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SUMS OF FOUR PELL NUMBERS AS POWERS OF 3

JACOB JUILLERAT[1]

ABSTRACT. Using lower bounds for the absolute value of linear forms in logarithms, a version of the Baker-Davenport reduction method, and properties of continued fractions of irrational numbers, we find all solutions to the Diophantine equation $P_{n_1}+P_{n_2}+P_{n_3}+P_{n_4} = 3^a$.

1. INTRODUCTION

The Pell sequence $\{P_n\}_{n\geq 0}$ is the binary recurrence sequence given by $P_0 = 0$, $P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$. In 1991, A. Pethő [20] found all the perfect powers (with exponent greater than 1) in the Pell sequence. That is, Pethő found all positive solutions (n, q, x) with $q \geq 2$ to the Diophantine equation

$$P_n = x^q.$$

The study of Diophantine equations involving binary recurrence sequences has since expanded. In 2014 J.J. Bravo and F. Luca [6] found all positive solutions to $P_n + P_m = 2^a$, and in 2016 [5] found all positive solutions to $F_n + F_m = 2^a$ where $\{F_n\}_{n\geq 0}$ is the Fibonacci sequence. In 2015 E.F. Bravo and J.J. Bravo [2] found all positive solutions to $F_n + F_m + F_k = 2^a$ and then in 2017 [3] found all positive solutions to $P_n + P_m + P_k = 2^a$. In 2021 Tiebekabe and Diouf [23] extended the Fibonacci case to the sum of four terms, and then the sum of five terms in 2022 [25]. In 2021, A. Çağman [7] moved away from 2^a and found all positive solutions to $P_n + P_m + P_k = 3^a$. Tiebekabe and Diouf [24] one year later shifted to the Lucas sequence given by $L_0 = 2$, $L_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$ for $n \geq 0$ and found all solutions to $L_n + L_m = 3^a$.

Alongside the sum of terms of various Lucas sequences has been a study of the difference of two terms. In 2017 Z. Şiar and R. Keskin [21] studied the Diophantine equation $F_n - F_m = 2^a$, which has since been extended by F. Erduvan and R. Keskin [13] to look at powers of 5 and by G. Anouar and M. Soufiane in 2023 [1] to look at powers of 7 and powers of 13. In 2021

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S. Kebli, O. Kihel, J. Larone, and F. Luca [15] showed that there are only finitely many solutions to $F_n \pm F_m = y^a$ with $n \geq m \geq 0$, $y \geq 2$ and $a \geq 2$. In 2021 A. Çağman and K. Polat [10] extended the difference of Fibonacci numbers to a difference of Pell numbers and found all positive solutions to $P_n - P_m = 3^a$.

Another large area of focus has been on repdigits (short for repeated digits) and their correspondance to binary recurrence sequences. For example, repdigits that are sums of three Fibonacci numbers were found in [16] by F. Luca, and was later extended to four Fibonacci or Lucas numbers in [19] by B.V. Normenyo, F. Luca, and A. Togbé. Repdigits that can be expressed as the sum of three Half-companion Pell numbers were found in [9] by A. Çağman who in 2023 [8] went on to find all repdigits expressible as a product of a Fibonacci number and a Pell number. Lucas numbers that are concatenations of two repdigits were investigated in [27] by B.P. Tripathy and B.K. Patel and by Y. Qu and J. Zeng in [29]. Padovan numbers which are palindromic concatenations of two distinct repdigits were found in [11] by T.P. Chalebgwa and M. Ddamulira.

Various problems have been investigated involving Fibonacci and Lucas sequences. Factorials which are sums of at most three Fibonacci numbers were found in [17] by F. Luca and S. Siksek. The sum of two Fibonacci numbers that are close to a power of 2 were studied by E. Hasanalizade in 2022 [14]. This was extended to the sum of three Fibonacci numbers in 2023 by B.P. Tripathy and B.K. Patel [28]. B.P. Tripathy and B.K. Patel also found the common terms between a generalized Pell sequence and Narayana's cows sequences, a ternary recurrent sequence given by $N_{m+3} = N_{m+2} + N_m$ with initial conditions $N_0 = N_1 = N_2 = 1$ in [26].

In this paper, we continue looking at sums of Pell numbers which can be expressed as a power of three and prove the following.

Theorem 1.1. *Let $\{P_n\}_{n\geq 0}$ be the Pell sequence defined by $P_0 = 0$, $P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$, and let n_1, n_2, n_3, n_4, and a be nonnegative integers such that $n_1 \geq n_2 \geq n_3 \geq n_4$. Then the Diophantine equation*

$$P_{n_1} + P_{n_2} + P_{n_3} + P_{n_4} = 3^a \tag{1.1}$$

has exactly 10 solutions, which are as follows

$$\begin{aligned}(n_1, n_2, n_3, n_4, a) \in \{&(1,0,0,0,0), (1,1,1,0,1), (2,1,0,0,1), (3,2,1,1,2),\\ &(3,2,2,0,2), (4,3,3,3,3), (4,4,2,1,3),\\ &(6,3,3,1,4), (7,6,2,2,5), (12,11,6,4,9)\}.\end{aligned}$$

Section 2 discusses some properties of the Pell sequence, some necessary results on upper bounds for the absolute value of linear forms in logarithms, and properties of convergents of a continued fraction of irrational numbers. Section 3 starts the proof of Theorem 1.1 by finding an upper bound on values of n_1 that can satisfy (1.1). Section 4 reduces this upper bound to the point that we can run a brute force check to find all the solutions to (1.1). Throughout the paper, computations were made in Maple 2019 and verified in Mathematica 13.0.

