Turkish J. Ineq., 7 (2) (2023), Pages 15 –28 .

Turkish Journal of
INEQUALITIES

Available online at www.tjinequality.com

SOME NEW APPLICATIONS FOR WEIGHTED HERMITE-HADAMARD TYPE INEQUALITIES

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ABSTRACT. For convex and $s-$convex functions, the Hermite-Hadamard inequality is already well known in the theory of inequalities. In this regard, this work presents new inequalities associated with the Weighted Hermite-Hadamard type inequalities for convex and $s-$convex functions by utilizing a different technique. Also, Hölder, Young, and power-mean inequalities are used to obtain these new inequalities.

1. INTRODUCTION

A fundamental concept in both applied and pure mathematics, convexity may be used as a potent tool for evaluating functions and sets, demonstrating inequalities, and modeling and resolving real-world problems. This concept is crucial for estimating integrals and defining limits in many branches of mathematics and beyond (see [1–10]).

Thus, we recall the elementary notation in convex analysis:

Definition 1.1. ([3]) A set $\mathcal{I} \subset \mathbb{R}$ is said to be convex function if

$$\zeta\varpi + (1-\zeta)\phi \in \mathcal{I}$$

for each $\varpi, \phi \in \mathcal{I}$ and $\zeta \in [0,1]$.

Definition 1.2. ([3]) The mapping $\wp : \mathcal{I} \to \mathbb{R}$, is said to be convex function if the following inequality holds:

$$\wp(\xi\varpi + (1-\xi)\phi) \leq \xi\wp(\varpi) + (1-\xi)\wp(\phi)$$

for all $\varpi, \phi \in \mathcal{I}$ and $\xi \in [0,1]$. If $(-\wp)$ is convex, then $\wp$ is said to be concave. In terms of geometry, this indicates that if $\mathcal{B}$, $\mathcal{U}$, and $\mathcal{Z}$ are three separate locations on the graph of $\wp$, with $\mathcal{U}$ between $\mathcal{B}$ and $\mathcal{Z}$, then $\mathcal{U}$ is on or below chord $\mathcal{BZ}$.

Key words and phrases. Young Inequality, Hölder Inequality, Power-mean inequality, Convex Functions, Hermite-Hadamard Inequality, $s-$Convex Functions.

2010 *Mathematics Subject Classification.* Primary: 26D15. Secondary: 26A55.

Received: 11/07/2023 *Accepted*: 30/11/2023.

Cited this article as: M.Ç. Tatar, M. Çoşkun, Ç. Yıldız, Some New Applications for Weighted Hermite-Hadamard Type Inequalities, Turkish Journal of Inequalities, 7(2) (2023), 15-28.

Convex functions are essential in many different areas. For instance, the theory of probability states that a convex function applied to the expected value of a random variable is never constrained below the expected value of the convex function. This result, called Jensen's inequality, may be used to generate additional inequalities, such as Hölder's inequality and the geometric-arithmetic mean inequality. The idea of convexity has developed into a rich source of inspiration and a fascinating topic for scholars because of its widespread viewpoints, resilience, and plenty of applications. Mathematicians have developed incredible tools and numerical methods using the notion of convexity to deal with and resolve an enormous number of issues that emerge in the pure and applied sciences. Because of its many views, adaptability, and wide range of applications, the concept of convexity has grown to be a rich source of inspiration and an intriguing subject for researchers. Mathematicians have developed incredible tools and numerical methods using the notion of convexity to deal with and resolve an enormous number of issues that emerge in the pure and applied sciences. This theory has a long and important history, and for more than a century, mathematics has focused on and concentrated on it. On the other hand, there are a lot of new issues in applied mathematics where the idea of convexity is insufficient to adequately characterize them in order to have beneficial consequences. Because of this, the idea of convexity has been expanded upon and developed in various research; see [11–20].

Convex functions have the following inequality properties.

Theorem 1.1. *([21, 22]) If $\wp$ is a convex function on I, then Jensen's inequality*

$$\wp\left(\frac{1}{n}\sum_{i=1}^{n}\mu_i\right) \le \frac{1}{n}\sum_{i=1}^{n}\wp(\mu_i)$$

and weighted Jensen's inequality

$$\wp\left(\frac{1}{\Im_n}\sum_{i=1}^{n}\hbar_i\mu_i\right) \le \frac{1}{\Im_n}\sum_{i=1}^{n}\hbar_i\wp(\mu_i)$$

are valid for $\mu_i \in I$ and $\hbar_i \ge 0$ with $i \in \mathbb{N}$ and $\Im_n = \sum_{i=1}^{n}\hbar_i > 0$.

In [23], the following integral form of Jensen's inequality is given.

Theorem 1.2. *Let $\wp : [a, b] \subset I \to \mathbb{R}$ be a convex function, and let $h : I \to (0, \infty)$ and $u : I \to \mathbb{R}_+ = [0, \infty)$ be integrable functions. Then*

$$\wp\left(\frac{\int_a^b h(t)u(t)dt}{\int_a^b h(t)dt}\right) \le \frac{\int_a^b h(t)\wp(u(t))dt}{\int_a^b h(t)dt} \tag{1.1}$$

provided that all the integrals in (1.1) are meaningful.

