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ON A SUBCLASS OF BI-UNIVALENT FUNCTIONS APPLYING Q-RUSCHEWEYH DIFFERENTIAL OPERATOR

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ABSTRACT. In this current research, we introduced a new subclass of holomorphic and bi-univalent functions using q-Ruscheweyh differential operator in $\mathfrak{U}$. For functions in the class $\Sigma_q^{\tau,\varphi}(\eta,\sigma,\gamma)$, we determine estimates on the first two Taylor-Maclaurin coefficients. Also, we derive another subclass of holomorphic and bi-univalent functions as a special consequences of the results.

1. INTRODUCTION

Indicate by $\mathfrak{A}$ the class of all holomorphic functions of the form

$$f(z) = z + \sum_{j=2}^{\infty} b_j z^j \tag{1.1}$$

in the open unit disk $\mathfrak{U} = \{z \in \mathcal{C} : |z| < 1\}$.

The famous Koebe one-quarter theorem [27] make sure that the image of $\mathfrak{U}$ under all univalent function $f \in \mathfrak{A}$ contains a disk of radius $1/4$. Therefore, all univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$, $(z \in \mathbb{U})$ and

$$f(f^{-1}(w)) = w, (|w| < r_0(f), r_0(f) \geq 1/4),$$

where

$$f^{-1}(w) = w - b_2 w^2 + (2b_2^2 - b_3)w^3 - (5b_2^3 - 5b_2 b_3 + b_4)w^4 + \cdots . \tag{1.2}$$

Let $\Sigma = \{f \in \mathfrak{A} : f(z)$ and $f^{-1}(z)$ are univalent in the unit disk $\mathfrak{U}\}$ denote the class of bi-univalent functions. We have some of the functions in the class Σ which are given below (for more details see Srivastava et al. [13]):

$$-\log(1-z), \quad \frac{z}{1-z}, \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

Key words and phrases. Holomorphic function, bi-univalent function, q-derivative, coefficient estimates, subordination, q-Ruscheweyh differential operator.

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The certain subclases of the bi-univalent function class Σ which is similar to the familiar subclasses of starlike and convex functions of order μ was introduced by Brannan and Taha [12], then in 2012, Ali et al. [9] extend the result of Brannan and Taha [12] by subordination. Since then, numerous subclasses of the bi-univalent function class Σ were introduced. Several studies has been done on the first two coefficients $|b_2|$ and $|b_3|$ in the series expansion (1.1) (see [11–18, 20–24, 29–32]).

Jackson [25, 26] introduced the q-derivative operator $\mathfrak{D}_q$ of a function as follows:

$$\mathfrak{D}_q f(z) = \frac{f(qz) - f(z)}{(q-1)z} \tag{1.3}$$

and $\mathfrak{D}_q f(z) = f'(0)$. In case $f(z) = z^{\phi}$ for ϕ is a positive integer, the q-derivative of $f(z)$ is given by

$$\mathfrak{D}_q z^{\phi} = \frac{z^{\phi} - (zq)^{\phi}}{(q-1)z} = [\phi]_q z^{\phi-1}.$$

As $q \longrightarrow 1^-$ and $\phi \in \mathcal{N}$, we get

$$[\phi]_q = \frac{1-q^{\phi}}{1-q} = 1 + q + \cdots + q^{\phi} \longrightarrow \phi \tag{1.4}$$

where $(z \neq 0,\ q \neq 0)$, for more details on the concepts of q-derivative (see [1, 2, 5–8, 10, 15, 28]).

