u}=\frac{3\acute{a}+\acute{e}}{4}$ in Theorem 4.1, we have*

$$\begin{aligned}&\left|\frac{1}{8}\left[F\left(\frac{15\acute{a}+\acute{e}}{16}\right)+F\left(\frac{7\acute{a}+\acute{e}}{8}\right)+2F\left(\frac{3\acute{a}+\acute{e}}{4}\right)+2F\left(\frac{\acute{a}+3\acute{e}}{4}\right)\right.\right.\\&\left.\left.+F\left(\frac{\acute{a}+7\acute{e}}{8}\right)+F\left(\frac{\acute{a}+15\acute{e}}{16}\right)\right]-\frac{\acute{e}-\ddot{E}\left(X\right)}{\acute{e}-\acute{a}}\right|\\&\leq\frac{1}{8}.\end{aligned}$$

5. Some New Trapezoid And Midpoint Inequalities

In this section, we introduce linear integral inequalities by using $\acute{a}\leq\check{c}<\partial\leq\acute{e}$ in $\mathbb{R}$ with $\acute{a}+\acute{e}=\check{c}+\partial$.

Theorem 5.1. *Let $\mathcal{L}:[\acute{a},\acute{e}]\to\mathbb{R}$ be a function of bounded variation on $[\acute{a},\acute{e}]$. Then we obtain*

$$\begin{aligned}&\left|\int_{\acute{a}}^{\acute{e}}\mathcal{L}\left(\acute{u}\right)d\acute{u}-\left\{\left(\frac{\check{c}-\acute{a}}{2}\right)\left[\mathcal{L}\left(\frac{3\acute{a}+\check{c}}{4}\right)+\mathcal{L}\left(\frac{\acute{a}+\check{c}}{2}\right)\right.\right.\right.\\&\left.\left.\left.+\mathcal{L}\left(\frac{\acute{e}+\partial}{2}\right)+\mathcal{L}\left(\frac{3\acute{e}+\partial}{4}\right)\right]+\left(\frac{\acute{a}+\acute{e}}{2}-\check{c}\right)\left[\mathcal{L}\left(\check{c}\right)+\mathcal{L}\left(\partial\right)\right]\right\}\right|\\&\leq\max\left\{\left(\frac{\check{c}-\acute{a}}{2}\right),\left(\frac{\acute{a}+\acute{e}}{2}-\check{c}\right),\left(\partial-\frac{\acute{a}+\acute{e}}{2}\right),\left(\frac{\acute{e}-\partial}{2}\right)\right\}\bigvee_{\acute{a}}^{\acute{e}}(\mathcal{L}),\end{aligned}\tag{5.1}$$

for all $\acute{u}\in\left[\acute{a},\frac{\acute{a}+\acute{e}}{2}\right]$.

Proof. Consider the kernel $P_2\left(\acute{u}\right)$ as follows

$$P_2\left(\acute{u}\right)=\begin{cases}\acute{a}-\acute{u} & , \quad \acute{u}\in\left[\acute{a},\frac{3\acute{a}+\check{c}}{4}\right]\\ \frac{\acute{a}+\check{c}}{2}-\acute{u} & , \quad \acute{u}\in\left(\frac{3\acute{a}+\check{c}}{4},\frac{\acute{a}+\check{c}}{2}\right]\\ \check{c}-\acute{u} & , \quad \acute{u}\in\left(\frac{\acute{a}+\check{c}}{2},\check{c}\right]\\ \frac{\acute{a}+\acute{e}}{2}-\acute{u} & , \quad \acute{u}\in\left(\check{c},\partial\right]\\ \partial-\acute{u} & , \quad \acute{u}\in\left(\partial,\frac{\acute{e}+\partial}{2}\right]\\ \frac{\acute{e}+\partial}{2}-\acute{u} & , \quad \acute{u}\in\left(\frac{\acute{e}+\partial}{2},\frac{3\acute{e}+\partial}{4}\right]\\ \acute{e}-\acute{u} & , \quad \acute{u}\in\left(\frac{3\acute{e}+\partial}{4},\acute{e}\right]\end{cases}\tag{5.2}$$