Convex mappings and sets have been improved and expanded in many disciplines of mathematics due to their robustness (as was described above); in particular, convexity theory has been used to prove a number of inequalities that are prevalent in the literature. In the practical literature on mathematical inequalities, the Hermite-Hadamard type integral inequality (or Hadamard inequality) is, to the best of our knowledge, a well-known, important, and incredibly helpful inequality. There are several classical inequalities that are closely associated with the classical Hermite-Hadamard type integral inequality, such as Ostrowski, Hardy, Simpson, Opial, Hölder, Minkowski, Grüss, arithmetic-geometric and Young inequalities. These inequalities are of pivotal significance. Following is a statement of this double inequality: Assume that $\wp$ is a convex mapping on $[\varpi, \phi] \subset \mathbb{R}$, where $\varpi \neq \phi$. Therefore

$$\wp\left(\frac{\varpi+\phi}{2}\right) \leq \frac{1}{\phi-\varpi}\int_{\varpi}^{\phi} \wp(\varkappa)d\varkappa \leq \frac{\wp(\varpi)+\wp(\phi)}{2}.$$

The reader who is interested is referred to [24–29] for a number of recent findings pertaining to Hermite-Hadamard inequality.

Utilizing various forms of convexity, some important inequalities have been observed. s-convexity is one of several varieties of convexity. Hudzik and Maligranda in paper [30] took into account, among other things, the class of functions that are s-convex in the second sense. The following is the definition of this class:

Definition 1.3. A function $\wp : [0, \infty) \to \mathbb{R}$ is $s-$convex in the second sense if

$$\wp(\xi\varpi + (1-\xi)\phi) \leq \xi^s \wp(\varpi) + (1-\xi)^s \wp(\phi)$$

holds for all $\varpi, \phi \in [0, \infty), \xi \in [0, 1]$ and for some fixed $s \in (0, 1]$. The class of $s-$convexity is frequently denoted by the symbol K_s^2. It is obvious that $s = 1$ converts $s-$convexity into the typical convexity of functions defined on $[0, \infty)$.

The authors of the same paper, namely [30], demonstrated that all functions from K_s^2, $s \in (0, 1)$, are nonnegative if $\wp \in K_s^2$ implies $\wp([0, \infty)) \subseteq [0, \infty)$.

Example 1.1. ([30]) Let $s \in (0, 1)$ and $\ell, \hbar, \gamma \in \mathbb{R}$. We define function $\wp : [0, \infty) \to \mathbb{R}$ as

$$\wp(\zeta) = \begin{cases} \ell, & \zeta = 0, \\ \hbar\zeta^s + \gamma, & \zeta > 0. \end{cases}$$

It can be simply confirmed that

(i) If $\hbar \geq 0$ and $0 \leq \gamma \leq \ell$, then $\wp \in K_s^2$,

(ii) If $\hbar > 0$ and $\gamma < 0$, then $\wp \notin K_s^2$.

Recently, there are many studies on s-convexity in the literature. A few new general Hermite-Hadamard type inequalities for s-convex mappings were demonstrated in [31] by Özdemir *et al.* The Hölder inequality, the power-mean integral inequality, and certain extensions connected to these inequalities were utilized to establish these inequalities. Additionally, they compared some inequalities. In [32], a new definition for s-convex functions is given, and some properties of this definition are investigated. In addition, extended versions of the previously well-known conclusions for harmonically convex functions, such as Hadamard, various Hermite-Hadamard refinements, and Ostrowski-type inequalities, are developed. In [33], the expression "extended s-convex functions" was introduced by the authors, who also developed some inequalities of the Hermite-Hadamard type for extended s-convex functions. The authors then used these newly discovered integral inequalities to deduce certain specific mean inequalities. In study [34], the authors establish an equation for a function whose third derivative is integrable, develop some novel integral inequalities of the

Hermite-Hadamard type for extended s-convex mappings using the Hölder inequality, and then use these integral inequalities to produce inequalities for various kinds of special means. In paper [35], the authors established some new inequalities of the Hermite-Hadamard type for extended s-convex mappings and obtained new inequalities with respect to λ and μ using the Lemma 2.1. Finally, utilizing the s-convexity for the Raina function, different inequalities are obtained with fractional integral operators in [36].

In [37], researchers proved a different form of Hermite-Hadamard inequality, which holds for s-convex mappings in the second sense:

Theorem 1.3. *Suppose that $\wp : [0,\infty) \to [0,\infty)$ is an $s-$convex function in the second sense, where $s \in (0,1)$, and let $\varpi, \phi \in [0,\infty)$, $\varpi < \phi$. If $\wp \in L([\varpi,\phi])$, then the following inequalities hold:*

$$2^{s-1}\wp\left(\frac{\varpi+\phi}{2}\right) \leq \frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\varkappa)d\varkappa \leq \frac{\wp(\varpi)+\wp(\phi)}{s+1}. \tag{1.2}$$

In the second inequality, the constant $\alpha = 1/(s+1)$ is the best possibility.

For these reasons, the different form of Hermite-Hadamard inequalities has attracted a lot of interest from researchers, and there are many papers and monographs dedicated to its development; here, we mention, e.g., [39–44].

