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SOME GENERALIZED OPIAL TYPE INEQUALITIES FOR INTERVAL-VALUED FUNCTION

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Abstract. In this paper, we first present some Opial type inequalities for real valued functions and prove a new Opial type inequalities. Then by using these Oial inequalities and the definitions of the gH-derivatives, we establish some generalization of Opial type inequalities for interval valued functions

1. Introduction

In the year 1960, Opial established the following interesting integral inequality[19]:

Theorem 1.1. *Let $x(t) \in C^{(1)}[0,h]$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0,h)$. Then, the following inequality holds*

$$\int_0^h |x(t)x'(t)|\, dt \le \frac{h}{4}\int_0^h (x'(t))^2\, dt \tag{1.1}$$

The constant $h/4$ is best possible.

Several integral inequalities involving integrable functions and their derivatives, such as Wirtinger's inequality, Ostrowski's inequality and Opial's inequality, among others, have been well studied during the past century (see [1–3, 6, 11–14, 17] and their references [20–24, 27, 28]). All these works have provided fundamental tools to the development of many areas in mathematical analysis. Interval analysis was introduced as an attempt to handle interval uncertainty that appears in many mathematical or computer models of some deterministic real-world phenomena. The first monograph dealing with interval analysis was given by Moore [18]. Moore is recognized to be the first to use intervals in computational mathematics, now called numerical analysis. He also extended and implemented the arithmetic of intervals

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to computers. The main theoretical and practical results in the interval analysis can be found in several works .

The concept of Hukuhara derivative is old and well known [15], but the concept of G-derivative was recently introduced by Bede and Gal [4]. Using this new concept of derivative, the class of fuzzy differential equations has been extended and studied in some papers such as: [16, 18, 26]. The major shortcomings of the H-derivative are well known. To eliminate these shortcomings, several notions of derivative of an interval-valued function were introduced. We remark that H-derivative is a very restrictive concept (see [4, 10]). On the other hand, it is well-known that gH-derivative is a very general concept on derivative for interval-valued functions, see [4, 10, 26]. Some concepts of derivatives for an interval-valued function, like G-derivative [4], gH-derivative [16, 26], and $\pi-$derivative [25], were analyzed in [16, 18]. It is known that, if the set of switching points is finite, then the notions of G-differentiability, gH-differentiability and π-differentiability coincide for an interval-valued function (see [16, 18, 26]). For this reason, in this paper we use only the notion of gH-differentiability.

In this direction, motivated by [3, 7–9] and by [4, 10, 16, 26], we establish some Opial-type integral inequalities for gH-differentiable interval-valued functions which is the main objective of this article. The structure of this paper is as follows. In Section 2, we give the definitions of the gH-derivatives and introduce several useful notations interval valued function used our main results. Moreover the algebraic meaning of the square of an interval is established which becomes a fundamental part for obtaining the Opial-type integral inequalities for interval-valued functions. Also the concept of piecewise continuously gH-differentiable interval-valued function is introduced in Section 2. In Section 3, the main result is presented.

2. Definitions and properties for interval valued functions of derivative

Let $\mathbf{R}$ be the one-dimensional Euclidean space. Let K_C denote the family of all bounded closed intervals of R, that is,

$$K_C = \{[a,b]\,|a,b \in R \text{ and } a \leq b\}.$$

The space (K_C, d_H), where d_H is the Pompei- Hausdorff metric given by

$$d_H([a_1,a_2],[b_1,b_2]) = \max\{|a_1-b_1|,|a_2-b_2|\}$$

for all $[a_1,a_2],[b_1,b_2] \in K_C$, is a complete metric space. A quasinorm $\|.\|$ in K_C is defined by $\|A\| = d_H(A,[0,0])$ for all $A \in K_C$. The equality $\|A^2\| = \|A\|^2$ holds for all $A \in K_C$ and is used throughout this article.

