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GENERALIZATION OF BULLEN TYPE, TRAPEZOID TYPE, MIDPOINT TYPE AND SIMPSON TYPE INEQUALITIES FOR S-CONVEX IN THE FOURTH SENSE

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Abstract. The main purpose of this article is to present the Bullen, Midpoint, Trapezoid and Simpson type inequalities, respectively, for s-convex in the fourth sense, with the help of identities existing in the literature.

1. Introduction

Convexity is a basic notion in geometry but also is widely used in other areas of mathematics. It is often hidden in other areas of mathematics: functional analysis, complex analysis, calculus of variations, graph theory, algebraic geometry and many other fields. Many scientists engaged attention and studied this field. Therefore there are many types of convexity in the literature.

The concept of the s-convex function was introduced in Breckners paper [1] and a number of properties and connections with s-convexity in the first sense are discussed in the paper [2]. In this article, s-convex in the fourth sense will be used in this article are as follows.

Many integral inequalities have been developed so far by different researchers in the due course of time. In the literature, we have many types of inequalities that involve convex functions, such as Bullen inequality [3], Hermite–Hadamard-Fejer inequality, Simpson type inequality [4], and Ostrowski type inequalities [5]. In the same way, There are a lot of well-know inequalities but the most notable one is Hermite-Hadamard type integral inequality. Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be an integrable convex function with $a < b$. Then, the Hermite-Hadamard inequality is expressed as follows:

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$$f(\frac{a+b}{2}) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}.$$

The left-hand side of inequality, proved in 1893 by Hadamard [6] before convex functions had been formally introduced, for functions f with f' increasing on $[a,b]$, is sometimes called the Hadamard inequality and the right-hand side is known as the Jensen inequality.

Trapezoid and midpoint inequalities are another known inequalities related to the right and left sides of the Hermite-Hadamard inequality. In order to obtain these inequalities, as stated below, identities play an important role.

In [7], Dragomir and Fitzpatrick proved a variant of Hermite-Hadamard inequality which holds for s-convex function.

Definition 1.1. A function $f : [0,\infty) \longrightarrow \mathbb{R}$ is said to be s-convex function if

$$f(\lambda x + (1-\lambda)y) \le \lambda^s f(x) + (1-\lambda)^s f(y) \tag{1.1}$$

for all $x, y \in [0,\infty), \lambda \in [0,1]$ and for some fixed $s \in (0,1]$.

Definition 1.2. [8] A function $f : [0,\infty) \longrightarrow \mathbb{R}$ is said to be s-convex function in the first sense if

$$f(\lambda x + \gamma y) \le \lambda^s f(x) + \gamma^s f(y) \tag{1.2}$$

for all $x, y \in [0,\infty), \lambda, \gamma \in [0,1]$ and $\lambda^s + \gamma^s = 1$ for some fixed $s \in (0,1]$.

Definition 1.3. [1] A function $f : [0,\infty) \longrightarrow \mathbb{R}$ is said to be s-convex function in the second sense if

$$f(\lambda x + \gamma y) \le \lambda^s f(x) + \gamma^s f(y) \tag{1.3}$$

for all $x, y \in [0,\infty), \lambda, \gamma \in [0,1]$ and $\lambda + \gamma = 1$ for some fixed $s \in (0,1]$.

Definition 1.4. [9] A function $f : [0,\infty) \longrightarrow \mathbb{R}$ is said to be s-convex function in the third sense if

$$f(\lambda x + \gamma y) \le \lambda^{\frac{1}{s}} f(x) + \gamma^{\frac{1}{s}} f(y) \tag{1.4}$$

for all $x, y \in [0,\infty), \lambda, \gamma \in [0,1]$ and $\lambda^s + \gamma^s = 1$ for some fixed $s \in (0,1]$.

Definition 1.5. [10] Let U be a subset of vector space X and let $s \in (0,1]$. A function $f : U \longrightarrow \mathbb{R}$ is said to be s-convex function in the fourth sense if the inequality

$$f(\lambda x + \gamma y) \le \lambda^{\frac{1}{s}} f(x) + \gamma^{\frac{1}{s}} f(y) \tag{1.5}$$

is satisfied for each $x, y \in U$ and for all $\lambda, \gamma \in [0,1]$ such that $\lambda + \gamma = 1$.