The famous Young inequality is defined as follows:

Theorem 1.4. *([38]) Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\varpi\phi \leq \frac{1}{p}\varpi^p + \frac{1}{q}\phi^q \tag{1.3}$$

where ϖ and ϕ are nonnegative numbers. The reversed version of inequality (1.3) reads

$$\varpi\phi \geq \frac{1}{p}\varpi^p + \frac{1}{q}\phi^q, \quad \varpi,\phi > 0, \quad 0 < p < 1, \quad \frac{1}{p}+\frac{1}{q} = 1.$$

The well-known Hölder inequality, one of the most significant inequalities in analysis, was demonstrated in this way using inequality (1.3). It makes a significant contribution to many fields of applied and pure mathematics and is essential in helping to solve several issues in the social, cultural, and natural sciences.

The most popular form of Young's inequality, which is frequently used to demonstrate the well-known inequality for L_p functions, is as follows:

$$\varpi^{\xi}\phi^{1-\xi} \leq \xi\varpi + (1-\xi)\phi,$$

where $\varpi, \phi > 0$ and $0 \leq \xi \leq 1$.

Theorem 1.5. *(Hölder Inequality) Let $p > 1$and $\frac{1}{p} + \frac{1}{q} = 1$. If $\wp$ and κ are real functions defined on $[\varpi,\phi]$ such that $|\wp|^p$ and $|\kappa|^q$ are integrable functions on $[\varpi,\phi]$, then*

$$\int_{\varpi}^{\phi}|\wp(x)\kappa(x)|\,dx \leq \left(\int_{\varpi}^{\phi}|\wp(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_{\varpi}^{\phi}|\kappa(x)|^q\,dx\right)^{\frac{1}{q}}.$$

Theorem 1.6. *(Power-mean Inequality) Let $q \geq 1$. If $\wp$ and κ are real functions defined on $[\varpi, \phi]$ such that $|\wp|$ and $|\kappa|^q$ are integrable functions on $[\varpi, \phi]$, then*

$$\int_{\varpi}^{\phi} |\wp(x)\kappa(x)|\, dx \leq \left(\int_{\varpi}^{\phi} |\wp(x)|\, dx \right)^{1-\frac{1}{q}} \left(\int_{\varpi}^{\phi} |\wp(x)|\, |\kappa(x)|^q\, dx \right)^{\frac{1}{q}}.$$

2. Preliminaries

This paper uses a relatively new approach based on weighted Hermite-Hadamard inequalities for convex and s-convex functions. First, we will start by giving the inequality a weighted form of the Hermite-Hadamard inequalities obtained by Vasić and Lacković ([45]).

Theorem 2.1. *Let p,q be given positive numbers and $(\varpi, \phi) \subset I$. Then the inequalities*

$$\wp\left(\frac{\alpha\varpi + \beta\phi}{\alpha+\beta}\right) \leq \frac{1}{2y}\int_{A-y}^{A+y} \wp(t)dt \leq \frac{\alpha\wp(\varpi)+\beta\wp(\phi)}{\alpha+\beta} \tag{2.1}$$

hold for

$$A = \frac{\alpha\varpi + \beta\phi}{\alpha+\beta},\ y > 0,$$

and all continuous convex functions $\wp : [\varpi, \phi] \to \mathbb{R}$ iff

$$y \leq \frac{\phi - \varpi}{\alpha+\beta} \leq \min\{\alpha, \beta\}.$$

Remark 2.1. For $\alpha = \beta = 1$ and $y = \frac{\phi-\varpi}{2}$, (2.1) is the Hermite-Hadamard inequality.

In following theorem, Xiao *et.al* proveded a new form of the weighted Hermite-Hadamard inequalities for convex function.

Theorem 2.2. *([46])For evevry convex function $\wp$ on $[\varpi, \phi] \subseteq I$ and $2\beta \geq \alpha \geq \frac{\beta}{2} > 0$, we have,*

$$\begin{aligned} & \wp\left(\frac{\alpha\varpi + \beta\phi}{\alpha+\beta}\right) \\ \leq\ & \frac{2}{(\alpha+\beta)(\phi-\varpi)^2}\int_{\varpi}^{\phi}\left[(\alpha-2\beta)\varpi + (2\alpha-\beta)\phi + 3(\beta-\alpha)x\right]\wp(x)dx \\ =\ & \frac{2}{\alpha+\beta}\int_0^1 \left[2\alpha - \beta + 3(\beta-\alpha)\xi\right]\wp((1-\xi)\varpi + \xi\phi)d\xi \leq \frac{\alpha\wp(\varpi)+\beta\wp(\phi)}{\alpha+\beta}. \end{aligned} \tag{2.2}$$

The aim of this study is to obtain new general integral inequalities with the use of different convex functions for weighted Hermite-Hadamard inequalities. In addition to the definition of convex function, Young, Hölder, and power-mean inequalities are used to obtain these new identities. As a consequence, these inequalities are associated with Hermite-Hadamard inequality.