Stefani and Bede introduced the concept of generalized Hukuhara difference of two sets $A, B \in K_C$ (gH difference for short) as follows

$$A \ominus_{gH} B = C \Longleftrightarrow \left\{ \begin{array}{c} \text{(a)}\;\; A = B + C \\ or\ \text{(b)}\;\; B = A + (-1)C \end{array} \right\}$$

In case(a), the gH difference is coincident with the H difference. Thus, the gH difference is a generalization of the H difference. On the other hand, gH difference exists for any two intervals $A=\left[\underline{a},\overline{a}\right], B=\left[\underline{b},\overline{b}\right]\in K_C$ and

$$A\ominus_{gH}B=\left[\min\left\{\underline{a}-\underline{b}\right\},\max\left\{\overline{a}-\overline{b}\right\}\right].$$

Using the gH difference, Stefanini and Bede introduced a differentiability concept for interval valued functions, which is more suitable than the H- differentiability. The following definitions and theorems with respect to H derivative and gH derivative were referred in (see [26]).

$F:T\subseteq R\rightarrow K_C$ given by $F(x)=[f(x),g(x)]$ for all $x\in T$, where $f,g:T\rightarrow \mathbf{R}$ are real valued functions, with $f(x)\leq g(x)$ for all $x\in T$, it is called an interval function. The functions f and g are called the lower and the upper(endpoint) functions of F, respectively.

Definition 2.1. Let $F:T\rightarrow K_C$ be an interval valued function. $L\in K_C$ is called a limit of F at $x_0\in T$ if for every $\epsilon>0$ there exists $\delta(\epsilon,x_0)=\delta>0$ such that $H(F(x),L)<\epsilon$ for all $x\in T$ with $0<|x-x_0|<\delta$.This is denoted by $\lim\limits_{x\rightarrow x_0}F(x)=L$.

Theorem 2.1. *Let $F:T\rightarrow K_C$ be an interval valued function such that $F(x)=[f(x),g(x)]$ for all $x\in T$. Then $L=[l_1,l_2]\in K_C$ is a limit of F at $x_0\in T$ if and only if l_i is the limit of f_i at x_0, $i\in\{1,2\}$ .Besides if L is limit of F at x_0, then*

$$\lim_{x\rightarrow x_0}F(x)=\left[\lim_{x\rightarrow x_0}f(x),\lim_{x\rightarrow x_0}g(x)\right].$$

Definition 2.2. Let $F:T\subseteq\mathbf{R}\rightarrow K_C$ be an interval valued function. F is said to be continuous at $x_0\in T$ if $\lim\limits_{x\rightarrow x_0}F(x)=F(x_0)$.

Theorem 2.2. *Let $F:T\subseteq\mathbf{R}\rightarrow K_C$ be an interval valued function such that $F(x)=[f(x),g(x)]$ for all $x\in T$. Then F is continuous at $x_0\in T$ if and only if f and g are continuous at x_0. Besides, F is continuous at x_0,then*

$$\lim_{x\rightarrow x_0}F(x)=[f(x_0),g(x_0)].$$

Definition 2.3. Let $F:T\rightarrow K_C$ be an interval valued function. We say that F is H-differentiable at $x_0\in T$ if there exists an element $F_H'(x_0)\in K_C$ such that the limits

$$\lim_{h\rightarrow 0^+}\frac{F(x_0+h)-F_H(x_0)}{h}$$

and

$$\lim_{h\rightarrow 0^+}\frac{F(x_0)-F_H(x_0-h)}{h}$$

exist and are equal to $F_H'(x_0)$ .In this case $F_H'(x_0)$ is called the $H-$derivative of F at x_0.

Definition 2.4. The gH-derivative of an interval-valued function $F:T\rightarrow K_C$ at $x_0\in T$ is defined as

$$F_{gH}'(x_0)=\lim_{h\rightarrow 0}\frac{F(x_0+h)\ominus F_{gH}(x_0-h)}{h}\tag{2.1}$$

If $F_{gH}'(x_0)\in K_C$ satisfying is differentiable, then (2.1) exist, then we say that F is generalized Hukuhara differentiable (gH-differentiable, for short) at x_0.