Corollary 1.1. [10] *If $f : U \longrightarrow \mathbb{R}$ is s-convex function in the fourth sense, then $f \le 0$.*

Theorem 1.1. [11] *Let $f : U \le \mathbb{R}$ is s-convex function in the fourth sense, where $s \in (0,1]$ and $a, b \in \mathbb{R},\ a < b$. If $f \in L[a,b]$, then the following inequalities hold*

$$2^{\frac{1}{s}-1} f\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{s}{s+1}\big(f(a)+f(b)\big)\ . \tag{1.6}$$

In recent years, many research scholars have focused their great attention on the study of this inequality. The aim of this paper is to establish some new Hermite-Hadamard type inequalities and Simpson type inequalities for s-convex function in the fourth sense.

2. Generalized Bullen Type Inequalities

Bullen-type inequalities for generalized convex functions were obtained in the paper [12].

Theorem 2.1. *Let $f : U \leq \mathbb{R}$ is s-convex function in the fourth sense, where $s \in (0,1]$ and $a, b \in \mathbb{R},\ a < b$. If $f \in L[a,b]$, then the following inequalities hold*

$$\begin{aligned} 2^{\frac{1}{s}-1}\left(f\left(\frac{a+x}{2}\right)+f\left(\frac{x+b}{2}\right)\right) &\leq \frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt \\ &\leq \frac{s}{s+1}\left(f(a)+f(b)+2f(x)\right) \ . \end{aligned}$$

for all $x \in [a,b]$.

Proof. Since $f : U \leq \mathbb{R}$ is s-convex function in the fourth sense on $[a,x] \subset [a,b]$, by using inequalities (1.6) we get

$$I_1 := 2^{\frac{1}{s}-1} f\left(\frac{a+x}{2}\right) \leq \frac{1}{x-a}\int_a^x f(x)dx \leq \frac{s}{s+1}\left(f(a)+f(x)\right)$$

By similar way for $[x,b] \subset [a,b]$, it follows that

$$I_2 := 2^{\frac{1}{s}-1} f\left(\frac{b+x}{2}\right) \leq \frac{1}{b-x}\int_x^b f(x)dx \leq \frac{s}{s+1}(f(b)+f(x))$$

As consequence, by adding I_1 and I_2, we have,

$$\begin{aligned} 2^{\frac{1}{s}-1}\left(f\left(\frac{a+x}{2}\right)+f\left(\frac{x+b}{2}\right)\right) &\leq \frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt \\ &\leq \frac{s}{s+1}\left(f(a)+f(b)+2f(x)\right) \end{aligned}$$

So,this proof completed. □

3. Trapezoid Type Inequalities

Lemma 3.1. [12] *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. $f' \in L[a,b]$, then the following equality holds,*

$$\begin{aligned} &f(x)+\frac{f(a)+f(b)}{2}-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right] \\ &= \frac{x-a}{2}\int_0^1 (1-2\lambda)f'(\lambda a+(1-\lambda)x)d\lambda+\frac{b-x}{2}\int_0^1 (1-2\lambda)f'(\lambda x+(1-\lambda)b)d\lambda \ . \end{aligned}$$

for all $x \in (a,b)$.

Theorem 3.1. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If f' is s-convex function in the fourth sense on $[a, b]$, then*

$$\left| f(x) + \frac{f(a)+f(b)}{2} - \left[\frac{1}{x-a}\int_a^x f(t)dt + \frac{1}{b-x}\int_x^b f(t)dt \right] \right|$$
$$\leq (b-a)\left[\frac{1}{2^{\frac{1}{s}+1}} \frac{s^2}{(s+1)(2s+1)} + \frac{s}{(s+1)(2s+1)} \right] |f'(x)|$$
$$+ \left[\frac{1}{2^{\frac{1}{s}+1}} \frac{s^2}{(s+1)(2s+1)} + \frac{s}{(s+1)(2s+1)} \right] \left((b-x)\,|f'(b)| + (x-a)\,|f'(a)| \right) .$$

for all $x \in [a, b]$.