3. New Inequalities on the Weighted Hermite-Hadamard Type for Convex Functions

Theorem 3.1. *Let $\wp : I \subset [0,\infty) \to \mathbb{R}$ be a positive mapping and $\wp \in L[\varpi,\phi]$, where $\varpi,\phi \in I$ with $\varpi < \phi$, $\xi \in [0,1]$. If $|\wp|^q$ is a convex function on $[\varpi,\phi]$, for $p > 1$ with $\frac{1}{p}$ $+\frac{1}{q} = 1$ and $2\beta \geq \alpha \geq \frac{\beta}{2} > 0$, then the following inequality holds:*

$$\begin{aligned}
& \left|\wp\left(\frac{\alpha\varpi+\beta\phi}{\alpha+\beta}\right)\right| \\
\leq\ & \frac{2}{\alpha+\beta}\int_0^1 \left[2\alpha-\beta+3(\beta-\alpha)\xi\right]\wp((1-\xi)\varpi+\xi\phi)d\xi \\
\leq\ & \frac{2}{\alpha+\beta}\left[\frac{(2\beta-\alpha)^{p+1}-(2\alpha-\beta)^{p+1}}{3(\beta-\alpha)(p+1)}\right]^{\frac{1}{p}}\left[\frac{|\wp(\varpi)|^q+|\wp(\phi)|^q}{2}\right]^{\frac{1}{q}}
\end{aligned}$$

Proof. Taking absolute values on both sides of (2.2), we have

$$\begin{aligned}
\left|\wp\left(\frac{\alpha\varpi+\beta\phi}{\alpha+\beta}\right)\right| &\leq \left|\frac{2}{\alpha+\beta}\int_0^1 \left[2\alpha-\beta+3(\beta-\alpha)\xi\right]\wp((1-\xi)\varpi+\xi\phi)d\xi\right| \\
&\leq \frac{2}{\alpha+\beta}\int_0^1 \left|2\alpha-\beta+3(\beta-\alpha)\xi\right|\left|\wp((1-\xi)\varpi+\xi\phi)\right|d\xi \\
&= \frac{2}{\alpha+\beta}\int_0^1 \left[2\alpha-\beta+3(\beta-\alpha)\xi\right]\left|\wp((1-\xi)\varpi+\xi\phi)\right|d\xi.
\end{aligned}$$

Using the Hölder inequality and definition of convex function, we obtain

$$\begin{aligned}
& \left|\wp\left(\frac{\alpha\varpi+\beta\phi}{\alpha+\beta}\right)\right| \\
\leq\ & \frac{2}{\alpha+\beta}\int_0^1 \left[2\alpha-\beta+3(\beta-\alpha)\xi\right]\left|\wp((1-\xi)\varpi+\xi\phi)\right|d\xi \\
\leq\ & \frac{2}{\alpha+\beta}\left(\int_0^1 \left[2\alpha-\beta+3(\beta-\alpha)\xi\right]^p d\xi\right)^{\frac{1}{p}}\left(\int_0^1 \left|\wp((1-\xi)\varpi+\xi\phi)\right|^q d\xi\right)^{\frac{1}{q}} \\
\leq\ & \frac{2}{\alpha+\beta}\left(\int_0^1 \left[2\alpha-\beta+3(\beta-\alpha)\xi\right]^p d\xi\right)^{\frac{1}{p}}\left[\int_0^1 \left[(1-\xi)\left|\wp(\varpi)\right|^q+\xi\left|\wp(\phi)\right|^q\right]d\xi\right]^{\frac{1}{q}} \\
=\ & \frac{2}{\alpha+\beta}\left[\frac{(2\beta-\alpha)^{p+1}-(2\alpha-\beta)^{p+1}}{3(\beta-\alpha)(p+1)}\right]^{\frac{1}{p}}\left[\frac{|\wp(\varpi)|^q+|\wp(\phi)|^q}{2}\right]^{\frac{1}{q}}
\end{aligned}$$

where

$$\int_0^1 \left[2\alpha-\beta+3(\beta-\alpha)\xi\right]^p d\xi = \frac{(2\beta-\alpha)^{p+1}-(2\alpha-\beta)^{p+1}}{3(\beta-\alpha)(p+1)}.$$

This completes the proof. □

Theorem 3.2. *Let $\wp : I \subset [0,\infty) \to \mathbb{R}$ be a positive mapping and $\wp \in L[\varpi,\phi]$, where $\varpi,\phi \in I$ with $\varpi < \phi$, $\xi \in [0,1]$. If $|\wp|^q$ is a convex function on $[\varpi,\phi]$, for $q \geq 1$ and $2\beta \geq \alpha \geq \frac{\beta}{2} > 0$, then the followimg inequality holds:*

$$\begin{aligned}
& \left|\wp\left(\frac{\alpha\varpi+\beta\phi}{\alpha+\beta}\right)\right| \\
\leq\ & \frac{2}{\alpha+\beta}\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\left|\wp((1-\xi)\varpi+\xi\phi)\right| d\xi \\
\leq\ & \left(\frac{2}{\alpha+\beta}\right)^{\frac{1}{q}}\left[\frac{\alpha\left|\wp(\varpi)\right|^q+\beta\left|\wp(\phi)\right|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}$$