Theorem 2.3. *Let $F : T \to K_C$ be an interval valued function such that $F(x) = [f(x), g(x)]$ for all $x \in T$. Then F is gH-differentiable at $x_0 \in T$ if and only if one of the following cases holds*

(i) f and g are differentiable at x_0 and

$$F'_{gH}(x_0) = \left[\min\left\{f'(x_0), g'(x_0)\right\}, \max\left\{f'(x_0), g'(x_0)\right\}\right];$$

(ii) $f'_-(x_0), g'_-(x_0), f'_+(x_0)$ and $g'_+(x_0)$ exist and satisfy $f'_-(x_0) = g'_+(x_0)$ and $g'_-(x_0) = f'_+(x_0)$. Moreover

$$\begin{aligned} F'_{gH}(x_0) &= \left[\min\left\{f'_-(x_0), g'_-(x_0)\right\}, \max\left\{f'_-(x_0), g'_-(x_0)\right\}\right] \\ &= \left[\min\left\{f'_+(x_0), g'_+(x_0)\right\}, \max\left\{f'_+(x_0), g'_+(x_0)\right\}\right]. \end{aligned}$$

Theorem 2.4. *Let $F : [a, b] \to K_C$ be a continuous interval valued function with $F(x) = [f(x), g(x)]$ for all $x \in [a, b]$. If F is piecewise continuously gH differentiable on $[a, b]$ and it has (if there exists) a finite number of switching points on (a, b), then f and g are absolutely continuous on $[a, b]$.*

Lemma 2.1. [26] *Let A and B be two intervals in K_C. Then we have*

(a) $H(A, B) = \|A \ominus B\|$

(b) $\|A^2\| = \max\left\{\left|\underline{a}^2\right|, \left|\overline{a}^2\right|\right\} = \max\left\{\left|\underline{a}\right|^2, \left|\overline{a}\right|^2\right\} = \|A\|^2$

Theorem 2.5. ***(Maroni' s Generalization)*** *Let p be positive and continuous on $[\alpha, \tau]$ with $\int_\alpha^\tau p^{1-\mu}(t)\,dt < \infty$, where $\mu > 1$. Further, let $x(t)$ be absolutely continuous on $[\alpha, \tau]$, and $x(\alpha) = 0$. Then, the following inequality holds*

$$\int_a^\tau \left|x(t)\,x'(t)\right| dt \le \frac{1}{2}\left(\int_a^\tau p(t)^{1-\mu} dt\right)^{\frac{2}{\mu}} \left(\int_a^\tau p(t)\left|x'(t)\right|^{\upsilon} dt\right)^{\frac{2}{\upsilon}},$$

where $\frac{1}{\mu} + \frac{1}{\upsilon} = 1$.

Theorem 2.6. *Assume that*

(i) l, m, μ and υ are non-negative real numbers such that $\frac{1}{\mu} + \frac{1}{\upsilon} = 1$, and $l\mu \ge 1$,

(ii) q is non-negative continuous function on $[0, h]$,

(iii) x_1, x_2 are absolutely continuous function on $[0, h]$, with $x_1(0) = x_2(0) = x_1(h) = x_2(h) = 0$. Then, the following inequality holds

$$\begin{aligned} &\int_0^h q(t)\left[\left|x_1(t)\right|^l \left|x_2'(t)\right|^m + \left|x_2(t)\right|^l \left|x_1'(t)\right|^m\right] dt \\ \le\ & \frac{1}{2}\left(\int_0^h \left[t(h-t)\right]^{\frac{l\mu-1}{2}} q^\mu(t)\,dt\right)^{\frac{1}{\mu}} \\ & \times\left[\int_0^h \left\{\frac{1}{\mu}\left(\left|x_1'(t)\right|^{l\mu} + \left|x_2'(t)\right|^{l\mu}\right) + \frac{1}{\upsilon}\left(\left|x_1'(t)\right|^{\mu\upsilon} + \left|x_2'(t)\right|^{\mu\upsilon}\right)\right\} dt\right]. \end{aligned}$$

Corollary 2.1. *If we choose $q(t) = 1$ and $m = 1$ in theorem 2.6,we have the following inequality*

$$
\begin{aligned}
& \int_0^h q(t) \left[|x_1(t)|^l |x_2'(t)| + |x_2(t)|^l |x_1'(t)| \right] dt \\
\leq\ & \frac{1}{2} \left(\int_0^h [t(h-t)]^{\frac{l\mu-1}{2}} dt \right)^{\frac{1}{\mu}} \\
& \times \left[\int_0^h \left\{ \frac{1}{\mu} \left(|x_1'(t)|^{l\mu} + |x_2'(t)|^{l\mu} \right) + \frac{1}{\upsilon} \left(|x_1'(t)|^{\mu\upsilon} + |x_2'(t)|^{\mu\upsilon} \right) \right\} dt \right].
\end{aligned}
$$