Proof. From Lemma 3.1, by using the properties of modulus and f' is s-convex function in the fourth sense on $[a, b]$, we have,

$$\left| f(x) + \frac{f(a)+f(b)}{2} - [\frac{1}{x-a}\int_a^x f(t)dt + \frac{1}{b-x}\int_x^b f(t)dt] \right|$$
$$\leq \frac{x-a}{2}\int_0^1 |1-2\lambda||f'(\lambda a + (1-\lambda)x|d\lambda$$
$$+ \frac{b-x}{2}\int_0^1 |1-2\lambda||f'(\lambda x + (1-\lambda)b)|d\lambda$$
$$\leq \frac{x-a}{2}\int_0^1 |1-2\lambda||\lambda^{\frac{1}{s}} f'(a) + (1-\lambda)^{\frac{1}{s}} f'(x)|d\lambda$$
$$+ \frac{b-x}{2}\int_0^1 |1-2\lambda||\lambda^{\frac{1}{s}} f'(x) + (1-\lambda)^{\frac{1}{s}} f'(b)|d\lambda$$
$$\leq \frac{x-a}{2}\int_0^1 |1-2\lambda| \left[\lambda^{\frac{1}{s}}|f'(a)| + (1-\lambda)^{\frac{1}{s}}|f'(x)| \right] d\lambda$$
$$+ \frac{b-x}{2}\int_0^1 |1-2\lambda| \left[\lambda^{\frac{1}{s}}|f'(x)| + (1-\lambda)^{\frac{1}{s}}|f'(b)| \right] d\lambda$$
$$\leq \frac{x-a}{2}\int_0^{\frac{1}{2}} |1-2\lambda| \left[\lambda^{\frac{1}{s}}|f'(a)| + (1-\lambda)^{\frac{1}{s}}|f'(x)| \right] d\lambda$$
$$+ \frac{x-a}{2}\int_{\frac{1}{2}}^1 |2\lambda-1| \left[\lambda^{\frac{1}{s}}|f'(a)| + (1-\lambda)^{\frac{1}{s}}|f'(x)| \right] d\lambda$$
$$+ \frac{b-x}{2}\int_0^{\frac{1}{2}} |1-2\lambda| \left[\lambda^{\frac{1}{s}}|f'(x)| + (1-\lambda)^{\frac{1}{s}}|f'(b)| \right] d\lambda$$
$$+ \frac{b-x}{2}\int_{\frac{1}{2}}^1 |2\lambda-1| \left[\lambda^{\frac{1}{s}}|f'(x)| + (1-\lambda)^{\frac{1}{s}}|f'(b)| \right] d\lambda$$
$$= (b-a)\left[\frac{1}{2^{\frac{1}{s}+1}} \frac{s^2}{(s+1)(2s+1)} + \frac{s}{(s+1)(2s+1)} \right] |f'(x)|$$
$$+ \left[\frac{1}{2^{\frac{1}{s}+1}} \frac{s^2}{(s+1)(2s+1)} + \frac{s}{(s+1)(2s+1)} \right] \left((b-x)\,|f'(b)| + (x-a)\,|f'(a)| \right) .$$

So,this proof completed. □

Theorem 3.2. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If f' is s-convex function in the fourth sense on $[a, b]$ for some $q > 1$, then*

$$\left| f(x) + \frac{f(a)+f(b)}{2} - \left[\frac{1}{x-a}\int_a^x f(t)dt + \frac{1}{b-x}\int_x^b f(t)dt \right] \right|$$

$$\leq \frac{x-a}{2(p+1)^{\frac{1}{p}}} \left[\frac{s}{s+1}|f'(a)|^q + \frac{s}{s+1}|f'(x)|^q \right]^{\frac{1}{q}} + \frac{b-x}{2(p+1)^{\frac{1}{p}}} \left[\frac{s}{s+1}|f'(a)|^q + \frac{s}{s+1}|f'(x)|^q \right]^{\frac{1}{q}}.$$

where $x \in [a, b]$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 3.1, by using Hölder inequality and f' s-convex function in the fourth sense on $[a, b]$, we have