Proof. Suppose that $q \geq 1$. From Theorem 2.2 and using the power-mean inequality, we have

$$\begin{aligned}
\left|\wp\left(\frac{\alpha\varpi+\beta\phi}{\alpha+\beta}\right)\right| &\leq \left|\frac{2}{\alpha+\beta}\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\,\wp((1-\xi)\varpi+\xi\phi)d\xi\right| \\
&\leq \frac{2}{\alpha+\beta}\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\left|\wp((1-\xi)\varpi+\xi\phi)\right| d\xi \\
&\leq \frac{2}{\alpha+\beta}\left(\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\, d\xi\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\left|\wp((1-\xi)\varpi+\xi\phi)\right|^q d\xi\right)^{\frac{1}{q}}.
\end{aligned}$$

Because $|\wp|^q$ is a convex function on $[\varpi,\phi]$, we have

$$\begin{aligned}
& \left|\wp\left(\frac{\alpha\varpi+\beta\phi}{\alpha+\beta}\right)\right| \\
\leq\ & \frac{2}{\alpha+\beta}\left(\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\, d\xi\right)^{1-\frac{1}{q}} \\
& \times\left(\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\left[(1-\xi)\left|\wp(\varpi)\right|^q+\xi\left|\wp\left(\phi\right)\right|^q\right] d\xi\right)^{\frac{1}{q}} \\
=\ & \left(\frac{2}{\alpha+\beta}\right)^{\frac{1}{q}}\left[\frac{\alpha\left|\wp(\varpi)\right|^q+\beta\left|\wp(\phi)\right|^q}{2}\right]^{\frac{1}{q}}
\end{aligned}$$

where

$$\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\left\{(1-\xi)\left|\wp(\varpi)\right|^q+\xi\left|\wp\left(\phi\right)\right|^q\right\} d\xi=\frac{\alpha\left|\wp(\varpi)\right|^q+\beta\left|\wp(\phi)\right|^q}{2}$$

and

$$\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\, d\xi=\frac{\alpha+\beta}{2}$$

which completes the proof. □

Theorem 3.3. *Let $\wp : I \subseteq [0,\infty) \to \mathbb{R}$ be a positive mapping, $\wp \in L[\varpi,\phi]$, where $\forall$ $\varpi,\phi \geq 0$. If $|\wp|^q$ is a convex function on $[\varpi,\phi]$ and $2\beta \geq \alpha \geq \frac{\beta}{2} > 0$, then the following* *inequality holds:*

$$\begin{aligned}
&\left|\wp\left(\frac{\alpha\varpi+\beta\phi}{\alpha+\beta}\right)\right| \\
\leq\ & \frac{2}{\alpha+\beta}\int_0^1 \left[2\alpha-\beta+3(\beta-\alpha)\xi\right]\left|\wp((1-\xi)\varpi+\xi\phi)\right| d\xi \\
\leq\ & \frac{2}{\alpha+\beta}\left[\frac{(2\beta-\alpha)^{p+1}-(2\alpha-\beta)^{p+1}}{p\,(p+1)\,3(\beta-\alpha)}+\frac{1}{q}\left(\frac{|\wp(\varpi)|^q+|\wp(\phi)|^q}{2}\right)\right]
\end{aligned}$$

where $\xi\in[0,1]$ *and* $\frac{1}{p}+\frac{1}{q}=1$, $p>1$.

Proof. From Theorem 2.2 and using the Young Inequality, we obtain

$$\begin{aligned}
\left|\wp\left(\frac{\alpha\varpi+\beta\phi}{\alpha+\beta}\right)\right| &\leq \left|\frac{2}{\alpha+\beta}\int_0^1 \left[2\alpha-\beta+3(\beta-\alpha)\xi\right]\wp((1-\xi)\varpi+\xi\phi)d\xi\right| \\
&\leq \frac{2}{\alpha+\beta}\int_0^1 \left[2\alpha-\beta+3(\beta-\alpha)\xi\right]\left|\wp((1-\xi)\varpi+\xi\phi)\right| d\xi \\
&\leq \frac{2}{\alpha+\beta}\left[\frac{1}{p}\int_0^1 \left[2\alpha-\beta+3(\beta-\alpha)\xi\right]^p d\xi+\frac{1}{q}\int_0^1 \left|\wp((1-\xi)\varpi+\xi\phi)\right|^q d\xi\right].
\end{aligned}$$

So, $|\wp|^q$is convex function on $[\varpi,\phi]$, we get

$$\begin{aligned}
&\left|\wp\left(\frac{\alpha\varpi+\beta\phi}{\alpha+\beta}\right)\right| \\
\leq\ & \frac{2}{\alpha+\beta}\int_0^1 \left[2\alpha-\beta+3(\beta-\alpha)\xi\right]\left|\wp((1-\xi)\varpi+\xi\phi)\right| d\xi \\
\leq\ & \frac{2}{\alpha+\beta}\left[\frac{1}{p}\int_0^1 \left[2\alpha-\beta+3(\beta-\alpha)\xi\right]^p d\xi+\frac{1}{q}\int_0^1 \left[(1-\xi)\left|\wp(\varpi)\right|^q+\xi\left|\wp(\phi)\right|^q\right] d\xi\right] \\
=\ & \frac{2}{\alpha+\beta}\left[\frac{(2\beta-\alpha)^{p+1}-(2\alpha-\beta)^{p+1}}{p\,(p+1)\,3(\beta-\alpha)}+\frac{1}{q}\left(\frac{|\wp(\varpi)|^q+|\wp(\phi)|^q}{2}\right)\right].
\end{aligned}$$