Theorem 2.7. ***(Hua' s Genaralization)*** *Let x be absolutely continuous on $[0, h]$, and $x(0) = 0$. Further, let l be a positive integer. Then the following inequality holds*

$$\int_0^h \left| x^l(t) x'(t) \right| dt \leq \frac{h^l}{l+1} \int_0^h |x'(t)|^{l+1} dt.$$

Theorem 2.8. *Let p be positive and continuous on $[\alpha, \beta]$ with $\int_\alpha^\tau \frac{dt}{p(t)} < \infty$, and let q be positive, bounded and non-increasing on $[\alpha, \beta]$. Further, let x_1, x_2 be absolutely continuous on $[\alpha, \beta]$,and $x_1(\alpha) = x_2(\alpha) = 0$. Then, we have the following inequality*

$$
\begin{aligned}
& \int_\alpha^\beta [|x_1'(t) x_2(t)| + |x_1(t) x_2'(t)| dt] \\
\leq\ & \frac{1}{2} \left(\int_\alpha^\beta p^{\frac{-\mu}{\upsilon}}(t) dt \right)^{\frac{2}{\mu}} \left[\left(\int_\alpha^\beta p(t) |x_1'(t)|^\upsilon dt \right)^{\frac{2}{\upsilon}} + \left(\int_\alpha^\beta p(t) |x_2'(t)|^\upsilon dt \right)^{\frac{2}{\upsilon}} \right].
\end{aligned}
$$

Proof. For $i = 1, 2$ let $y_i(t) = \int_\alpha^t |x_i'(t)| dt$ so that $y_i(t) = |x_i'(t)|$, and $|x_i(t)| \leq y_i(t)$.

$$
\begin{aligned}
& \int_\alpha^\beta [|x_1'(t) x_2(t)| + |x_1(t) x_2'(t)| dt] \\
\leq\ & \int_\alpha^\beta [y_1'(t) y_2(t) + y_1(t) y_2'(t) dt] \\
=\ & y_1(\beta) y_2(\beta).
\end{aligned}
$$

Since $ab \leq \frac{1}{2}(a^2 + b^2)$,

$$
\begin{aligned}
& \int_\alpha^\beta [|x_1'(t) x_2(t)| + |x_1(t) x_2'(t)| dt] \\
\leq\ & \frac{1}{2} \left(y_1^2(\beta) + y_2^2(\beta) \right) \\
=\ & \frac{1}{2} \left(\int_\alpha^\beta p^{\frac{1}{\upsilon}}(t) p^{\frac{-1}{\upsilon}}(t) |x_1'(t)| dt \right)^2 + \frac{1}{2} \left(\int_\alpha^\beta p^{\frac{1}{\upsilon}}(t) p^{\frac{-1}{\upsilon}}(t) |x_2'(t)| dt \right)^2.
\end{aligned}
$$

By using the Hölder inequality for indices μ and υ, it follows that

$$\begin{aligned}&\int_{\alpha}^{\beta}\left[\left|x_1'(t)\,x_2(t)\right|+\left|x_1(t)\,x_2'(t)\right|dt\right]\\ \leq\ &\frac{1}{2}\left(\int_{\alpha}^{\beta}p^{\frac{-\mu}{\upsilon}}(t)dt\right)^{\frac{2}{\mu}}\left[\left(\int_{\alpha}^{\beta}p(t)\left|x_1'(t)\right|^{\upsilon}dt\right)^{\frac{2}{\upsilon}}+\left(\int_{\alpha}^{\beta}p(t)\left|x_2'(t)\right|^{\upsilon}dt\right)^{\frac{2}{\upsilon}}\right].\end{aligned}$$

This completes the proof. □

Now we present the main results.