2. Preliminary Results

Let $\{P_n\}_{n\geq 0}$ be the Pell sequence defined by $P_0 = 0$, $P_1 = 1$, and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$. The proof of Theorem 1.1 is broken into two sections. Section 3 begins with a brute force check to find the small values of n_1 that can satisfy (1.1). Next, we find a relationship between a and n_1 using the well-known inequalities

$$\alpha^{n-2} \leq P_n \leq \alpha^{n-1} \tag{2.1}$$

which hold for $n \geq 1$. To simplify the arguments to come, notice that the roots of the characteristic polynomial of the Pell sequence, $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$, satisfy

$$2 < \alpha = 1 + \sqrt{2} < 3 \qquad \text{and} \qquad |\beta|^m < \frac{\sqrt{2}}{3} \qquad \text{for all } m > 1. \tag{2.2}$$

To get a rather large upper bound on the values of n_1 that can satisfy (1.1), we employ Binet's formula for Pell numbers

$$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}} \text{ for all } n \geq 0, \tag{2.3}$$

to manipulate (1.1) by first rewriting P_{n_1} using Binet's formula; then rewriting P_{n_1} and P_{n_2}; then P_{n_1}, P_{n_2}, and P_{n_3}; and finally rewriting each P_{n_i} using Binet's formula. For each manipulation of (1.1) we use a version of the Baker-Davenport reduction method that requires the following definition.

Definition 2.1. For a non-zero algebraic number of degree d whose minimal polynomial in $\mathbb{Z}$ is $f(x) = a_d \prod_{i=1}^{d}(x - \xi_i)$, we define the *logarithmic height of* ξ to be

$$h(\xi) = \frac{1}{d}\left(\log|a_d| + \sum_{i=1}^{d} \log(\max\{|\xi_1|, 1\})\right)$$

where $\log(\cdot)$ denotes the natural logarithm.

The following properties found in [22] will assist in the calculations of logarithmic heights.

Proposition 2.1. *Let $\xi_1, \xi_2, \ldots, \xi_t$ be elements of an algebraic closure of $\mathbb{Q}$ and $m \in \mathbb{Z}$. Then*

(a) $h(\xi_1 \cdots \xi_t) \leq \sum_{i=1}^{t} h(\xi_i)$

(b) $h(\xi_1 + \cdots + \xi_t) \leq \log t + \sum_{i=1}^{t} h(\xi_i)$

(c) $h(\xi^m) = |m| h(\xi)$.

Each time we rewrite (1.1) using Binet's formula, we use the following theorem due to Matveev [18] to eliminate the dependence on a.

Theorem 2.1. *Let $\gamma_1, \gamma_2, \ldots, \gamma_s$ be nonzero elements of a real algebraic number field $\mathbb{F}$ of degree D, and let $b_1, b_2, \ldots, b_s$ be rational integers. Set*

$$B := \max\{|b_1|, \ldots, |b_s|\}$$

and

$$\Lambda := \gamma_1^{b_1}\gamma_2^{b_2}\cdots\gamma_s^{b_s} - 1.$$

Let $A_1, \ldots, A_s$ be real numbers such that

$$A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}$$

for all $1 \leq i \leq s$ where $h(\gamma_i)$ is the logarithmic height of γ_i. Then if Λ is nonzero, then

$$\log|\Lambda| > -3 \cdot 30^{s+4} \cdot (s+1)^{5.5} \cdot D^2 \cdot (1+\log D) \cdot (1+\log sB) \cdot A_1 \cdots A_s.$$

Furthermore, if $\mathbb{F} = \mathbb{R}$, then

$$\log|\Lambda| > -1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot D^2 \cdot (1+\log D) \cdot (1+\log B) \cdot A_1 \cdots A_s.$$

In Section 4, we reduce this large upper bound on n_1 by rewriting the inequalities found in Section 3 when manipulating (1.1) using Binet's formula. For each new inequality, we use one of two methods. We either use the following result on convergnets of the continued fraction of an irrational number due to Bravo and Luca [4], a variation of a result of Dujella and Pethő [12], to find a reduced upper bound on n_1.

Lemma 2.1. *Let A, B, and μ be some real numbers with $A > 0$ and $B > 1$, and let γ be an irrational number and M be a positive integer. Take p/q as a convergent of the continued fraction of γ such that $q > 6M$. Set $\epsilon := ||\mu q|| - M||\gamma q|| > 0$ where $||\cdot||$ denotes the distance from the nearest integer. Then there is no solution to the inequality*

$$0 < |u\gamma - v + \mu| < AB^{-w}$$

in positive integers u, v, and w with

$$u \leq M \qquad \text{and} \qquad w \geq \frac{\log \frac{AB}{\epsilon}}{\log B}.$$

When $\epsilon < 0$ in Lemma 2.1, we use the following result due to Legendre.

Lemma 2.2. *Let τ be a real number with x, y integers such that*

$$\left|\tau - \frac{x}{y}\right| < \frac{1}{2y^2},$$

then $\dfrac{x}{y} = \dfrac{p_n}{q_n}$ is a convergent of τ.

Knowing that x/y is a convergent of τ, we use the lower bound

$$\left|\tau - \frac{p_n}{q_n}\right| > \frac{1}{(a_{\max}+2)q_n^2}, \tag{2.4}$$

to find a reduced upper bound on n_1, where $p_n/q_n = [a_0; a_1, a_2, \ldots, a_n]$ and $a_{\max} = \max\{a_i\}$ for $0 \leq i \leq n$.