□

4. New Inequalities on the Weighted Hermite-Hadamard Type for $s-$Convex Functions

Theorem 4.1. *Let* $\wp : I\subseteq[0,\infty)\to\mathbb{R}$ *be a positive mapping and* $\wp\in L[\varpi,\phi]$ *,where* $\varpi,\phi\in I$ *with* $\varpi<\phi$, $\xi\in[0,1]$. *If* $|\wp|$ *is* $s-$*convex function on* $[\varpi,\phi]$, *for some fixed* $s\in(0,1]$ *and* $2\beta\geq\alpha\geq\frac{\beta}{2}>0$, *then the following inequality holds:*

$$\begin{aligned}
&\left|\wp\left(\frac{\alpha\varpi+\beta\phi}{\alpha+\beta}\right)\right| \qquad (4.1)\\
\leq\ & \frac{2}{\alpha+\beta}\int_0^1 \left[2\alpha-\beta+3(\beta-\alpha)\xi\right]\left|\wp((1-\xi)\varpi+\xi\phi)\right| d\xi \\
\leq\ & \frac{2}{\alpha+\beta}\left\{\frac{s\,(2\alpha-\beta)+(\alpha+\beta)}{(s+1)(s+2)}\left|\wp\left(\varpi\right)\right|+\frac{s\,(2\beta-\alpha)+(\alpha+\beta)}{(s+1)(s+2)}\left|\wp\left(\phi\right)\right|\right\}.
\end{aligned}$$

Proof. From Theorem 2.2 and using the s-convex in the second sense, we have

$$\begin{aligned}\left|\wp\left(\frac{\alpha\varpi+\beta\phi}{\alpha+\beta}\right)\right| &\leq \left|\frac{2}{\alpha+\beta}\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\,\wp((1-\xi)\varpi+\xi\phi)d\xi\right| \\ &\leq \frac{2}{\alpha+\beta}\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\,|\wp((1-\xi)\varpi+\xi\phi)|\,d\xi \\ &\leq \frac{2}{\alpha+\beta}\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\,[(1-\xi)^s|\wp(\varpi)|+\xi^s|\wp(\phi)|]\,d\xi \\ &= \frac{2}{\alpha+\beta}\left\{\frac{s(2\alpha-\beta)+(\alpha+\beta)}{(s+1)(s+2)}|\wp(\varpi)|+\frac{s(2\beta-\alpha)+(\alpha+\beta)}{(s+1)(s+2)}|\wp(\phi)|\right\}\end{aligned}$$

which is the desired inequality. □

Corollary 4.1. *If we choose $s=1$ in inequality (4.1), we obtain the following result;*

$$\begin{aligned}\left|\wp\left(\frac{\alpha\varpi+\beta\phi}{\alpha+\beta}\right)\right| &\leq \frac{2}{\alpha+\beta}\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\,|\wp((1-\xi)\varpi+\xi\phi)|\,d\xi \\ &\leq \frac{2}{\alpha+\beta}\left\{\frac{\alpha|\wp(\varpi)|+\beta|\wp(\phi)|}{2}\right\}.\end{aligned}$$

Theorem 4.2. *Let $\wp : I\subseteq[0,\infty)\to\mathbb{R}$ be a positive mapping and $\wp\in L[\varpi,\phi]$, where $\varpi,\phi\in I$ with $\varpi<\phi$, $\xi\in[0,1]$. If $|\wp|^q$ is $s-$ convex function on $[\varpi,\phi]$, for $\frac{1}{p}+\frac{1}{q}=1$ and $2\beta\geq\alpha\geq\frac{\beta}{2}>0$, then the following inequality holds:*

$$\begin{aligned}&\left|\wp\left(\frac{\alpha\varpi+\beta\phi}{\alpha+\beta}\right)\right| \qquad (4.2)\\ &\leq \frac{2}{\alpha+\beta}\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\,|\wp((1-\xi)\varpi+\xi\phi)|\,d\xi \\ &\leq \frac{2}{\alpha+\beta}\left[\frac{(2\beta-\alpha)^{p+1}-(2\alpha-\beta)^{p+1}}{3(\beta-\alpha)(p+1)}\right]^{\frac{1}{p}}\left[\frac{|\wp(\varpi)|^q+|\wp(\phi)|^q}{s+1}\right]^{\frac{1}{q}}.\end{aligned}$$

Proof. From Theorem 2.2 and using Hölder inequality, we get

$$\begin{aligned}\left|\wp\left(\frac{\alpha\varpi+\beta\phi}{\alpha+\beta}\right)\right| &\leq \left|\frac{2}{\alpha+\beta}\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\,\wp((1-\xi)\varpi+\xi\phi)d\xi\right| \\ &\leq \frac{2}{\alpha+\beta}\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\,|\wp((1-\xi)\varpi+\xi\phi)|\,d\xi \\ &\leq \frac{2}{\alpha+\beta}\left(\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]^p\,d\xi\right)^{\frac{1}{p}}\left(\int_0^1 |\wp((1-\xi)\varpi+\xi\phi)|^q\,dt\right)^{\frac{1}{q}}.\end{aligned}$$