3. Opial type inequalities for interval-valued functions

Theorem 3.1. *Let $F:[\alpha,\beta]\to K_C$ be a continuous interval valued function such that $F(t)=[f(t),g(t)]$ for all $t\in[\alpha,\beta]$, F is gH-differentiable at $t_0\in(\alpha,\beta)$ and $F(\alpha)=0$. Let p be positive and continuous on $[\alpha,\beta]$ with $\int_{\alpha}^{\beta}p^{1-\mu}(t)\,dt<\infty$, where $\mu>1$. Then, the following inequality holds*

$$\int_{\alpha}^{\beta}\left\|F(t)F_{gH}'(t)\right\|dt\leq 2K\left(\int_{a}^{\beta}p(t)\left\|F_{gH}'(t)\right\|^{\upsilon}dt\right)^{\frac{2}{\upsilon}}$$

where $K=\left(\int_{\alpha}^{\beta}p^{\frac{-\mu}{\upsilon}}(t)dt\right)^{\frac{2}{\mu}}$ and $\frac{1}{\mu}+\frac{1}{\upsilon}=1$.

Proof. From the hypotheses we have that $F_{gH}'(t)=[f'(t),g'(t)]$ or $F_{gH}'(t)=[g'(t),f'(t)]$ for almost every $t\in[\alpha,\beta]$. Then from Lemma 2.1, we have $\left\|F_{gH}'(t)\right\|^2=\max\left\{\left|f'(t)\right|^2,\left|g'(t)\right|^2\right\}$ for almost every $t\in[\alpha,\beta]$. Moreover,

$$\begin{aligned}F(t)F_{gH}'(t)\ =\ &[\min\left\{f(t)f'(t),g(t)g'(t),f'(t)g(t),f(t)g'(t)\right\}\\ &\max\left\{f(t)f'(t),g(t)g'(t),f'(t)g(t),f(t)g'(t)\right\}]\end{aligned}\tag{3.1}$$

for almost every $t\epsilon\,[\alpha,\beta]$. Thus from (3.1), it follows that

$$\begin{aligned}\left\|F(t)F_{gH}'(t)\right\|\ =\ &\max\{\left|\min\left\{f(t)f'(t),g(t)g'(t),f'(t)g(t),f(t)g'(t)\right\}\right|,\\ &\left|\max\left\{f(t)f'(t),g(t)g'(t),f'(t)g(t),f(t)g'(t)\right\}\right|\}\end{aligned}$$

and thus,

$$\left\|F(t)F_{gH}'(t)\right\|\leq\left|f(t)f'(t)\right|+\left|g(t)g'(t)\right|+\left|f'(t)g(t)\right|+\left|f(t)g'(t)\right|$$

for almost every $t\in[\alpha,\beta]$. Consequently,

$$\begin{aligned}&\int_{\alpha}^{\beta}\left\|F(t)F_{gH}'(t)\right\|dt\\ \leq\ &\int_{\alpha}^{\beta}\left|f(t)f'(t)\right|dt+\int_{\alpha}^{\beta}\left|g(t)g'(t)\right|dt+\int_{\alpha}^{\beta}\left|f'(t)g(t)\right|+\left|f(t)g'(t)\right|dt\end{aligned}$$

By using Maroni inequality and Theorem 2.8,

$$\begin{aligned}
&\int_{\alpha}^{\beta}\left\|F(t)F'_{gH}(t)\right\|dt \\
\leq\ & \frac{1}{2}\left(\int_{a}^{\beta}p(t)^{1-\mu}dt\right)^{\frac{2}{\mu}}\left(\int_{a}^{\beta}p(t)\left|f'(t)\right|^{\upsilon}dt\right)^{\frac{2}{\upsilon}} \\
&+\frac{1}{2}\left(\int_{a}^{\beta}p(t)^{1-\mu}dt\right)^{\frac{2}{\mu}}\left(\int_{a}^{\beta}p(t)\left|g'(t)\right|^{\upsilon}dt\right)^{\frac{2}{\upsilon}} \\
&+\frac{1}{2}\left(\int_{\alpha}^{\beta}p^{1-\mu}(t)dt\right)^{\frac{2}{\mu}}\left[\left(\int_{\alpha}^{\beta}p(t)\left|f'(t)\right|^{\upsilon}dt\right)^{\frac{2}{\upsilon}}+\left(\int_{\alpha}^{\beta}p(t)\left|g'(t)\right|^{\upsilon}dt\right)^{\frac{2}{\upsilon}}\right].
\end{aligned}$$