3. First Bound on n_1

Proof of Theorem 1.1. Observe that if $n_4 = 0$, equation (1.1) becomes

$$P_{n_1} + P_{n_2} + P_{n_3} = 3^a,$$

which is found to have solutions

$$(n_1, n_2, n_3, a) \in \{(1,0,0,0), (1,1,1,1), (2,1,0,1), (3,2,2,2)\}$$

in [7]. Assume that $n_1 \geq n_2 \geq n_3 \geq n_4 \geq 1$. Using Maple, a brute force search on all $n_1 \leq 197$ found that (1.1) has solutions precisely those stated in Theorem 1.1.

Assume that $n_1 > 197$. We now find a relationship between n_1 and a. From (2.1) and (2.2), equation (1.1) gives

$$\begin{aligned}
3^a &= P_{n_1} + P_{n_2} + P_{n_3} + P_{n_4} \\
&\leq \alpha^{n_1-1} + \alpha^{n_2-1} + \alpha^{n_3-1} + \alpha^{n_4-1} \\
&< 3^{n_1-1}\left(1 + 3^{n_2-n_1} + 3^{n_3-n_1} + 3^{n_4-n_1}\right) \\
&< 3^{n_1-1} \cdot 4.
\end{aligned}$$

Solving for a, we get $a < n_1 + \log_3 4 - 1 < n_1 + 0.27$. Since $a, n_1 \in \mathbb{Z}$, we have

$$a \leq n_1.$$

A first bound on $n_1 - n_2$:

Using Binet's formula, (2.3), on P_{n_1}, we rewrite (1.1) as

$$\frac{\alpha^{n_1}}{2\sqrt{2}} - 3^a = \frac{\beta^{n_1}}{2\sqrt{2}} - (P_{n_2} + P_{n_3} + P_{n_4}).$$

Taking the absolute value of both sides and using (2.2) and (2.1), we obtain

$$\left|\frac{\alpha^{n_1}}{2\sqrt{2}} - 3^a\right| \leq \frac{|\beta|^{n_1}}{2\sqrt{2}} + P_{n_2} + P_{n_3} + P_{n_4} < \frac{1}{6} + \alpha^{n_2} + \alpha^{n_3} + \alpha^{n_4}.$$

Dividing both sides by $\alpha^{n_1}/2\sqrt{2}$ and noting that $n_1 \geq n_2 \geq n_3 \geq n_4 \geq 1$, we have

$$\begin{aligned}
|\Delta_1| := \left|1 - 3^a\alpha^{-n_1}2\sqrt{2}\right| &< \frac{2\sqrt{2}}{\alpha^{n_1}}\left(\frac{1}{6} + \alpha^{n_2} + \alpha^{n_3} + \alpha^{n_4}\right) \\
&= \frac{2\sqrt{2}}{\alpha^{n_1-n_2}}\left(\frac{1}{6}\alpha^{-n_2} + 1 + \alpha^{n_3-n_2} + \alpha^{n_4-n_2}\right) \\
&\leq \frac{2\sqrt{2}}{\alpha^{n_1-n_2}} \cdot \frac{19}{6} < \frac{9}{\alpha^{n_1-n_2}} \qquad (3.1)
\end{aligned}$$

Our aim is to now apply Theorem 2.1 with Δ_1 defined above. Note that $\Delta_1 \neq 0$, for if it was zero, then $3^a 2\sqrt{2} = \alpha^{n_1}$. Squaring both sides results in $\alpha^{2n_1} = 3^{2a} \cdot 8 \in \mathbb{Z}$. Looking at the binomial expansion of $\alpha^k = (1+\sqrt{2})^k$, we see that $\alpha^k \notin \mathbb{Z}$ for any positive integer k, providing a contradiction. Hence, we have that $\Delta_1 \neq 0$. Applying Theorem 2.1, set $s := 3$, $(\gamma_1, \gamma_2, \gamma_3) := (3, \alpha, 2\sqrt{2})$, and $(b_1, b_2, b_3) := (a, -n_1, 1)$. Then $|\Delta_1| = \left|1 - \gamma_1^{b_1}\gamma_2^{b_2}\gamma_3^{b_3}\right|$. Since each $\gamma_i \in \mathbb{Q}(\sqrt{2})$, we can take $D := 2$. Since $a \le n_1$, take $B := \max\{|b_i|\} = n_1$. From the definition of $h(\gamma_i)$,

$$\begin{aligned} Dh(3) &= 2\log 3, \\ Dh(\alpha) &= 2\cdot\frac{1}{2}\left(\log 1 + \log\alpha + \log\max\left(|\beta|, 1\right)\right) = \log\alpha, \\ Dh(2\sqrt{2}) &= 2\cdot\frac{1}{2}\left(\log 1 + \log 2\sqrt{2} + \log|-2\sqrt{2}|\right) = 2\log 2\sqrt{2}. \end{aligned}$$

Take

$$A_1 := 2.2 > Dh(3), \quad A_2 := 0.9 > Dh(\alpha), \quad \text{and} \quad A_3 := 2.1 > Dh(2\sqrt{2}).$$

Noting that $\mathbb{Q}(\sqrt{2})$ is real, Theorem 2.1 along with (3.1) gives

$$\frac{9}{\alpha^{n_1-n_2}} > |\Delta_1| > \exp(-C_1(1+\log n_1))$$

where $C_1 := 4.04\cdot 10^{12} > 1.4\cdot 30^6\cdot 3^{4.5}\cdot 4\cdot(1+\log 2)\cdot 2.2\cdot 0.9\cdot 2.1$. Applying a logarithm and solving for $(n_1-n_2)\log\alpha$ using $1+\log n_1 < 2\log n_1$ for $n_1 \ge 3$, we acquire

$$(n_1-n_2)\log\alpha < \log 9 + 2C_1\log n_1 < 8.1\cdot 10^{12}\log n_1 \tag{3.2}$$

A first bound on $n_1 - n_3$:

Rewrite (1.1) using Binet's formula on P_{n_1} and P_{n_2} to get

$$\frac{\alpha^{n_1}+\alpha^{n_2}}{2\sqrt{2}} - 3^a = \frac{\beta^{n_1}+\beta^{n_2}}{2\sqrt{2}} - (P_{n_3}+P_{n_4}).$$

Taking the absolute value of both sides and using (2.2) and (2.1), we obtain

$$\left|\frac{\alpha^{n_1}}{2\sqrt{2}}\left(1+\alpha^{n_2-n_1}\right) - 3^a\right| \le \frac{|\beta|^{n_1}+|\beta|^{n_2}}{2\sqrt{2}} + P_{n_3} + P_{n_4} < \frac{1}{3} + \alpha^{n_3} + \alpha^{n_4}.$$

Dividing both sides by $\alpha^{n_1}(1+\alpha^{n_2-n_1})/2\sqrt{2}$ we have

$$\begin{aligned} |\Delta_2| &:= \left|1 - 3^a\alpha^{-n_1}2\sqrt{2}\left(1+\alpha^{n_2-n_1}\right)^{-1}\right| \\ &< \frac{2\sqrt{2}}{\alpha^{n_1}\left(1+\alpha^{n_2-n_1}\right)}\left(\frac{1}{3}+\alpha^{n_3}+\alpha^{n_4}\right) \\ &= \frac{2\sqrt{2}}{\alpha^{n_1-n_3}\left(1+\alpha^{n_2-n_1}\right)}\left(\frac{1}{3}\alpha^{-n_3}+1+\alpha^{n_4-n_3}\right) \\ &\le \frac{2\sqrt{2}}{\alpha^{n_1-n_3}\left(1+\alpha^{n_2-n_1}\right)}\cdot\frac{7}{3} < \frac{7}{\alpha^{n_1-n_3}} \end{aligned} \tag{3.3}$$

where the last inequality follows from the fact that $0 < \alpha^{n_2-n_1} < 1$.

Note that $\Delta_2 \ne 0$, for if it was zero, then $3^a 2\sqrt{2} = \alpha^{n_1}+\alpha^{n_2}$. Conjugating in $\mathbb{Q}(\sqrt{2})$, we also have $-3^a 2\sqrt{2} = \beta^{n_1}+\beta^{n_2}$. Subtracting the two we obtain

$$2\cdot 3^a = \frac{\alpha^{n_1}-\beta^{n_1}}{2\sqrt{2}} + \frac{\alpha^{n_2}-\beta^{n_2}}{2\sqrt{2}} = P_{n_1}+P_{n_2}.$$

Thus, equation (1.1) becomes $P_{n_3} + P_{n_4} = -3^a$, a contradiction. Hence, $\Delta_2 \neq 0$ and we can apply Theorem 2.1 with $s := 3$, $(\gamma_1, \gamma_2, \gamma_3) := \left(3, \alpha, 2\sqrt{2}\,(1+\alpha^{n_2-n_1})^{-1}\right)$, and $(b_1, b_2, b_3) := (a, -n_1, 1)$. We also have $D := 2$ and $B := n_1$. Using Proposition 2.1, we find that

$$h(\gamma_3) = h\left(\frac{2\sqrt{2}}{1+\alpha^{n_2-n_1}}\right) \leq h(2\sqrt{2}) + h\left(1+\alpha^{n_2-n_1}\right)$$
$$\leq \log 2\sqrt{2} + \log 2 + \log 1 + h\left(\alpha^{n_2-n_1}\right)$$
$$= \log 4\sqrt{2} + (n_1 - n_2)\frac{\log \alpha}{2}.$$

Using $n_2 - n_1 < 0$, we see that both $\gamma_3 < 2\sqrt{2}$ and $\gamma_3^{-1} < 2\sqrt{2}$. Hence $|\log \gamma_3| < \log 4\sqrt{2}$. Taking

$$A_1 := 2.2, \quad A_2 := 0.9, \quad \text{and} \quad A_3 := 3.47 + (n_1 - n_2)\log\alpha,$$

we see that $A_3 \geq \max\{Dh(\gamma_3), |\log\gamma_3|, 0.16\}$. Applying Theorem 2.1 along with (3.3), we have

$$\frac{7}{\alpha^{n_1-n_3}} > |\Delta_2| > \exp(-C_2(1+\log n_2)(3.47 + (n_1-n_2)\log\alpha)$$

where $C_2 := 1.93 \cdot 10^{12} > 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 4 \cdot (1+\log 2) \cdot 2.2 \cdot 0.9$. Solving for $(n_1 - n_3)\log\alpha$ using (3.2) and considering that $1 + \log n_1 < 2\log n_1$ for $n_1 > 3$, one can see that

$$(n_1 - n_3)\log\alpha < 3.2 \cdot 10^{25} \log^2 n_1. \tag{3.4}$$

A first bound on $n_1 - n_4$:

Once again, rewrite (1.1) using Binet's formula on P_{n_1}, P_{n_2}, and P_{n_3} to obtain, after similar manipulations,

$$|\Delta_3| := \left|1 - 3^a\alpha^{-n_1}2\sqrt{2}\left(1+\alpha^{n_2-n_1}+\alpha^{n_3-n_1}\right)^{-1}\right| < \frac{3\sqrt{2}}{\alpha^{n_1-n_4}} < \frac{5}{\alpha^{n_1-n_4}} \tag{3.5}$$