For two analytic functions n_1 and n_2 we say that n_1 is subordinated to n_2 and symbolically is written as $n_1 \prec n_2$, if there is an analytic function $\nu(z)$ in $\mathfrak{U}$, with the properties that $\nu(0) = 0$ and $|\nu(z)| < 1$, such that $n_1(z) = n_2(\nu(z))$, $z \in \mathfrak{U}$. In case if $n_2(z) \in \mathfrak{A}$ then the following relation holds;

$$n_1(z) \prec n_2(z), \quad z \in \mathfrak{U} \iff n_1(0) = n_2(0) \quad \& \quad n_1(\mathfrak{U}) \subset n_2(\mathfrak{U}).$$

In 2014, Aldweby and Darus [3], introduced the q-analogue of Ruscheweyh operator which can be defined has follows:

Definition 1.1. Let $f \in \mathfrak{A}$. Denote by $\mathfrak{R}_q^{\sigma}$ the q-analogue of Ruscheweyh operator defined by

$$\mathfrak{R}_q^{\sigma} f(z) = z + \sum_{j=2}^{\infty} \frac{[j+\sigma-1]_q!}{[\sigma]_q![j-1]_q!} b_j z^j,$$

where $[j]_q!$ given by

$$[j]_q! = \begin{cases} [j]_q[j-1]_q \cdots [1]_q, & j = 1, 2, \cdots, \\ 1, & j = 0. \end{cases} \tag{1.5}$$

From the definition we see that if $q \longrightarrow 1$, we get

$$\lim_{q \longrightarrow 1} \mathfrak{R}_q^{\sigma} f(z) = z + \lim_{q \longrightarrow 1} \left[\sum_{j=2}^{\infty} \frac{[j+\sigma-1]_q!}{[\sigma]_q![j-1]_q!} b_j z^j \right] = z + \sum_{j=2}^{\infty} \frac{(j+\sigma-1)!}{(\sigma)!(j-1)!} b_j z^j = \mathfrak{R}^{\sigma} f(z),$$

where $\mathfrak{R}^{\sigma} f(z)$ is Ruscheweyh differential operator defined in [4].

Let γ be an holomorphic function with positive real part in $\mathfrak{U}$ such that $\gamma(0)=1$, $\gamma'(0)>0$ and $\gamma(\mathfrak{U})$ is symmetric with respect to real axis. Such a function has a series expansion of the form

$$\gamma(z)=1+\mathfrak{B}_1 z+\mathfrak{B}_2 z^2+\mathfrak{B}_3 z^3+\cdots, \quad (\mathfrak{B}>0). \tag{1.6}$$

From our introduction above, we introduce the following class of bi-univalent functions and obtain the coefficients estimates with the use of q-derivative.

Definition 1.2. Let $\tau \geq 1$, $\varphi \geq 0$, $\eta \neq 0$, $\sigma > -1$. A function $f \in \Sigma$ is said to be in te class $\Sigma_q^{\tau,\varphi}(\eta,\sigma,\gamma)$, if each of the following subordition condition holds true:

$$1+\frac{1}{\eta}\left[(1-\tau)\frac{\mathfrak{R}_q^\sigma f(z)}{z}+\tau\mathfrak{D}_q(\mathfrak{R}_q^\sigma f(z))+\varphi z(\mathfrak{D}_q(\mathfrak{R}_q^\sigma f(z)))'-1\right]\prec\gamma(z), \quad z\in\mathfrak{U},$$

and

$$1+\frac{1}{\eta}\left[(1-\tau)\frac{\mathfrak{R}_q^\sigma g(w)}{z}+\tau\mathfrak{D}_q(\mathfrak{R}_q^\sigma g(w))+\varphi z(\mathfrak{D}_q(\mathfrak{R}_q^\sigma g(w)))'-1\right]\prec\gamma(w), \quad w\in\mathfrak{U},$$

where $g(w)=f^{-1}(w)$.

Lemma 1.1. *Let the function $r\in\mathfrak{P}$ be given by the following series:*

$$r(z)=1+r_1z+r_2z^2+r_3z^3+\cdots, \quad (z\in\mathfrak{U}).$$

The sharp estimates yields

$$|r_n|\leq 2, \quad (n\in\mathfrak{N}),$$

holds true.