Then it follows that

$$\begin{aligned}
&\int_{\alpha}^{\beta}\left\|F(t)F'_{gH}(t)\right\|dt \\
=\ & \frac{1}{2}K\left[2\left(\int_{a}^{\beta}p(t)\left|f'(t)\right|^{\upsilon}dt\right)^{\frac{2}{\upsilon}}+2\left(\int_{a}^{\beta}p(t)\left|g'(t)\right|^{\upsilon}dt\right)^{\frac{2}{\upsilon}}\right] \\
\leq\ & K\left[\left(\int_{a}^{\beta}p(t)\left\|F'_{gH}(t)\right\|^{\upsilon}dt\right)^{\frac{2}{\upsilon}}+\left(\int_{a}^{\beta}p(t)\left\|F'_{gH}(t)\right\|^{\upsilon}dt\right)^{\frac{2}{\upsilon}}\right] \\
=\ & 2K\left(\int_{a}^{\beta}p(t)\left\|F'_{gH}(t)\right\|^{\upsilon}dt\right)^{\frac{2}{\upsilon}}.
\end{aligned}$$

This completes the proof of the inequality. □

Corollary 3.1. *If we choose $p(t)=1$ in Theorem 3.1, we have the following inequality*

$$\int_{\alpha}^{\beta}\left\|F(t)F'_{gH}(t)\right\|dt\leq 2\left(\beta-\alpha\right)^{\frac{2}{\mu}}\left(\int_{a}^{\beta}\left\|F'_{gH}(t)\right\|^{\upsilon}dt\right)^{\frac{2}{\upsilon}}.$$

Remark 3.1. If we take $\mu=\upsilon=2$ in Corollary 3.1, then we have the following inequality

$$\int_{\alpha}^{\beta}\left\|F(t)F'_{gH}(t)\right\|dt\leq 2\left(\beta-\alpha\right)\int_{a}^{\beta}\left\|F'_{gH}(t)\right\|^{2}dt$$

which is proved by Costa et al. in [13].

Theorem 3.2. *Let $F:[0,h]\rightarrow K_C$ be a continuous interval valued function such that $F(t)=[f(t),g(t)]$ for all $t\in[\alpha,\beta]$, F is gH-differentiable at $t_0\in(0,h)$, and $F(0)=0$. Further, let l be a positive integer with μ and υ are non-negative real numbers such that $\frac{1}{\mu}+\frac{1}{\upsilon}=1$, and $l\mu\geq 1$. Then, the following inequality holds*

$$\begin{aligned}
\int_{0}^{h}\left\|F^{l}(t)F'_{gH}(t)\right\|dt\ \leq\ & K_1\int_{0}^{h}\left\|F'_{gH}(t)\right\|^{l+1}dt \\
&+K_2\left(\frac{1}{\mu}\int_{0}^{h}\left\|F'_{gH}(t)\right\|^{l\mu}dt+\frac{1}{\upsilon}\int_{0}^{h}\left\|F'_{gH}(t)\right\|^{\mu\upsilon}dt\right)
\end{aligned}$$

where $K_1 = \frac{2h^l}{l+1}$ *and* $K_2 = \left(\int_0^h [t(h-t)]^{\frac{l\mu-1}{2}} dt\right)^{\frac{1}{\mu}}$.

Proof. By using similar method used in proof of Theorem 3.1, we have $\left\|F'_{gH}(t)\right\|^l = \max\left\{|f'(t)|^l, |g'(t)|^l\right\}$ for almost every $t \in [0,h]$. Then we have,

$$\left\|F^l(t)F'_{gH}(t)\right\| \leq \left|f^l(t)f'(t)\right| + \left|g^l(t)g'(t)\right| + \left|f'(t)g^l(t)\right| + \left|f^l(t)g'(t)\right|$$

for almost every $t \in [0,h]$. Thus,

$$\begin{aligned} &\int_0^h \left\|F^l(t)F'_{gH}(t)\right\| dt \\ \leq\ &\int_0^h \left|f^l(t)f'(t)\right| dt + \int_0^h \left|g^l(t)g'(t)\right| dt + \int_0^h \left(\left|f'(t)g^l(t)\right| + \left|f^l(t)g'(t)\right|\right) dt \end{aligned}$$