Note that $\Delta_3 \neq 0$, for if it was zero, then $3^a 2\sqrt{2} = \alpha^{n_1} + \alpha^{n_2} + \alpha^{n_3}$. Conjugating in $\mathbb{Q}(\sqrt{2})$, we also have $-3^a 2\sqrt{2} = \beta^{n_1} + \beta^{n_2} + \beta^{n_3}$. Subtracting the two we obtain

$$2 \cdot 3^a = \frac{\alpha^{n_1} - \beta^{n_1}}{2\sqrt{2}} + \frac{\alpha^{n_2} - \beta^{n_2}}{2\sqrt{2}} + \frac{\alpha^{n_3} - \beta^{n_3}}{2\sqrt{2}} = P_{n_1} + P_{n_2} + P_{n_3}.$$

Thus, equation (1.1) becomes $P_{n_4} = -3^a$, a contradiction. Hence, $\Delta_3 \neq 0$ and we can apply Theorem 2.1 with $s := 3$, $(\gamma_1, \gamma_2, \gamma_3) := \left(3, \alpha, 2\sqrt{2}\,(1+\alpha^{n_2-n_1}+\alpha^{n_3-n_1})^{-1}\right)$, and $(b_1, b_2, b_3) := (a, -n_1, 1)$. We also have $D := 2$ and $B := n_1$. Using Proposition 2.1, we find that

$$h(\gamma_3) = h\left(\frac{2\sqrt{2}}{1+\alpha^{n_2-n_1}+\alpha^{n_3-n_1}}\right) \leq h(2\sqrt{2}) + h\left(1+\alpha^{n_2-n_1}+\alpha^{n_3-n_1}\right)$$
$$\leq \log 2\sqrt{2} + \log 3 + \log 1 + h\left(\alpha^{n_2-n_1}\right) + h\left(\alpha^{n_3-n_1}\right)$$
$$= \log 6\sqrt{2} + ((n_1 - n_2) + (n_1 - n_3))\frac{\log\alpha}{2}.$$

Notice that both $\gamma_3 < 2\sqrt{2}$ and $\gamma_3^{-1} < 2\sqrt{2}$, thus $|\log \gamma_3| < \log 6\sqrt{2}$. Taking

$$A_1 := 2.2, \quad A_2 := 0.9, \quad \text{and} \quad A_3 := 4.28 + ((n_1 - n_2) + (n_1 - n_3)) \log \alpha,$$

we see that $A_3 \geq \max\{Dh(\gamma_3), |\log \gamma_3|, 0.16\}$. Applying Theorem 2.1 along with (3.5), we have

$$\frac{5}{\alpha^{n_1 - n_3}} > |\Delta_3| > \exp(-C_2(1 + \log n_1)(4.28 + ((n_1 - n_2) + (n_1 - n_3)) \log \alpha)$$

where C_2 is as defined previously with $C_2 < 1.93 \cdot 10^{12}$. Solving for $(n_1 - n_4) \log \alpha$ using (3.2) and (3.4), we have for $n_1 \geq 3$,

$$\begin{aligned}(n_1 - n_4) \log \alpha &< \log 5 + 2C_2 \log n_1(4.28 + ((n_1 - n_2) + (n_1 - n_3)) \log \alpha)\\ &< \log 5 + 2C_2 \log n_1(4.28 + 8.2 \cdot 10^{12} \log n_1 + 3.2 \cdot 10^{25} \log^2 n_1)\\ &< 3.86 \cdot 10^{12} \log n_1(6.4 \cdot 10^{25} \log^2 n_1)\\ &< 2.5 \cdot 10^{38} \log^3 n_1. \end{aligned} \tag{3.6}$$

A first bound on n_1:

In a similar manner, we rewrite (1.1) using Binet's formula on each P_{n_i} to obtain

$$|\Delta_4| := \left|1 - 3^a \alpha^{-n_1} 2\sqrt{2}\left(1 + \alpha^{n_2 - n_1} + \alpha^{n_3 - n_1} + \alpha^{n_4 - n_1}\right)^{-1}\right| < \frac{4\sqrt{2}}{3\alpha^{n_1}} < \frac{2}{\alpha^{n_1}} \tag{3.7}$$

Note that $\Delta_4 \neq 0$, for if it was zero, then we would have

$$2\sqrt{2} \cdot 3^a = \alpha^{n_1} + \alpha^{n_2} + \alpha^{n_3} + \alpha^{n_4}.$$

Taking the conjugate in $\mathbb{Q}(\sqrt{2})$ and subtracting the result from the above, we would obtain

$$2 \cdot 3^a = P_{n_1} + P_{n_2} + P_{n_3} + P_{n_4},$$

contradicting (1.1). Hence, $\Delta_4 \neq 0$ and we can apply Theorem 2.1 with $s := 3$, $(\gamma_1, \gamma_2, \gamma_3) := \left(3, \alpha, 2\sqrt{2}\left(1 + \alpha^{n_2 - n_1} + \alpha^{n_3 - n_1} + \alpha^{n_4 - n_1}\right)^{-1}\right)$, and $(b_1, b_2, b_3) := (a, -n_1, 1)$. We also have $D := 2$ and $B := n_1$. Using Proposition 2.1, we find that

$$h(\gamma_3) \leq \log 8\sqrt{2} + ((n_1 - n_2) + (n_1 - n_3) + (n_1 - n_4)) \frac{\log \alpha}{2}.$$