$$\begin{aligned}
&\left| f(x) + \frac{f(a)+f(b)}{2} - \left[\frac{1}{x-a}\int_a^x f(t)dt + \frac{1}{b-x}\int_x^b f(t)dt \right] \right| \\
&\leq \frac{x-a}{2}\int_0^1 |1-2\lambda||f'(\lambda a + (1-\lambda)x|d\lambda \\
&+ \frac{b-x}{2}\int_0^1 |1-2\lambda||f'(\lambda x + (1-\lambda)b)|d\lambda \\
&\leq \frac{x-a}{2}\Big(\int_0^1 |1-2\lambda|^p d\lambda\Big)^{\frac{1}{p}}\Big(\int_0^1 |f'(\lambda a + (1-\lambda)x)|^q d\lambda\Big)^{\frac{1}{q}} \\
&+ \frac{b-x}{2}\Big(\int_0^1 |1-2\lambda|^p d\lambda\Big)^{\frac{1}{p}}\Big(\int_0^1 |f'(\lambda x + (1-\lambda)b)|^q d\lambda\Big)^{\frac{1}{q}} \\
&\leq \frac{x-a}{2}\Big(\int_0^1 |1-2\lambda|^p d\lambda\Big)^{\frac{1}{p}}\Big(\int_0^1 |\lambda^{\frac{1}{s}} f'(a) + (1-\lambda)^{\frac{1}{s}} f'(x)|^q d\lambda\Big)^{\frac{1}{q}} \\
&+ \frac{b-x}{2}\Big(\int_0^1 |1-2\lambda|^p d\lambda\Big)^{\frac{1}{p}}\Big(\int_0^1 |\lambda^{\frac{1}{s}} f'(x) + (1-\lambda)^{\frac{1}{s}} f'(b)|^q d\lambda\Big)^{\frac{1}{q}} \\
&\leq \frac{x-a}{2(p+1)^{\frac{1}{p}}}\Big(\int_0^1 (\lambda^{\frac{1}{s}}|f'(a)|^q + (1-\lambda)^{\frac{1}{s}}|f'(x)|^q)d\lambda\Big)^{\frac{1}{q}} \\
&+ \frac{b-x}{2(p+1)^{\frac{1}{p}}}\Big(\int_0^1 (\lambda^{\frac{1}{s}}|f'(x)|^q + (1-\lambda)^{\frac{1}{s}}|f'(b)|^q)d\lambda\Big)^{\frac{1}{q}} \\
&= \frac{x-a}{2(p+1)^{\frac{1}{p}}} \left[\frac{s}{s+1}|f'(a)|^q + \frac{s}{s+1}|f'(x)|^q \right]^{\frac{1}{q}} \\
&+ \frac{b-x}{2(p+1)^{\frac{1}{p}}} \left[\frac{s}{s+1}|f'(a)|^q + \frac{s}{s+1}|f'(x)|^q \right]^{\frac{1}{q}} .
\end{aligned}$$

So, this proof completed. □

4. Midpoint Type Inequalities

Lemma 4.1. [12] *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. $f' \in L[a, b]$, then the following equality holds,*

$$f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]$$
$$=(x-a)\int_0^{\frac{1}{2}}\lambda\left[f'(\lambda x+(1-\lambda)a)-f'(\lambda a+(1-\lambda)x)\right]d\lambda$$
$$+(b-x)\int_{\frac{1}{2}}^1(1-\lambda)\left[f'(\lambda x+(1-\lambda)b)-f'(\lambda b+(1-\lambda)x)\right]d\lambda\ .$$

Theorem 4.1. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If f' is s-convex function in the fourth sense on $[a, b]$, then*

$$\left|f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right|$$
$$\leq\frac{s^2}{(s+1)(2s+1)}\left(1-\frac{1}{2^{\frac{1}{s}+1}}\right)\left[(x-a)|f'(a)|+(b-x)|f'(b)|\right]$$
$$+\frac{s^2}{(s+1)(2s+1)}\left(1-\frac{1}{2^{\frac{1}{s}+1}}\right)\left[(b-a)|f'(x)|\right].$$

for all $x \in [a, b]$.

Proof. From Lemma 4.1 by using the properties of modulus and f' is s-convex function in the fourth sense on $[a, b]$, we have,