By using Hua' generalization and Corollary 2.1, we get

$$\begin{aligned} &\int_0^h \left\|F^l(t)F'_{gH}(t)\right\| dt \\ \leq\ &\frac{h^l}{l+1}\int_0^h |f'(t)|^{l+1} dt + \frac{h^l}{l+1}\int_0^h |g'(t)|^{l+1} dt + \frac{1}{2}\left(\int_0^h [t(h-t)]^{\frac{l\mu-1}{2}} dt\right)^{\frac{1}{\mu}} \\ &\times\left[\int_0^h \left\{\frac{1}{\mu}\left(|x_1'(t)|^{l\mu} + |x_2'(t)|^{l\mu}\right) + \frac{1}{\upsilon}\left(|x_1'(t)|^{\mu\upsilon} + |x_2'(t)|^{\mu\upsilon}\right)\right\} dt\right]. \end{aligned}$$

Hence, we have

$$\begin{aligned} &\int_0^h \left\|F^l(t)F'_{gH}(t)\right\| dt \\ \leq\ &K_1\int_0^h \left\|F'_{gH}(t)\right\|^{l+1} + K_2\left(\int_0^h \frac{1}{\mu}\left\|F'_{gH}(t)\right\|^{l\mu} dt + \int_0^h \frac{1}{\upsilon}\left\|F'_{gH}(t)\right\|^{\mu\upsilon} dt\right). \end{aligned}$$

This completes the proof of the inequality. □

Corollary 3.2. *If we choose* $l = 1$ *and* $\mu = \upsilon = 2$ *in Theorem* 3.2, *then*

$$\begin{aligned} \int_0^h \left\|F^l(t)F'_{gH}(t)\right\| dt \ &\leq\ h\int_0^h \left\|F'_{gH}(t)\right\|^2 dt + \frac{1}{2}\left(\int_0^h [t(h-t)]^{\frac{1}{2}} dt\right)^{\frac{1}{2}} \\ &\times\left(\int_0^h \left\|F'_{gH}(t)\right\|^2 dt + \int_0^h \left\|F'_{gH}(t)\right\|^4 dt\right). \end{aligned}$$

Example 3.1. Let F:$[0,1] \to K_C$ be the interval-valued function given by $F(t) = [-1,1]\left(t-t^2\right)$ for all $t \in [0,1]$. Since $h(t) = t - t^2$ is continuously gH-differentiable on $(0,1)$. Further,F has only one switching point at $t = \frac{1}{2}$, $F(0) = F(1) = [0,0]$, and

$$F'_{gH}(t) = \begin{cases} [2t-1, -2t+1], \text{ if } t \in \left(0, \frac{1}{2}\right), \\ [-2t+1, 2t-1], \text{ if } t \in \left(\frac{1}{2}, 0\right), \end{cases}$$

equivalently

$$F'_{gH}(t) = [-1, 1] \, |1 - 2t| \, .$$

Thus,

$$\left\| F'_{gH}(t) \right\|^{\upsilon} = |2t - 1|^{\upsilon}$$

and

$$\begin{aligned}
\int_0^1 \left\| F'_{gH}(t) \right\|^{\upsilon} dt &= \int_0^{\frac{1}{2}} (2t-1)^{\upsilon} \, dt + \int_{\frac{1}{2}}^1 (1-2t)^{\upsilon} \, dt \\
&= \frac{(-1)^{\upsilon+2}}{\upsilon+1}.
\end{aligned}$$

Also, $\left\| F(t) F'_{gH}(t) \right\| = |2x^3 - 3x^2 + x|$ such that

$$\int_0^1 \left\| F(t) F'_{gH}(t) \right\| dt = \frac{1}{32}.$$

Then from 3.1, it follows that

$$\begin{aligned}
\int_0^1 \left\| F(t) F'_{gH}(t) \right\| dt &\leq 2 \left(\int_0^1 \left\| F'_{gH}(t) \right\|^{\upsilon} dt \right)^{\frac{2}{\upsilon}} \\
&\leq \frac{2}{(\upsilon+1)^{\frac{2}{\upsilon}}}.
\end{aligned}$$

4. Conclusions

In this paper, by utilizing some Opial type inequalities for real valued functions, we proved some Opial inequalities for interval valued functions. In the future works, authors can try to generalize our results for fractional integrals of interval valued functions.

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