Notice that both $\gamma_3 < 2\sqrt{2}$ and $\gamma_3^{-1} < 2\sqrt{2}$, thus $|\log \gamma_3| < \log 8\sqrt{2}$. Taking

$$A_1 := 2.2, \quad A_2 := 0.9, \quad \text{and} \quad A_3 := 5 + ((n_1 - n_2) + (n_1 - n_3) + (n_1 - n_4)) \log \alpha,$$

we see that $A_3 \geq \max\{Dh(\gamma_3), |\log \gamma_3|, 0.16\}$. Applying Theorem 2.1 along with (3.7), we have

$$\frac{2}{\alpha^{n_1}} > |\Delta_4| > \exp(-C_2(1 + \log n_1)(5 + ((n_1 - n_2) + (n_1 - n_3) + (n_1 - n_4)) \log \alpha)$$

where C_2 is as defined previously with $C_2 < 1.93 \cdot 10^{12}$. Solving for $n_1 \log \alpha$ using (3.2), (3.4), and (3.6), we have for $n_1 \geq 3$,

$$n_1 \log \alpha < 2.9 \cdot 10^{51} \log^4 n_1.$$

Using Maple and verifying the computations in Mathematica, one checks that the above inequality holds for

$$n_1 < 1.20518... \cdot 10^{60} < 1.21 \cdot 10^{60}. \tag{3.8}$$

4. Reducing the Bound on n_1

We now work on reducing the bounds found in Section 3 to be more manageable. To do so, we will rewrite equations (3.1), (3.3), (3.5), and (3.7) in such a way to utilize Lemmas 2.1 and 2.2.

Reducing the bound on $n_1 - n_2$:

Notice that since $|\beta|^{n_1} < \sqrt{2}/3 < 1$ and $n_1 \geq n_2 \geq n_3 \geq n_4 \geq 1$, we have

$$\frac{\alpha^{n_1}}{2\sqrt{2}} = \frac{\beta^{n_1}}{2\sqrt{2}} + P_{n_1} \leq P_{n_1} + 1 \leq P_{n_1} + P_{n_2} + P_{n_3} + P_{n_4} = 3^a.$$

Thus, we see that $3^a\alpha^{-n_1}2\sqrt{2} > 1$, so

$$z_1 := a\log 3 - n_1 \log\alpha + \log 2\sqrt{2} > 0.$$

Using (3.1), we have

$$0 < z_1 < e^{z_1} - 1 < \frac{9}{\alpha^{n_1-n_2}}.$$

Dividing by $\log\alpha$, we obtain

$$0 < \frac{z_1}{\log\alpha} = a\,\frac{\log 3}{\log\alpha} - n_1 + \frac{\log 2\sqrt{2}}{\log\alpha} < \frac{9}{\alpha^{n_1-n_2}\log\alpha} < \frac{11}{\alpha^{n_1-n_2}}. \tag{4.1}$$

We now apply Lemma 2.1 to (4.1) with the parameters

$$\gamma := \frac{\log 3}{\log\alpha}, \quad \mu := \frac{\log 2\sqrt{2}}{\log\alpha}, \quad A := 11, \quad B := \alpha, \quad w := n_1 - n_2, \quad \text{and} \quad v := n_1.$$

Set $M := 1.21 \cdot 10^{60}$ so that by (3.8) we have $a < n_1 < M$. Notice that γ is irrational, for if it was rational then there would be integers p and $q \neq 0$ such that $\gcd(p,q) = 1$ with $\gamma = p/q$. Rearranging, we would have $3^q = \alpha^p$ contradicting the fact that $\alpha^p \notin \mathbb{Z}$ for any positive integer k. Hence, γ is irrational and we can apply Lemma 2.1 to (4.1). If we take the denominator of the 114th convergent of γ, denoted q_{114}, then we get $q_{114} > 6M$ and $\epsilon := ||\mu q|| - M||\gamma q|| > 0$. Lemma 2.1 states that there is no solution to (4.1) with $a \leq M$ and $n_1 - n_2 \geq \log(AB/\epsilon)/\log B$. Since $a < M$, we must therefore have

$$n_1 - n_2 < \frac{\log AB/\epsilon}{\log B} < 164.$$

Reducing the bound on $n_1 - n_3$:

We now aim to rewrite $|\Delta_2|$ as $e^{z_2} - 1$ for some positive z_2. Using Binet's formula on P_{n_1} and P_{n_2} in a similar manner to that above when considering z_1, we see that $3^a\alpha^{n_1}2\sqrt{2}(1 + \alpha^{n_2-n_1})^{-1}$ is > 1. Hence

$$z_2 := a\log 3 - n_1\log\alpha + \log\left(2\sqrt{2}(1+\alpha^{n_2-n_1})^{-1}\right) > 0$$

and $z_2 < e^{z_2} - 1$. Dividing (3.3) by $\log\alpha$, we obtain

$$0 < a\,\frac{\log 3}{\log\alpha} - n_1 + \frac{\log\left(2\sqrt{2}(1+\alpha^{n_2-n_1})^{-1}\right)}{\log\alpha} < \frac{7}{\alpha^{n_1-n_3}\log\alpha} < \frac{8}{\alpha^{n_1-n_3}}. \tag{4.2}$$

Denote the parameters

$$\gamma := \frac{\log 3}{\log\alpha}, \quad \mu := \frac{\log\left(2\sqrt{2}(1+\alpha^{n_2-n_1})^{-1}\right)}{\log\alpha}, \quad \text{and} \quad A := 8.$$

Notice that μ depends on $n_1 - n_2 \in [0, 163]$. Set $M := 1.21 \cdot 10^{60}$ so that $a < n_1 < M$. Applying Lemma 2.1, if we take the the denominator of the 114th convergent of γ then we get $q_{114} > 6M$; however, $\epsilon < 0$ for several values of $n_1 - n_2 \in [0, 163]$. If instead we take the denominator of the 116th convergent of γ, then $\epsilon > 0$ for all $n_1 - n_2 \in [0, 163]$ except $n_1 - n_2 = 2$. Applying Lemma 2.1 for all $n_1 - n_2 \in [0, 163]$ except $n_1 - n_2 = 2$, we see that $n_1 - n_3 < 171$.