Since $|\wp|^q$ is $s-$convex function, then we have

$$
\begin{aligned}
&\left|\wp\left(\frac{\alpha\varpi+\beta\phi}{\alpha+\beta}\right)\right| \\
\leq\ & \frac{2}{\alpha+\beta}\left(\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]^p\, d\xi\right)^{\frac{1}{p}}\left[\int_0^1 [(1-\xi)^s\left|\wp(\varpi)\right|^q+\xi^s\left|\wp\left(\phi\right)\right|^q]\, d\xi\right]^{\frac{1}{q}} \\
=\ & \frac{2}{\alpha+\beta}\left[\frac{(2\beta-\alpha)^{p+1}-(2\alpha-\beta)^{p+1}}{3(\beta-\alpha)(p+1)}\right]^{\frac{1}{p}}\left[\frac{|\wp(\varpi)|^q+|\wp(\phi)|^q}{s+1}\right]^{\frac{1}{q}}.
\end{aligned}
$$

This completes the proof. □

Corollary 4.2. *If we take $s=1$ in inequality (4.2), we obtain the following inequalities;*

$$
\begin{aligned}
\left|\wp\left(\frac{\alpha\varpi+\beta\phi}{\alpha+\beta}\right)\right| &\leq \frac{2}{\alpha+\beta}\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\left|\wp((1-\xi)\varpi+\xi\phi)\right| d\xi \\
&\leq \frac{2}{\alpha+\beta}\left[\frac{(2\beta-\alpha)^{p+1}-(2\alpha-\beta)^{p+1}}{3(\beta-\alpha)(p+1)}\right]^{\frac{1}{p}}\left[\frac{|\wp(\varpi)|^q+|\wp(\phi)|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}
$$

Theorem 4.3. *Let $\wp: I\subseteq[0,\infty)\to\mathbb{R}$ be a positive mapping and $\wp\in L\left[\varpi,\phi\right]$,where $\varpi,\phi\in I$ with $\varpi<\phi$, $\xi\in[0,1]$. If $|\wp|^q$is $s-$convex function on $[\varpi,\phi]$, $q\geq 1$ and $2\beta\geq\alpha\geq\frac{\beta}{2}>0$, then the followimg inequality holds:*

$$
\begin{aligned}
&\left|\wp\left(\frac{\alpha\varpi+\beta\phi}{\alpha+\beta}\right)\right| \\
\leq\ & \frac{2}{\alpha+\beta}\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\left|\wp((1-\xi)\varpi+\xi\phi)\right| d\xi \\
\leq\ & \left(\frac{2}{\alpha+\beta}\right)^{\frac{1}{q}}\left\{\frac{s\left(2\alpha-\beta\right)+(\alpha+\beta)}{(s+1)(s+2)}\left|\wp(\varpi)\right|^q+\frac{s\left(2\beta-\alpha\right)+(\alpha+\beta)}{(s+1)(s+2)}\left|\wp(\phi)\right|^q\right\}^{\frac{1}{q}}.
\end{aligned} \tag{4.3}
$$

Proof. Suppose that $q\geq 1$. Taking absolute values on both sides of (2.2), we obtain

$$
\begin{aligned}
\left|\wp\left(\frac{\alpha\varpi+\beta\phi}{\alpha+\beta}\right)\right| &\leq \left|\frac{2}{\alpha+\beta}\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\,\wp((1-\xi)\varpi+\xi\phi)d\xi\right| \\
&\leq \frac{2}{\alpha+\beta}\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\left|\wp((1-\xi)\varpi+\xi\phi)\right| dt.
\end{aligned}
$$

From s-convexity and using the power-mean inequality, we have

$$\begin{aligned}
&\left|\wp\left(\frac{\alpha\varpi+\beta\phi}{\alpha+\beta}\right)\right| \\
\leq\ & \frac{2}{\alpha+\beta}\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\left|\wp((1-\xi)\varpi+\xi\phi)\right| d\xi \\
\leq\ & \frac{2}{\alpha+\beta}\left(\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\, d\xi\right)^{1-\frac{1}{q}} \\
& \times\left(\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\left|\wp((1-\xi)\varpi+\xi\phi)\right|^q d\xi\right)^{\frac{1}{q}} \\
\leq\ & \frac{2}{\alpha+\beta}\left[\frac{\alpha+\beta}{2}\right]^{1-\frac{1}{q}} \\
& \times\left[\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\left[(1-\xi)^s\left|\wp(\varpi)\right|^q+\xi^s\left|\wp(\phi)\right|^q\right] d\xi\right]^{\frac{1}{q}} \\
=\ & \left(\frac{2}{\alpha+\beta}\right)^{\frac{1}{q}}\left\{\frac{s(2\alpha-\beta)+(\alpha+\beta)}{(s+1)(s+2)}\left|\wp(\varpi)\right|^q+\frac{s(2\beta-\alpha)+(\alpha+\beta)}{(s+1)(s+2)}\left|\wp(\phi)\right|^q\right\}^{\frac{1}{q}}.
\end{aligned}$$