2. A set of main results

Theorem 2.1. *Let $f\in\Sigma_q^{\tau,\varphi}(\eta,\sigma,\gamma)$ be of the form (1.5). Then*

$$|b_2|\leq\frac{\eta^{3/2}[2]_q\mathfrak{B}_1^{3/2}}{\sqrt{\eta[1+\sigma]_q\left[\eta\mathfrak{B}_1^2[2+\sigma]_q(\tau q[2]_q+2\cdot[3]_q\varphi+1)+[2]_q[1+\sigma]_q(q\tau+[2]_q\varphi+1)^2(\mathfrak{B}_1-\mathfrak{B}_2)\right]}} \tag{2.1}$$

and

$$|b_3|\leq\frac{\mathfrak{B}_1\eta}{[1+\sigma]_q}\left[\frac{\mathfrak{B}_1\eta}{[1+\sigma]_q(q\tau+[2]_q\varphi+1)^2}+\frac{[2]_q}{[2+\sigma]_q(\tau q[2]_q+2\cdot[3]_q\varphi+1)}\right] \tag{2.2}$$

where the coefficients $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are given as in (1.6).

Proof. Let $f\in\Sigma_q^{\tau,\varphi}(\eta,\sigma,\gamma)$ and $g=f^{-1}$. Then there are holomorphic functions $s,t:\mathfrak{U}\longrightarrow\mathfrak{U}$ with $s(0)=t(0)=0$, satisfying the following conditions:

$$1+\frac{1}{\eta}\left[(1-\tau)\frac{\mathfrak{R}_q^\sigma f(z)}{z}+\tau\mathfrak{D}_q(\mathfrak{R}_q^\sigma f(z))+\varphi z(\mathfrak{D}_q(\mathfrak{R}_q^\sigma f(z)))'-1\right]=\gamma(s(z)), \quad z\in\mathfrak{U}, \tag{2.3}$$

and

$$1+\frac{1}{\eta}\left[(1-\tau)\frac{\mathfrak{R}_q^\sigma g(w)}{z}+\tau\mathfrak{D}_q(\mathfrak{R}_q^\sigma g(w))+\varphi z(\mathfrak{D}_q(\mathfrak{R}_q^\sigma g(w)))'-1\right]=\gamma(t(w)),\quad w\in\mathfrak{U}. \tag{2.4}$$

Define the function u and v by

$$u(z)=\frac{1+s(z)}{1-s(z)}=1+u_1z+u_2z^2+\cdots,$$

and

$$v(w)=\frac{1+t(w)}{1-t(w)}=1+v_1w+v_2w^2+\cdots.$$

Then u and v are analytic in $\mathfrak{U}$ with $u(0)=v(0)=1$.

Since $s,t:\mathfrak{U}\longrightarrow\mathfrak{U}$, each of the functions u and v has a positive real part in $\mathfrak{U}$. Therefore, in view of the above lemma, we get

$$|u_n|\le 2\quad\text{and}\quad|v_n|\le 2,\quad(n\in\mathfrak{N}).$$

Solving for $s(z)$ and $t(w)$, we get

$$s(z)=\frac{u(z)-1}{u(z)+1}=\frac{1}{2}\left[u_1z+\left(u_2-\frac{u_1^2}{2}\right)z^2\right]+\cdots,\quad(z\in\mathfrak{U}), \tag{2.5}$$

and

$$t(w)=\frac{v(w)-1}{v(w)+1}=\frac{1}{2}\left[v_1z+\left(v_2-\frac{v_1^2}{2}\right)z^2\right]+\cdots,\quad(z\in\mathfrak{U}), \tag{2.6}$$