When $n_1 - n_2 = 2$, we have $\mu = 1$, so (4.2) becomes

$$0 < a\gamma - (n_1 - 1) < \frac{8}{\alpha^{n_1-n_3}}.$$

If $8/\alpha^{n_1-n_3} \geq 1/2a$, then $16a \geq \alpha^{n_1-n_3}$, and by (3.8),

$$n_1 - n_3 \leq \frac{\log 16a}{\log\alpha} \leq \frac{16n_1}{\log\alpha} < \frac{\log(16 \cdot 1.21 \cdot 10^{60})}{\log\alpha} < 161.$$

Instead, if $8/\alpha^{n_1-n_3} < 1/2a$, then we apply Lemma 2.2 to see that $(n_1-1)/a = p_k/q_k = [a_0; a_1, a_2, \ldots, a_k]$ is a convergent of γ. Knowing that $a < 1.21 \cdot 10^{60}$, we see that $q_{112} < 1.21 \cdot 10^{60} < q_{113}$. Furthermore, by (2.4) we have

$$\frac{1}{(a_{\max}+2)a} < \left| a\frac{\log 3}{\log\alpha} - (n_1 - 1) \right| < \frac{8}{\alpha^{n_1-n_3}}$$

where $a_{\max} = \max\{a_i : 0 \leq i \leq 113\} = 200$. Solving for $n_1 - n_3$ using (3.8) we see that for $n_1 - n_2 = 2$, we have $n_1 - n_3 < 166$. Thus, for every $n_1 - n_2 \in [0, 163]$, we have

$$n_1 - n_3 < 171.$$

Reducing the bound on $n_1 - n_4$:

We repeat the above process on $|\Delta_3|$ for some positive z_3. Using Binet's formula on P_{n_1}, P_{n_2}, and P_{n_3} we see that

$$z_3 := a\log 3 - n_1\log\alpha + \log\left(2\sqrt{2}(1+\alpha^{n_2-n_1}+\alpha^{n_3-n_1})^{-1}\right) > 0$$

and $z_3 < e^{z_3} - 1$. After dividing by $\log\alpha$, equation (3.5) gives

$$0 < a\,\frac{\log 3}{\log\alpha} - n_1 + \frac{\log\left(2\sqrt{2}(1+\alpha^{n_2-n_1}+\alpha^{n_3-n_1})^{-1}\right)}{\log\alpha} < \frac{6}{\alpha^{n_1-n_4}}. \tag{4.3}$$

Denote the parameters

$$\gamma := \frac{\log 3}{\log\alpha}, \quad \mu := \frac{\log\left(2\sqrt{2}(1+\alpha^{n_2-n_1}+\alpha^{n_3-n_1})^{-1}\right)}{\log\alpha}, \quad \text{and} \quad A := 6.$$

Set $M := 1.21 \cdot 10^{60}$ so that $a < n_1 < M$. Applying Lemma 2.1 to (4.3), we must go out to the denominator of the 121st convergent of γ to get $q_{121} > 6M$ with $\epsilon > 0$ for all $n_1 - n_2 \in [0, 163]$ and $n_1 - n_3 \in [0, 170]$. To reduce the computational load, notice that μ is symmetric in $n_1 - n_2$ and $n_1 - n_3$. Thus, we need only check $n_1 - n_3 \in [0, 170]$ and $n_1 - n_2 \in [0, \min\{n_1 - n_3, 163\}]$. Applying Lemma 2.1 for all $n_1 - n_3 \in [0, 170]$ and $n_1 - n_2 \in [0, \min\{n_1 - n_3, 163\}]$, we see that the maximum value of $n_1 - n_4$ occurs when $(n_1 - n_3, n_1 - n_2) = (101, 88)$ and, rounded to the nearest hundredth, is 181.22. Thus, take

$$n_1 - n_4 < 182.$$

Reducing the bound on n_1:

Let

$$z_4 := a \log 3 - n_1 \log \alpha + \log\left(2\sqrt{2}(1 + \alpha^{n_2 - n_1} + \alpha^{n_3 - n_1} + \alpha^{n_4 - n_1})^{-1}\right).$$

Notice that if $z_4 = 0$, then $\Delta_4 = 1 - e^{z_4} = 0$ a contradiction. So $z_4 \neq 0$. By (3.7), we acquire

$$|e^{z_4} - 1| < \frac{2}{\alpha^{n_1}}.$$

If $z_4 > 0$, then $0 < z_4 < e^{z_4} - 1 < 4/\alpha^{n_1}$. If instead $z_4 < 0$, then for $n_1 > 1$ we have,

$$0 < |e^{z_4} - 1| < \frac{2}{\alpha^{n_1}} < \frac{1}{2}.$$

Thus, we have $|e^{z_4} - 1| = 1 - e^{-|z_4|} < 1/2$, so $e^{|z_4|} < 2$. Hence,

$$0 < |z_4| \leq e^{|z_4|} - 1 = e^{|z_4|}\,|e^{z_4} - 1| < 2 \cdot \frac{2}{\alpha^{n_1}} = \frac{4}{\alpha^{n_1}}.$$

In either case, we have

$$0 < |z_4| < \frac{4}{\alpha^{n_1}}.$$

Dividing by $\log \alpha$, the inequality becomes

$$0 < \left| a\frac{\log 3}{\log \alpha} - n_1 + \frac{\log\left(2\sqrt{2}(1 + \alpha^{n_2 - n_1} + \alpha^{n_3 - n_1} + \alpha^{n_4 - n_1})^{-1}\right)}{\log \alpha} \right| < \frac{5}{\alpha^{n_1}}. \tag{4.4}$$