$$\left|f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right|$$
$$\leq(x-a)\int_0^{\frac{1}{2}}\lambda\left[|f'(\lambda x+(1-\lambda)a)|+|f'(\lambda a+(1-\lambda)x)|\right]d\lambda$$
$$+(b-x)\int_{\frac{1}{2}}^1(1-\lambda)\left[|f'(\lambda x+(1-\lambda)b)|+|f'(\lambda b+(1-\lambda)x)|\right]d\lambda$$
$$\leq(x-a)\int_0^{\frac{1}{2}}\lambda\left[|\lambda^{\frac{1}{s}}f'(x)+(1-\lambda)^{\frac{1}{s}}f'(a))|+|\lambda^{\frac{1}{s}}f'(a)+(1-\lambda)^{\frac{1}{s}}f'(x)|\right]d\lambda$$
$$+(b-x)\int_{\frac{1}{2}}^1(1-\lambda)\left[\left|\lambda^{\frac{1}{s}}f'(x)+(1-\lambda)^{\frac{1}{s}}f'(b)\right|+\left|\lambda^{\frac{1}{s}}f'(b)+(1-\lambda)^{\frac{1}{s}}f'(x)\right|\right]d\lambda$$
$$\leq(x-a)\int_0^{\frac{1}{2}}\lambda\left[\lambda^{\frac{1}{s}}|f'(x)|+(1-\lambda)^{\frac{1}{s}}|f'(a)|+\lambda^{\frac{1}{s}}|f'(a)|+(1-\lambda)^{\frac{1}{s}}|f'(x)|\right]d\lambda$$
$$+(b-x)\int_{\frac{1}{2}}^1\lambda\left[\lambda^{\frac{1}{s}}|f'(x)|+(1-\lambda)^{\frac{1}{s}}|f'(b)|+\lambda^{\frac{1}{s}}|f'(b)|+(1-\lambda)^{\frac{1}{s}}|f'(x)|\right]d\lambda$$

$$
\begin{aligned}
&= \frac{s^2}{(s+1)(2s+1)}\left(1-\frac{1}{2^{\frac{1}{s}+1}}\right)\left[(x-a)|f'(a)|+(b-x)|f'(b)|\right] \\
&+ \frac{s^2}{(s+1)(2s+1)}\left(1-\frac{1}{2^{\frac{1}{s}+1}}\right)\left[(b-a)|f'(x)|\ \right] \ .
\end{aligned}
$$

and completes the proof of this theorem.

□

Theorem 4.2. *Let $f:[a,b]\longrightarrow\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If f' is s-convex function in the fourth sense on $[a,b]$ for some $q>1$, then*

$$
\begin{aligned}
&\left|f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| \\
&\leq \frac{x-a}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}}\left[\left(\frac{s}{(1+s)2^{\frac{1}{s}+1}}|f'(x)|^q+\left(1-\frac{1}{2^{1+\frac{1}{s}}}\right)\frac{s}{s+1}|f'(a)|^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{s}{(1+s)2^{\frac{1}{s}+1}}|f'(a)|^q+\left(1-\frac{1}{2^{1+\frac{1}{s}}}\right)\frac{s}{s+1}|f'(x)|^q\right)^{\frac{1}{q}}\right] \\
&+\frac{b-a}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}}\left[\left(\frac{s}{(1+s)2^{\frac{1}{s}+1}}|f'(x)|^q+\left(1-\frac{1}{2^{1+\frac{1}{s}}}\right)\frac{s}{s+1}|f'(b)|^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{s}{(1+s)2^{\frac{1}{s}+1}}|f'(b)|^q+\left(1-\frac{1}{2^{1+\frac{1}{s}}}\right)\frac{s}{s+1}|f'(x)|^q\right)^{\frac{1}{q}}\right]
\end{aligned}
$$

where $x\in[a,b]$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 4.1, by using Hölder inequality and f' s-convex function in the fourth sense on $[a,b]$, we have

$$
\begin{aligned}
&\left|f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| \\
&\leq (x-a)\left(\int_0^{\frac{1}{2}}\lambda^p d\lambda\right)^{\frac{1}{p}}\left[\left(\int_0^{\frac{1}{2}}|f'(\lambda x+(1-\lambda)a)|^q d\lambda\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^{\frac{1}{2}}|f'(\lambda a+(1-\lambda)x)|^q d\lambda\right)^{\frac{1}{q}}\right] \\
&+(b-x)\left(\int_{\frac{1}{2}}^1(1-\lambda)^p d\lambda\right)^{\frac{1}{p}}\left[\left(\int_{\frac{1}{2}}^1|f'(\lambda x+(1-\lambda)b)|^q d\lambda\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1|f'(\lambda b+(1-\lambda)x)|^q d\lambda\right)^{\frac{1}{q}}\right]
\end{aligned}
$$