Hence, the proof is done. □

Corollary 4.3. *If we take $s=1$ in inequality (4.3), we obtain the following inequalities;*

$$\begin{aligned}
\left|\wp\left(\frac{\alpha\varpi+\beta\phi}{\alpha+\beta}\right)\right| &\leq \frac{2}{\alpha+\beta}\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\left|\wp((1-\xi)\varpi+\xi\phi)\right| dt \\
&\leq \left(\frac{2}{\alpha+\beta}\right)^{\frac{1}{p}}\left[\frac{\alpha\left|\wp(\varpi)\right|^q+\beta\left|\wp(\phi)\right|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}$$

Theorem 4.4. *Let $\wp : I \subseteq [0,\infty) \to \mathbb{R}$ be a positive mapping, $\wp \in L[\varpi,\phi]$, where $\forall$ $\varpi,\phi \geq 0$. If $|\wp|^q$ is $s-$convex on $[\varpi,\phi]$ and $2\beta \geq \alpha \geq \frac{\beta}{2} > 0$, then the following inequalities holds:*

$$\begin{aligned}
&\left|\wp\left(\frac{\alpha\varpi+\beta\phi}{\alpha+\beta}\right)\right| \qquad (4.4)\\
\leq\ & \frac{2}{\alpha+\beta}\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\left|\wp((1-\xi)\varpi+\xi\phi)\right| dt \\
\leq\ & \frac{2}{\alpha+\beta}\left[\frac{(2\beta-\alpha)^{p+1}-(2\alpha-\beta)^{p+1}}{3p\,(p+1)\,(\beta-\alpha)}+\frac{1}{q}\left(\frac{\left|\wp(\varpi)\right|^q+\left|\wp(\phi)\right|^q}{2(s+1)}\right)\right].
\end{aligned}$$

where $\xi \in [0,1]$ and $\frac{1}{p}+\frac{1}{q}=1$, $p>1$.

Proof. From Theorem 2.2 and using the Young Inequality, we have

$$\begin{aligned}\left|\wp\left(\frac{\alpha\varpi+\beta\phi}{\alpha+\beta}\right)\right| &\leq \left|\frac{2}{\alpha+\beta}\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\,\wp((1-\xi)\varpi+\xi\phi)d\xi\right| \\ &\leq \frac{2}{\alpha+\beta}\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\,|\wp((1-\xi)\varpi+\xi\phi)|\,d\xi \\ &\leq \frac{2}{\alpha+\beta}\left[\frac{1}{p}\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]^p\,d\xi+\frac{1}{q}\int_0^1 |\wp((1-\xi)\varpi+\xi\phi)|^q\,d\xi\right]\end{aligned}$$

Since $|\wp|^q$ is $s-$convex function, then we obtain

$$\begin{aligned}&\left|\wp\left(\frac{\alpha\varpi+\beta\phi}{\alpha+\beta}\right)\right| \\ \leq\ & \frac{2}{\alpha+\beta}\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\,|\wp((1-\xi)\varpi+\xi\phi)|\,d\xi \\ \leq\ & \frac{2}{\alpha+\beta}\left[\frac{1}{p}\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]^p\,d\xi+\frac{1}{k}\int_0^1 [(1-\xi)^s\,|\wp(\varpi)|^q+\xi^s\,|\wp\,(\phi)|^q]\,d\xi\right] \\ =\ & \frac{2}{\alpha+\beta}\left[\frac{(2\beta-\alpha)^{p+1}-(2\alpha-\beta)^{p+1}}{3p\,(p+1)\,(\beta-\alpha)}+\frac{1}{q}\left(\frac{|\wp(\varpi)|^q+|\wp(\phi)|^q}{2(s+1)}\right)\right].\end{aligned}$$

Thus, the desired result was obtained. □

Corollary 4.4. *If we choose $s=1$ in inequality (4.4), we obtain the following inequalities;*

$$\begin{aligned}\left|\wp\left(\frac{\alpha\varpi+\beta\phi}{\alpha+\beta}\right)\right| &\leq \frac{2}{\alpha+\beta}\int_0^1 [2\alpha-\beta+3(\beta-\alpha)\xi]\,|\wp((1-\xi)\varpi+\xi\phi)|\,d\xi \\ &\leq \frac{2}{\alpha+\beta}\left[\frac{(2\beta-\alpha)^{p+1}-(2\alpha-\beta)^{p+1}}{3p\,(p+1)\,(\beta-\alpha)}+\frac{1}{q}\left(\frac{|\wp(\varpi)|^q+|\wp(\phi)|^q}{4}\right)\right].\end{aligned}$$

5. Conclusions

In this paper, we have established some new Weighted Hermite-Hadamard-type inequalities by using convex and s-convex functions. Since the class of convex functions has large applications in many mathematical areas, these functions can be applied to obtain some new results in the theory of inequalities and may stimulate further research in different convex functions. In this sense, we hope that this study will inspire researchers to obtain further results.

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