Integration by parts gives us

$$\begin{aligned}&\int_{\acute{a}}^{\acute{e}} P_2(\acute{u})\, d\mathcal{L}(\acute{u}) \\ &= \int_{\acute{a}}^{\acute{e}} \mathcal{L}(\acute{u})\, d\acute{u} - \left\{ \left(\frac{\check{c}-\acute{a}}{2}\right) \left[\mathcal{L}\left(\frac{3\acute{a}+\check{c}}{4}\right) + \mathcal{L}\left(\frac{\acute{a}+\check{c}}{2}\right)\right.\right. \\ &\left.\left. + \mathcal{L}\left(\frac{\acute{e}+\partial}{2}\right) + \mathcal{L}\left(\frac{3\acute{e}+\partial}{4}\right)\right] + \left(\frac{\acute{a}+\acute{e}}{2} - \check{c}\right)\left[\mathcal{L}(\check{c}) + \mathcal{L}(\partial)\right]\right\}. \end{aligned} \tag{5.3}$$

From (5.2), we have

$$\begin{aligned}&\left|\int_{\acute{a}}^{\acute{e}} P_2(\acute{u})\, d\mathcal{L}(\acute{u})\right| \\ &\le \left|\int_{\acute{a}}^{\frac{3\acute{a}+\check{c}}{4}} (\acute{a}-\acute{u})\, d\mathcal{L}(\acute{u})\right| + \left|\int_{\frac{3\acute{a}+\check{c}}{4}}^{\frac{\acute{a}+\check{c}}{2}} \left(\frac{\acute{a}+\check{c}}{2} - \acute{u}\right) d\mathcal{L}(\acute{u})\right| \\ &+ \left|\int_{\frac{\acute{a}+\check{c}}{2}}^{\check{c}} (\check{c}-\acute{u})\, d\mathcal{L}(\acute{u})\right| + \left|\int_{\check{c}}^{\partial} \left(\frac{\acute{a}+\acute{e}}{2} - \acute{u}\right) d\mathcal{L}(\acute{u})\right| \\ &+ \left|\int_{\partial}^{\frac{\acute{e}+\partial}{2}} (\partial-\acute{u})\, d\mathcal{L}(\acute{u})\right| + \left|\int_{\frac{\acute{e}+\partial}{2}}^{\frac{3\acute{e}+\partial}{4}} \left(\frac{\acute{e}+\partial}{2} - \acute{u}\right) d\mathcal{L}(\acute{u})\right| \\ &+ \left|\int_{\frac{3\acute{e}+\partial}{4}}^{\acute{e}} (\acute{e}-\acute{u})\, d\mathcal{L}(\acute{u})\right|. \end{aligned} \tag{5.4}$$

By using (2.4) for each term in (5.4), we get

$$\begin{aligned}&\left|\int_{\acute{a}}^{\acute{e}} \mathcal{L}(\acute{u})\, d\acute{u} - \left\{ \left(\frac{\check{c}-\acute{a}}{2}\right) \left[\mathcal{L}\left(\frac{3\acute{a}+\check{c}}{4}\right) + \mathcal{L}\left(\frac{\acute{a}+\check{c}}{2}\right)\right.\right.\right. \\ &\left.\left.\left. + \mathcal{L}\left(\frac{\acute{e}+\partial}{2}\right) + \mathcal{L}\left(\frac{3\acute{e}+\partial}{4}\right)\right] + \left(\frac{\acute{a}+\acute{e}}{2} - \check{c}\right)\left[\mathcal{L}(\check{c}) + \mathcal{L}(\partial)\right]\right\}\right| \\ &\le \max\left\{\left(\frac{\check{c}-\acute{a}}{2}\right), \left(\frac{\acute{a}+\acute{e}}{2} - \check{c}\right), \left(\partial - \frac{\acute{a}+\acute{e}}{2}\right), \left(\frac{\acute{e}-\partial}{2}\right)\right\} \bigvee_{\acute{a}}^{\acute{e}}(\mathcal{L}) \end{aligned}$$

for all $\acute{u} \in \left[\acute{a}, \frac{\acute{a}+\acute{e}}{2}\right]$. Hence proved. □

Corollary 5.1. *With the assumption of Theorem* 5.1 *with* $\check{c} = (1-\lambda)\acute{a} + \lambda\acute{e}$ *and* $\partial = (1-\lambda)\acute{a} + \lambda\acute{e}$ *with* $0 \le \lambda < \frac{1}{2}$, *then*

$$\begin{aligned}&\left|\int_{\acute{a}}^{\acute{e}} \mathcal{L}(\acute{u})\, d\acute{u} - \left\{ \left(\frac{\lambda(\acute{e}-\acute{a})}{2}\right) \left[\mathcal{L}\left(\acute{a} + \frac{\lambda(\acute{e}-\acute{a})}{4}\right)\right.\right.\right. \\ &\left. + \mathcal{L}\left(\acute{a} + \frac{\lambda(\acute{e}-\acute{a})}{2}\right) + \mathcal{L}\left(\acute{e} + \frac{\lambda(\acute{a}-\acute{e})}{2}\right) + \mathcal{L}\left(\acute{e} + \frac{\lambda(\acute{a}-\acute{e})}{4}\right)\right] \\ &\left.\left. + \left(\frac{\acute{e}-\acute{a}}{2} + \lambda(\acute{a}-\acute{e})\right) + 2\left[\mathcal{L}((1-\lambda)\acute{a} + \lambda\acute{e})\right]\right\}\right| \\ &\le (\acute{e}-\acute{a})\left[\frac{1}{4} + \left|\lambda - \frac{1}{4}\right|\right] \bigvee_{\acute{a}}^{\acute{e}}(\mathcal{L}). \end{aligned}$$