By substituting from (2.5) and (2.6) into (2.3) and (2.4), respectively, suppose we make use of (1.6), we obtain

$$1+\frac{1}{\eta}\left[(1-\tau)\frac{\mathfrak{R}_q^\sigma f(z)}{z}+\tau\mathfrak{D}_q(\mathfrak{R}_q^\sigma f(z))+\varphi z(\mathfrak{D}_q(\mathfrak{R}_q^\sigma f(z)))'-1\right]$$
$$=\gamma\left(\frac{u(z)-1}{u(z)+1}\right)=1+\frac{1}{2}\mathfrak{B}_1u_1z+\left[\frac{1}{2}\mathfrak{B}_1\left(u_2-\frac{u_1^2}{2}\right)+\frac{1}{4}\mathfrak{B}_2u_1^2\right]z^2+\cdots,$$

and

$$1+\frac{1}{\eta}\left[(1-\tau)\frac{\mathfrak{R}_q^\sigma g(w)}{z}+\tau\mathfrak{D}_q(\mathfrak{R}_q^\sigma g(w))+\varphi z(\mathfrak{D}_q(\mathfrak{R}_q^\sigma g(w)))'-1\right]$$
$$=\gamma\left(\frac{v(w)-1}{v(w)+1}\right)=1+\frac{1}{2}\mathfrak{B}_1v_1w+\left[\frac{1}{2}\mathfrak{B}_1\left(v_2-\frac{v_1^2}{2}\right)+\frac{1}{4}\mathfrak{B}_2v_1^2\right]w^2+\cdots.$$

Also

$$1+\frac{1}{\eta}\left[(1-\tau)\frac{\mathfrak{R}_q^\sigma f(z)}{z}+\tau\mathfrak{D}_q(\mathfrak{R}_q^\sigma f(z))+\varphi z(\mathfrak{D}_q(\mathfrak{R}_q^\sigma f(z)))'-1\right]$$
$$=1+\frac{[1+\sigma]_q(q\tau+[2]_q\varphi+1)}{\eta}b_2z+\frac{[2+\sigma]_q[1+\sigma]_q(\tau q[2]_q+2\cdot[3]_q\varphi+1)}{\eta[2]_q}b_3z^2+\cdots$$

and

$$1+\frac{1}{\eta}\left[(1-\tau)\frac{\mathfrak{R}_q^\sigma g(w)}{z}+\tau\mathfrak{D}_q(\mathfrak{R}_q^\sigma g(w))+\varphi z(\mathfrak{D}_q(\mathfrak{R}_q^\sigma g(w)))'-1\right]$$
$$=1-\frac{[1+\sigma]_q(q\tau+[2]_q\varphi+1)}{\eta}b_2w+\frac{[2+\sigma]_q[1+\sigma]_q(\tau q[2]_q+2\cdot[3]_q\varphi+1)}{\eta[2]_q}(2b_2^2-b_3)w^2+\cdots$$

Now equating the coefficients in (2.3) and (2.4), we get

$$\frac{[1+\sigma]_q(q\tau+[2]_q\varphi+1)}{\eta}b_2=\frac{1}{2}\mathfrak{B}_1u_1 \tag{2.7}$$

$$\frac{[2+\sigma]_q[1+\sigma]_q(\tau q[2]_q+2\cdot[3]_q\varphi+1)}{\eta[2]_q}b_3=\frac{1}{2}\mathfrak{B}_1\left(u_2-\frac{u_1^2}{2}\right)+\frac{1}{4}\mathfrak{B}_2u_1^2 \tag{2.8}$$

$$-\frac{[1+\sigma]_q(q\tau+[2]_q\varphi+1)}{\eta}b_2=\frac{1}{2}\mathfrak{B}_1v_1 \tag{2.9}$$

$$\frac{[2+\sigma]_q[1+\sigma]_q(\tau q[2]_q+2\cdot[3]_q\varphi+1)}{\eta[2]_q}(2b_2^2-b_3)=\frac{1}{2}\mathfrak{B}_1\left(v_2-\frac{v_1^2}{2}\right)+\frac{1}{4}\mathfrak{B}_2v_1^2 \tag{2.10}$$