$$\leq (x-a)\left(\int_0^{\frac{1}{2}} \lambda^p d\lambda\right)^{\frac{1}{p}} \left[\left(\int_0^{\frac{1}{2}} |\lambda^{\frac{1}{s}} f'(x) + (1-\lambda)^{\frac{1}{s}} f'(a)|^q d\lambda\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\int_0^{\frac{1}{2}} |\lambda^{\frac{1}{s}} f'(a) + (1-\lambda)^{\frac{1}{s}} f'(x)|^q d\lambda\right)^{\frac{1}{q}}\right]$$
$$+ (b-x)\left(\int_{\frac{1}{2}}^1 (1-\lambda)^p d\lambda\right)^{\frac{1}{p}} \left[\left(\int_{\frac{1}{2}}^1 |\lambda^{\frac{1}{s}} f'(x) + (1-\lambda)^{\frac{1}{z}} f'(b)|^q d\lambda\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\int_{\frac{1}{2}}^1 |\lambda^{\frac{1}{s}} f'(b) + (1-\lambda)^{\frac{1}{s}} f'(x)|^q d\lambda\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{x-a}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}} \left[\left(\int_0^{\frac{1}{2}} \lambda^{\frac{1}{s}} |f'(x)|^q + (1-\lambda)^{\frac{1}{s}} |f'(a)|^q d\lambda\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\int_0^{\frac{1}{2}} \lambda^{\frac{1}{s}} |f'(a)|^q + (1-\lambda)^{\frac{1}{s}} |f'(x)|^q d\lambda\right)^{\frac{1}{q}}\right]$$
$$+ \frac{b-x}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}} \left[\left(\int_{\frac{1}{2}}^1 \lambda^{\frac{1}{s}} |f'(x)|^q + (1-\lambda)^{\frac{1}{s}} |f'(b)|^q d\lambda\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\int_{\frac{1}{2}}^1 \lambda^{\frac{1}{s}} |f'(b)|^q + (1-\lambda)^{\frac{1}{s}} |f'(x)|^q d\lambda\right)^{\frac{1}{q}}\right]$$
$$= \frac{x-a}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}} \left[\left(\frac{s}{(1+s)2^{\frac{1}{s}+1}} |f'(x)|^q + \left(1 - \frac{1}{2^{1+\frac{1}{s}}}\right) \frac{s}{s+1} |f'(a)|^q\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\frac{s}{(1+s)2^{\frac{1}{s}+1}} |f'(a)|^q + \left(1 - \frac{1}{2^{1+\frac{1}{s}}}\right) \frac{s}{s+1} |f'(x)|^q\right)^{\frac{1}{q}}\right]$$
$$+ \frac{b-a}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}} \left[\left(\frac{s}{(1+s)2^{\frac{1}{s}+1}} |f'(x)|^q + \left(1 - \frac{1}{2^{1+\frac{1}{s}}}\right) \frac{s}{s+1} |f'(b)|^q\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\frac{s}{(1+s)2^{\frac{1}{s}+1}} |f'(b)|^q + \left(1 - \frac{1}{2^{1+\frac{1}{s}}}\right) \frac{s}{s+1} |f'(x)|^q\right)^{\frac{1}{q}}\right].$$

So, the proof completed. □

5. Simpson Type Inequalities

Lemma 5.1. [12] *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. $f' \in L[a,b]$, then the following equality holds,*

$$\frac{1}{3}\left[2f\left(\frac{a+x}{2}\right) + 2f\left(\frac{b+x}{2}\right) + f(x) + \frac{f(a)+f(b)}{2}\right]$$
$$- \left[\frac{1}{x-a}\int_a^x f(t)dt + \frac{1}{b-x}\int_x^b f(t)dt\right]$$

$$= (x-a)\int_0^{\frac{1}{2}}\left(\lambda-\frac{1}{6}\right)\left[f'(\lambda x+(1-\lambda)a)-f'(\lambda a+(1-\lambda)x)\right]d\lambda$$
$$+(b-x)\int_{\frac{1}{2}}^{1}\left(\frac{5}{6}-\lambda\right)\left[f'(\lambda x+(1-\lambda)b)-f'(\lambda b+(1-\lambda)x)\right]d\lambda\ .$$

for all $x\in[a,b]$.