Denote the parameters

$$\gamma := \frac{\log 3}{\log \alpha}, \quad \mu := \frac{\log\left(2\sqrt{2}(1 + \alpha^{n_2 - n_1} + \alpha^{n_3 - n_1} + \alpha^{n_4 - n_1})^{-1}\right)}{\log \alpha},$$

and $A := 5$. To reduce the number of instances where $\epsilon < 0$, we go out to the denominator of the 122nd convergent of γ to get $q_{122} > 6M$ with $\epsilon > 0$ for all $n_1 - n_2 \in [0, 163]$, $n_1 - n_3 \in [0, 170]$, and $n_1 - n_4 \in [0, 181]$ except

$$(n_1 - n_4, n_1 - n_3, n_1 - n_2) \in \{(1, 1, 0), (4, 3, 3), (5, 4, 1)\}.$$

To reduce the computations, we again note that μ is symmetric in $n_1 - n_2$, $n_1 - n_3$, and $n_1 - n_4$; thus, it suffices to check $n_1 - n_4 \in [0, 181]$, $n_1 - n_3 \in [0, \min\{170, n_1 - n_4\}]$, and $n_1 - n_2 \in [0, \min\{163, n_1 - n_3\}]$. Applying Lemma 2.1 for all instances except where $\epsilon < 0$, we see that the maximum value of n_1 occurs when $(n_1-n_4, n_1-n_3, n_1-n_2) = (134, 110, 62)$ and, rounded to the nearest hundredth, is 196.76. Thus, take

$$n_1 < 197.$$

We consider each case where $\epsilon < 0$ separately. If $(n_1 - n_4, n_1 - n_3, n_1 - n_2) = (1, 1, 0)$, then (1.1) reduces to

$$P_{n_1} + P_{n_1} + P_{n_1-1} + P_{n_1-1} = 2(P_{n_1} + P_{n_1-1}) = 3^a,$$

which has no solutions since the left side is even but the right side is odd.

If $(n_1-n_4, n_1-n_3, n_1-n_2) = (4, 3, 3)$, then, using $P_k = 2P_{k-1}+P_{k-2}$ for $k \geq 2$, equation (1.1) reduces to

$$P_{n_1} + P_{n_1-3} + P_{n_1-3} + P_{n_1-4} = P_{n_1} + P_{n_1-2} = 3^a,$$

which we already considered in Section 3 when looking at the case that $n_4 = 0$.

If $(n_1 - n_4, n_1 - n_3, n_1 - n_2) = (5, 4, 1)$, then $\mu = 2 - \gamma$, so (4.4) is equivalent to

$$0 < \left|(a-1)\frac{\log 3}{\log \alpha} - (n_1 - 2)\right| < \frac{5}{\alpha^{n_1}}.$$

If $5/\alpha^{n_1} \geq 1/2(a-1)$, then solving for n_1 using $a < 1.21 \cdot 10^{60}$, we see that $n_1 < 160$. If instead $5/\alpha^{n_1} < 1/2(a-1)$, then we apply Lemma 2.2 to see that $(n_1 - 2)/(a - 1) = p_k/q_k = [a_0; a_1, a_2, \ldots, a_k]$ is a convergent of γ. We see that $q_{112} < 1.21 \cdot 10^{60} - 1 < q_{113}$. Furthermore, by (2.4) we have

$$\frac{1}{(a_{\max}+2)a} < \frac{5}{\alpha^{n_1}}$$

where $a_{\max} = \max\{a_i : 0 \leq i \leq 113\} = 200$. Solving for n_1 we see that for $(n_1 - n_4, n_1 - n_3, n_1 - n_2) = (5, 4, 1)$, we have $n_1 < 165$. In every case, we have

$$n_1 < 197,$$

which contradicts our assumption that $n_1 > 197$. This completes the proof of Theorem 1.1. □

5. Conclusion

In conclusion, we rewrote equation (1.1) three times using Binet's formula for Pell numbers to get upper bounds on n_1-n_2, n_1-n_3, and n_1-n_4 in terms of n_1 by utilizing Theorem 2.1. Using these upper bounds, as well as Binet's formula one last time, we obtained a rather large upper bound for n_1. We then reduced the upper bounds for $n_1 - n_2$, $n_1 - n_3$, $n_1 - n_4$, and finally n_1 using properties of convergents of the continued fraction of an irrational number. Once the upper bound for n_1 was reduced to $n_1 < 197$, we ran a brute force check on all options for n_1, n_2, n_3, and n_4 in (1.1) to find all possible solutions to

$$P_{n_1} + P_{n_2} + P_{n_3} + P_{n_4} = 3^a,$$

completing the proof of Theorem 1.1.

In the concluding remarks of [25] the authors make the following conjecture.

Conjecture 5.1. *Consider the Diophantine equation*

$$F_{n_1} + F_{n_2} + F_{n_3} + F_{n_4} = p^a, p \geq 2, a \geq 2$$

where $n_1 \geq n_2 \geq n_3 \geq n_4$ *are positive integers,* p *is prime, and* F_k *represents the Fibonacci numbers. Then* $p = 2, 3, 5,$ *or 7.*

This conjecture does not hold if we replace the F_k with P_k, the Pell numbers, which can be seen by the counterexample

$$P_{15} + P_9 + P_7 + P_6 = 443^2 = P_{15} + P_8 + P_8 + P_8.$$

We conclude with a few open questions arising from this counterexample.

1. Is there a prime q such that for all primes $p > q$, Conjecture 5.1 holds if we replace the F_k with P_k? A quick check in Mathematica over the first 5000 primes with $n_1 \leq 20$ suggests that $q = 433$ might be sufficient.
2. What other values of p and a with p a prime and $a \geq 1$ are such that p^a can be represented as the sum of two *different* sets of four Pell numbers?

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[1]Department of Mathematics and Computer Science,
University of North Carolina at Pembroke,
Pembroke, NC 28372
Email address: juillera@uncp.edu