From (2.7) and (2.9), we have

$$u_1=-v_1 \tag{2.11}$$

and

$$\frac{2[1+\sigma]_q^2(q\tau+[2]_q\varphi+1)^2}{\eta^2}b_2^2=\frac{1}{4}\mathfrak{B}_1^2(u_1^2+v_1^2). \tag{2.12}$$

Now by adding (2.8) and (2.10), we have

$$\frac{2[2+\sigma]_q[1+\sigma]_q(\tau q[2]_q+2\cdot[3]_q\varphi+1)}{\eta[2]_q}b_2^2=\frac{1}{2}\mathfrak{B}_1\left[(u_2+v_2)-\left(\frac{u_1^2+v_1^2}{2}\right)\right]+\frac{1}{4}\mathfrak{B}_2[u_1^2+v_1^2]$$

By using (2.12), we have

$$b_2^2=\frac{\eta^3[2]_q^2\mathfrak{B}_1^3(u_2+v_2)}{4\eta[1+\sigma]_q\left[\eta\mathfrak{B}_1^2[2+\sigma]_q(\tau q[2]_q+2\cdot[3]_q\varphi+1)+[2]_q[1+\sigma]_q(q\tau+[2]_q\varphi+1)^2(\mathfrak{B}_1-\mathfrak{B}_2)\right]} \tag{2.13}$$

Applying Lemma 1.1, we have

$$|b_2|\leq\frac{\eta^{3/2}[2]_q\mathfrak{B}_1^{3/2}}{\sqrt{\eta[1+\sigma]_q\left[\eta\mathfrak{B}_1^2[2+\sigma]_q(\tau q[2]_q+2\cdot[3]_q\varphi+1)+[2]_q[1+\sigma]_q(q\tau+[2]_q\varphi+1)^2(\mathfrak{B}_1-\mathfrak{B}_2)\right]}}. \tag{2.14}$$

This gives us the bounds on $|b_2|$ as asserted in (2.1).

Furthermore, for us to get the bound on $|b_3|$, we substrate (2.10) from (2.8) and also from (2.11), we have $u_1^2=v_1^2$, hence

$$2[2+\sigma]_q[1+\sigma]_q(\tau q[2]_q+2\cdot[3]_q\varphi+1)(b_3-b_2^2)=\frac{\mathfrak{B}_1\eta[2]_q(u_2-v_2)}{2}$$

Using (2.12) and applying Lemma 1.1 once again, we have

$$|b_3| \leq \frac{\mathfrak{B}_1\eta}{[1+\sigma]_q}\left[\frac{\mathfrak{B}_1\eta}{[1+\sigma]_q(q\tau+[2]_q\varphi+1)^2}+\frac{[2]_q}{[2+\sigma]_q(\tau q[2]_q+2\cdot[3]_q\varphi+1)}\right]. \tag{2.15}$$

This completes the proof of Theorem 2.1. □

Putting $q \longrightarrow 1$ in Theorem 2.1, we have the following corollary.

Corollary 2.1. *Let $f \in \Sigma_q^{\tau,\varphi}(\eta,\sigma,\gamma)$ be of the form (1.5). Then*

$$|b_2| \leq \frac{2\eta^{3/2}\mathfrak{B}_1^{3/2}}{\sqrt{\eta(1+\sigma)\left[\eta\mathfrak{B}_1^2(2+\sigma)(2\tau+6\varphi+1)+2(1+\sigma)(\tau+2\varphi+1)^2(\mathfrak{B}_1-\mathfrak{B}_2)\right]}}$$

and

$$|b_3| \leq \frac{\mathfrak{B}_1\eta}{(1+\sigma)}\left[\frac{\mathfrak{B}_1\eta}{(1+\sigma)(\tau+2\varphi+1)^2}+\frac{2}{(2+\sigma)(2\tau+6\varphi+1)}\right]$$

where the coefficients $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are given as in (1.6).