Theorem 5.1. *Let* $f:[a,b]\longrightarrow\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a<b$. *If* f' *is s-convex function in the fourth sense on* $[a,b]$, *then*

$$\left|\frac{1}{3}\left[2f\left(\frac{a+x}{2}\right)+2f\left(\frac{b+x}{2}\right)+f(x)+\frac{f(a)+f(b)}{2}\right]\right.$$
$$\left.-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right|$$
$$\leq\frac{1}{6}\left[\frac{4s^2-s}{(2s+1)(s+1)}+\frac{3s^2}{2^{\frac{1}{s}}(2s+1)(s+1)}\right]\left((b-x)\left|f'(b)\right|+(x-a)\left|f'(a)\right|\right)$$
$$+\frac{1}{6}\left[\frac{4s^2-s}{(2s+1)(s+1)}+\frac{3s^2}{2^{\frac{1}{s}}(2s+1)(s+1)}\right](b-a)\left|f'(x)\right|\ .$$

for all $x\in[a,b]$.

Proof. From Lemma 5.1 by using the properties of modulus and f' is s-convex function in the fourth sense on $[a,b]$, we have

$$\left|\frac{1}{3}\left[2f\left(\frac{a+x}{2}\right)+2f\left(\frac{b+x}{2}\right)+f(x)+\frac{f(a)+f(b)}{2}\right]\right.$$
$$\left.-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right|$$
$$\leq(x-a)\int_0^{\frac{1}{2}}|\lambda-\frac{1}{6}|\left[|f'(\lambda x-(1-\lambda)a)|+|f'(\lambda a+(1-\lambda)x)|\right]d\lambda$$
$$+(b-x)\int_{\frac{1}{2}}^{1}|\frac{5}{6}-\lambda|\left[|f'(\lambda x-(1-\lambda)b)|+|f'(\lambda b+(1-\lambda)x)|\right]d\lambda$$

$$\leq(x-a)\int_0^{\frac{1}{2}}|\lambda-\frac{1}{6}|\left[\lambda^{\frac{1}{s}}|f'(x)|+(1-\lambda)^{\frac{1}{s}}|f'(a)|+\lambda^{\frac{1}{s}}|f'(a)|+(1-\lambda)^{\frac{1}{s}}|f'(x)|\right]$$
$$+(b-x)\int_{\frac{1}{2}}^{1}|\frac{5}{6}-\lambda|\left[\lambda^{\frac{1}{s}}|f'(x)|+(1-\lambda)^{\frac{1}{s}}|f'(b)|+\lambda^{\frac{1}{s}}|f'(b)|+(1-\lambda)^{\frac{1}{s}}|f'(x)|\right]$$
$$\leq\frac{1}{6}\left[\frac{4s^2-s}{(2s+1)(s+1)}+\frac{3s^2}{2^{\frac{1}{s}}(2s+1)(s+1)}\right]((b-x)|f'(b)|+(x-a)|f'(a)|)$$
$$+\frac{1}{6}\left[\frac{4s^2-s}{(2s+1)(s+1)}+\frac{3s^2}{2^{\frac{1}{s}}(2s+1)(s+1)}\right](b-a)\left|f'(x)\right|\ .$$

So, the proof completed. □

Theorem 5.2. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If f' is s-convex function in the fourth sense on $[a,b]$ for some $q > 1$, then*

$$
\begin{aligned}
&\left| \frac{1}{3}\left[2f\left(\frac{a+x}{2}\right) + 2f\left(\frac{b+x}{2}\right) + f(x) + \frac{f(a)+f(b)}{2}\right] \right. \\
&\left. - \left[\frac{1}{x-a}\int_a^x f(t)dt + \frac{1}{b-x}\int_x^b f(t)dt\right]\right| \\
&\le \frac{x-a}{(1+p)^{\frac{1}{p}}}\left(\frac{1}{3^{p+1}}\frac{1}{6^{p+1}}\right)\left\{\left(\frac{s}{s+1}\left(\frac{1}{2}^{1+\frac{1}{s}} - 1\right)(|f'(x)|^q - |f'(a)|^q)\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\frac{s}{s+1}\left(\frac{1}{2}^{1+\frac{1}{s}} - 1\right)(|f'(a)|^q - |f'(x)|^q)\right)^{\frac{1}{q}}\right\} \\
&+ \frac{b-x}{(1+p)^{\frac{1}{p}}}\left(\frac{1}{3^{p+1}}\frac{1}{6^{p+1}}\right)\left\{\left(\frac{s}{s+1}\left(\frac{1}{2}^{1+\frac{1}{s}} - 1\right)(|f'(x)|^q - |f'(b)|^q)\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\frac{s}{s+1}\left(\frac{1}{2}^{1+\frac{1}{s}} - 1\right)(|f'(b)|^q - |f'(x)|^q)\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