Corollary 5.2. *Choosing* $\lambda = \frac{1}{3}$ *in Corollary* 5.1*, we get*

$$\left|\int_{á}^{é} \mathcal{L}(ú)\,dú - \left\{\left(\frac{é-á}{6}\right)\left[\mathcal{L}\left(\frac{11á+é}{12}\right)+\mathcal{L}\left(\frac{5á+é}{6}\right)\right.\right.\right.$$
$$\left.\left.\left.+\mathcal{L}\left(\frac{á+5é}{6}\right)+\mathcal{L}\left(\frac{á+11é}{12}\right)\right]+\left(\frac{é-á}{3}\right)\mathcal{L}\left(\frac{2á+é}{3}\right)\right\}\right|$$
$$\leq \frac{1}{6}(é-á)\bigvee_{á}^{é}(\mathcal{L}).$$

Corollary 5.3. *Choosing* $\lambda = \frac{1}{4}$ *in Corollary* 5.1*, then*

$$\left|\int_{á}^{é} \mathcal{L}(ú)\,dú - \left\{\left(\frac{é-á}{8}\right)\left[\mathcal{L}\left(\frac{15á+é}{16}\right)+\mathcal{L}\left(\frac{7á+é}{8}\right)\right.\right.\right.$$
$$\left.\left.\left.+\mathcal{L}\left(\frac{á+7é}{8}\right)+\mathcal{L}\left(\frac{á+15é}{16}\right)\right]+\left(\frac{é-á}{2}\right)\mathcal{L}\left(\frac{3á+é}{4}\right)\right\}\right|$$
$$\leq \frac{1}{4}(é-á)\bigvee_{á}^{é}(\mathcal{L}).$$

Corollary 5.4. *By taking Theorem* 5.1*, suppose that* $\mathcal{L} \in C^1[á, é]$*, then we have*

$$\left|\int_{á}^{é} \mathcal{L}(ú)\,dú - \left\{\left(\frac{č-á}{2}\right)\left[\mathcal{L}\left(\frac{3á+č}{4}\right)+\mathcal{L}\left(\frac{á+č}{2}\right)\right.\right.\right.$$
$$\left.\left.\left.+\mathcal{L}\left(\frac{é+\partial}{2}\right)+\mathcal{L}\left(\frac{3é+\partial}{4}\right)\right]+\left(\frac{á+é}{2}-č\right)[\mathcal{L}(č)+\mathcal{L}(\partial)]\right\}\right|$$
$$\leq \max\left\{\left(\frac{č-á}{2}\right),\left(\frac{á+é}{2}-č\right),\left(\partial-\frac{á+é}{2}\right),\left(\frac{é-\partial}{2}\right)\right\}\left\|\mathcal{L}'\right\|_1.$$

Corollary 5.5. *By taking Theorem* 5.1*, let* $\mathcal{L} : [á, é] \to \mathbb{R}$ *be a Lipschitzian with the constant* $L > 0$*, then*

$$\left|\int_{á}^{é} \mathcal{L}(ú)\,dú - \left\{\left(\frac{č-á}{2}\right)\left[\mathcal{L}\left(\frac{3á+č}{4}\right)+\mathcal{L}\left(\frac{á+č}{2}\right)\right.\right.\right.$$
$$\left.\left.\left.+\mathcal{L}\left(\frac{é+\partial}{2}\right)+\mathcal{L}\left(\frac{3é+\partial}{4}\right)\right]+\left(\frac{á+é}{2}-č\right)[\mathcal{L}(č)+\mathcal{L}(\partial)]\right\}\right|$$
$$\leq \max\left\{\left(\frac{č-á}{2}\right),\left(\frac{á+é}{2}-č\right),\left(\partial-\frac{á+é}{2}\right),\left(\frac{é-\partial}{2}\right)\right\}(é-á)L.$$

6. Conclusion

In this paper, we presented some new Ostrowski type inequalities by using a special type of peano kernel. Some related results are also discussed in details.

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