3. Applications of the main result

If we

$$\gamma(z) = \frac{1+(1-2\delta)z}{z}, \quad (z \in \mathfrak{U}, 0 \leq \delta < 1),$$

in Definition 1.2 of the bi-univalent functions class $\Sigma_q^{\tau,\varphi}(\eta,\sigma,\gamma)$, we obtain a new class $\Sigma_q^{\tau,\varphi}(\eta,\sigma,\delta)$ given by Definition 3.1.

Definition 3.1. A function $f \in \Sigma$ is said to be in the class $\Sigma_q^{\tau,\varphi}(\eta,\sigma,\delta)$, if the following contions hold true:

$$\Re\left(1+\frac{1}{\eta}\left[(1-\tau)\frac{\mathfrak{R}_q^\sigma f(z)}{z}+\tau\mathfrak{D}_q(\mathfrak{R}_q^\sigma f(z))+\varphi z(\mathfrak{D}_q(\mathfrak{R}_q^\sigma f(z)))'-1\right]\right) > \delta$$

and

$$\Re\left(1+\frac{1}{\eta}\left[(1-\tau)\frac{\mathfrak{R}_q^\sigma g(w)}{z}+\tau\mathfrak{D}_q(\mathfrak{R}_q^\sigma g(w))+\varphi z(\mathfrak{D}_q(\mathfrak{R}_q^\sigma g(w)))'-1\right]\right) > \delta$$

where $z, w \in \mathfrak{U}$ and $g(w) = f^{-1}(w)$.

Using the parameter setting of Definition 3.1 in the Theorem 2.1, we get the following corollary:

Corollary 3.1. *Let the function $f \in \Sigma_q^{\tau,\varphi}(\eta,\sigma,\delta)$ be of the form (1.1). Then*

$$|b_2| \leq \begin{cases} 2\sqrt{\frac{|\eta|[2]_q(1-\delta)}{2[2+\sigma]_q[1+\sigma]_q(\tau q[2]_q+2\cdot[3]_q\varphi+1)}}, & 0 \leq \delta \leq 1-\frac{[1+\sigma]_q(q\tau+[2]_q\varphi+1)^2}{|\eta|[2+\sigma]_q(\tau q[2]_q+2\cdot[3]_q\varphi+1)}, \\ \frac{2|\eta|(1-\delta)}{[1+\sigma]_q(q\tau+[2]_q\varphi+1)}, & 1-\frac{[1+\sigma]_q(q\tau+[2]_q\varphi+1)^2}{|\eta|[2+\sigma]_q(\tau q[2]_q+2\cdot[3]_q\varphi+1)} \leq \delta < 1. \end{cases} \tag{3.1}$$

and

$$|b_3| \leq \frac{2\eta[2]_q(1-\delta)}{[2+\sigma]_q[1+\sigma]_q(\tau q[2]_q+2\cdot[3]_q\varphi+1)}.$$

Remark 3.1. When $\sigma = 0$, Corollary 3.1 gives us the following corollary below.

Corollary 3.2. *Let the function* $f \in \Sigma_q^{\tau,\varphi}(\eta, 0, \delta) := \Sigma_q^{\tau,\varphi}(\eta, \delta)$ *be of the form* (1.1). *Then*

$$|b_2| \leq \begin{cases} 2\sqrt{\frac{|\eta|(1-\delta)}{2[1]_q(\tau q[2]_q+2\cdot[3]_q\varphi+1)}}, & 0 \leq \delta \leq 1 - \frac{[1]_q(q\tau+[2]_q\varphi+1)^2}{|\eta|[2]_q(\tau q[2]_q+2\cdot[3]_q\varphi+1)}, \\ \frac{2|\eta|(1-\delta)}{[1]_q(q\tau+[2]_q\varphi+1)}, & 1 - \frac{[1]_q(q\tau+[2]_q\varphi+1)^2}{|\eta|[2]_q(\tau q[2]_q+2\cdot[3]_q\varphi+1)} \leq \delta < 1. \end{cases} \tag{3.2}$$

and

$$|b_3| \leq \frac{2\eta(1-\delta)}{[1]_q(\tau q[2]_q + 2 \cdot [3]_q\varphi + 1)}.$$

Remark 3.2. When $q \longrightarrow 1$, Corollary 3.1 gives us Theorem 2 [19], which was studied by Bulut and Wanas.