where $x \in [a,b]$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 5.1, by using Hölder inequality and f' s-convex function in the fourth sense on $[a,b]$, we have

$$
\begin{aligned}
&\left| \frac{1}{3}\left[2f\left(\frac{a+x}{2}\right) + 2f\left(\frac{b+x}{2}\right) + f(x) + \frac{f(a)+f(b)}{2}\right] \right. \\
&\left. - \left[\frac{1}{x-a}\int_a^x f(t)dt + \frac{1}{b-x}\int_x^b f(t)dt\right]\right| \\
&\le (x-a)\left(\int_0^{\frac{1}{2}} |\lambda - \frac{1}{6}|^p d\lambda\right)^{\frac{1}{p}}\left[\left(\int_0^{\frac{1}{2}} |f'(\lambda x + (1-\lambda)a)|^q d\lambda\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\int_0^{\frac{1}{2}} |f'(\lambda a + (1-\lambda)x)|^q d\lambda\right)^{\frac{1}{q}}\right] \\
&+ (b-x)\left(\int_0^{\frac{1}{2}} |\frac{5}{6} - \lambda|^p d\lambda\right)^{\frac{1}{p}}\left[\left(\int_{\frac{1}{2}}^1 |f'(\lambda x + (1-\lambda)b)|^q d\lambda\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\int_{\frac{1}{2}}^1 |f'(\lambda b + (1-\lambda)x)|^q d\lambda\right)^{\frac{1}{q}}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{(1+p)^{\frac{1}{p}}}\left(\frac{1}{3^{p+1}}\frac{1}{6^{p+1}}\right)\left\{(x-a)\left[\left(\int_0^{\frac{1}{2}} \lambda^{\frac{1}{s}}|f'(x)|^q+(1-\lambda)^{\frac{1}{s}}|f'(a)|^q d\lambda\right)^{\frac{1}{q}}\right.\right.\\
&\left.+\left(\int_0^{\frac{1}{2}} \lambda^{\frac{1}{s}}|f'(a)|^q+(1-\lambda)^{\frac{1}{s}}|f'(x)|^q d\lambda\right)^{\frac{1}{q}}\right]\\
&+(b-x)\left[\left(\int_{\frac{1}{2}}^1 \lambda^{\frac{1}{s}}|f'(x)|^q+(1-\lambda)^{\frac{1}{s}}|f'(b)|^q d\lambda\right)^{\frac{1}{q}}\right.\\
&\left.\left.+\left(\int_{\frac{1}{2}}^1 \lambda^{\frac{1}{s}}|f'(b)|^q+(1-\lambda)^{\frac{1}{s}}|f'(x)|^q d\lambda\right)^{\frac{1}{q}}\right]\right\}\\
&\leq \frac{x-a}{(1+p)^{\frac{1}{p}}}\left(\frac{1}{3^{p+1}}\frac{1}{6^{p+1}}\right)\left\{\left(\frac{s}{s+1}\left(\frac{1}{2}^{1+\frac{1}{s}}-1\right)(|f'(x)|^q-|f'(a)|^q)\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{s}{s+1}\left(\frac{1}{2}^{1+\frac{1}{s}}-1\right)(|f'(a)|^q-|f'(x)|^q)\right)^{\frac{1}{q}}\right\}\\
&+\frac{b-x}{(1+p)^{\frac{1}{p}}}\left(\frac{1}{3^{p+1}}\frac{1}{6^{p+1}}\right)\left\{\left(\frac{s}{s+1}\left(\frac{1}{2}^{1+\frac{1}{s}}-1\right)(|f'(x)|^q-|f'(b)|^q)\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{s}{s+1}\left(\frac{1}{2}^{1+\frac{1}{s}}-1\right)(|f'(b)|^q-|f'(x)|^q)\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

So, the proof completed. □

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