If we set

$$\gamma(z) = \left(\frac{1+z}{1-z}\right)^{\rho}, \quad (0 < \rho \leq 1, z \in \mathfrak{U}),$$

in Definition 1.2 of the bi-univalent functions class $\Sigma_q^{\tau,\varphi}(\eta, \sigma, \gamma)$, we obtain a new class $\Sigma_q^{\tau,\varphi}(\eta, \sigma, \rho)$ given by Definition 3.2.

Definition 3.2. A function $f \in \Sigma$ is said to be in the class $\Sigma_q^{\tau,\varphi}(\eta, \sigma, \rho)$, if the following contions hold true:

$$\left|\arg\left(1 + \frac{1}{\eta}\left[(1-\tau)\frac{\mathfrak{R}_q^{\sigma} f(z)}{z} + \tau\mathfrak{D}_q(\mathfrak{R}_q^{\sigma} f(z)) + \varphi z(\mathfrak{D}_q(\mathfrak{R}_q^{\sigma} f(z)))' - 1\right]\right)\right| < \frac{\rho\pi}{2}$$

and

$$\left|\arg\left(1 + \frac{1}{\eta}\left[(1-\tau)\frac{\mathfrak{R}_q^{\sigma} g(w)}{z} + \tau\mathfrak{D}_q(\mathfrak{R}_q^{\sigma} g(w)) + \varphi z(\mathfrak{D}_q(\mathfrak{R}_q^{\sigma} g(w)))' - 1\right]\right)\right| < \frac{\rho\pi}{2}$$

where $z, w \in \mathfrak{U}$ and $g(w) = f^{-1}(w)$.

Using the parameter setting of Defintion 3.2 in the Theorem 2.1, we get the corollary below:

Corollary 3.3. *Let the function* $f \in \Sigma_q^{\tau,\varphi}(\eta, \sigma, \rho)$ *be of the form* (1.1). *Then*

$$|b_2| \leq \sqrt{\frac{4\rho^2\eta^2[2]_q}{2\eta\rho[2+\sigma]_q[1+\sigma]_q(\tau q[2]_q + 2\cdot[3]_q\varphi + 1) + (1-\rho)[1+\sigma]_q^2(q\tau + [2]_q\varphi + 1)[2]_q}} \tag{3.3}$$

and

$$|b_3| \leq \frac{2\eta[2]_q\rho}{[2+\sigma]_q[1+\sigma]_q(\tau q[2]_q + 2\cdot[3]_q\varphi + 1)} + \frac{4\eta^2\rho^2}{[1+\sigma]_q^2(q\tau + [2]_q\varphi + 1)^2}.$$

Remark 3.3. When $q \longrightarrow 1$, Corollary 3.3 gives us Theorem 1 [19], which was studied by Bulut and Wanas.

4. Conclusion

The q-calculus is a broad area that has applications in mathematics, physics and other aspects, such as operator theory, quantum group, special functions theory, numerical analysis, differential equation and other similar concepts.

The main goal of this article is to generate the two initial Taylor-Maclaurin coefficient estimates for holomorphic and bi-univalent functions inside the novel subclass $\Sigma_q^{\tau,\varphi}(\eta,\sigma,\gamma)$ in the unit disk. The Ruscheweyh q-calculus operator operator is used to accomplish this. Furthermore, the article show that the results here strengthen and generalize some recent work by using consequences and corollaries, as discussed earlier by sufficient specialization of the parameters.

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