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MILNE-TYPE INEQUALITIES FOR DIFFERENT CLASSES OF MAPPING BASED ON PROPORTIONAL CAPUTO-HYBRID OPERATOR

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ABSTRACT. In this study, we prove many integral inequalities associated with the Milne-type integral inequalities for proportional Caputo-hybrid operator with the use of the identity given by Demir [12]. Firstly, we establish some Milne-type inequalities for bounded mappings with proportional Caputo-hybrid operator. Moreover, we give several Milne-type inequalities by using the properties of Lipschitz condition and bounded variation with the help of proportional Caputo-hybrid operator. We observe that the acquired outcomes improve and generalize certain of the previous findings in the field of integral inequalities.

1. Introduction

Convexity is a significant and interesting theory with many applications in classical analysis. In addition, the use of integral inequality and its applications has grown significantly, influencing not only the many current mathematical topics like measure theory, approximation theory and information theory but also a wide range of scientific and technical fields. Moreover, by using the integral inequalities, the error bounds of the numerical integration formulae for the differentiable mappings may be found. Many scholars have been interested in combining the theory of convexity and the theory of inequality due to their close association, which helps to construct and generalize integral inequalities.

Simpson's inequality, which is as follows, is among the most significant and frequently needed inequalities:

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^4}{2880}\left\|f^{(4)}\right\|_\infty,$$

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where $f:[a,b]\rightarrow\mathbb{R}$ is four times continuously differantiable mapping on (a,b) and $\left\|f^{(4)}\right\|_{\infty}=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. This inequality establishes an upper limit on the error that occurs when estimating a definite integral using Simpson's rule. Because of its widespread geometric significance and applications, a number of authors have focused on Simpson-type inequalities for different classes of mappings in recent years. For example, in [14], Dragomir et al. demonstrated some recent advancements in Simpson's inequality, where the remaining part is expressed in terms of derivatives lower than the fourth order. In [2], Alomari introduced Simpson's type inequalities for s-convex functions. Sarikaya et al. gave some Simpson's type inequalities via twice differentiable functions in [27]. For the other results, one can refer to [9], [17], [19], [26].

The Milne inequality provides estimates of the error boundaries for the Milne formula under circumstances similar to those obtained from the Simpson inequality:

$$\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|\leq\frac{7(b-a)^{4}}{23040}\left\|f^{(4)}\right\|_{\infty},$$

where $f:[a,b]\rightarrow\mathbb{R}$ is a four times differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_{\infty}=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. So, the Milne inequality has received a lot of attention from researchers recently. Alomari and Liu [3] established error estimations for the Milne's rule for mappings of bounded variation and for absolutely continuous mappings. Rom$\acute{a}$n-Flores et al. [21] proved some Milne type inequalities for interval-valued functions. Budak et al. [10] investigated Milne-type inequalities for bounded functions, Lipschitz functions and functions of bounded variation. Ali et al. [1] gave the fractional version of Milne's formula-type inequalities for differentiable convex functions and Riemann–Liouville fractional integrals. Many recent articles have been published on this subject, as in [7], [8], [18].

Meanwhile, a subfield of mathematics known as fractional calculus studies integrals and derivatives with non-integer order. Therefore, it is important for the generalization of classical calculus, complex system modeling, fractional differential equation solving, fractal geometry analysis, and many other applications in science and engineering. Moreover, it offers a framework for interpreting and assessing fractional dynamics systems, enabling a more thorough mathematical explanation of complicated circumstances. Thus, due to the new fractional integral and derivative such as Caputo-Fabrizio [11], Atangana-Baleanu [5] and tempered [22], this calculus has gained more importance and has found applications in various fields of science and engineering.

This is one of the important definitions of fractional analysis [23]:

Definition 1.1. Let $\alpha>0$ and $\alpha\notin\{1,2,...\}$, $n=[\alpha]+1$, $f\in AC^{n}[a,b]$, the space of functions having $n-th$ derivatives absolutely continuous. The left-sided and right-sided

Caputo fractional derivatives of order α are defined as follows:

$$^{C}D_{a^+}^{\alpha}\ f(x)=\frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}(x-t)^{n-\alpha-1}f^{(n)}(t)dt,\quad x>a$$

and

$$^{C}D_{b^-}^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\int_{x}^{b}(t-x)^{n-\alpha-1}f^{(n)}(t)dt,\quad x<b.$$

If $\alpha=n\in\{1,2,3,...\}$ and usual derivative $f^{(n)}(x)$ of order n exists, then Caputo fractional derivative $^{C}D_{a^+}^{\alpha}\ f(x)$ coincides with $f^{(n)}(x)$ whereas $^{C}D_{b^-}^{\alpha}\ f(x)$ with exactness to a constant multiplier $(-1)^n$. For $n=1$ and $\alpha=0$, we have $^{C}D_{a^+}^{\alpha}\ f(x)={}^{C}D_{b^-}^{\alpha}\ f(x)=f(x)$.

The Caputo derivative is defined as the application of a fractional integral to a standard derivative of the function whereas the Riemann-Liouville fractional derivative is obtained by differentiating the fractional integral of a function with respect to its independent variable of order n. The Caputo fractional derivative necessitates more suitable initial conditions in contrast to the conventional Riemann-Liouville fractional derivative considering fractional differential equations [13]. Therefore, the Caputo derivative is preferable when analyzing other fractional derivatives since it produces more physically significant answers for the particular issues. On the other hand, the operator of proportional derivative denoted as $^{P}D_{\alpha}f(x)$ is given by the equation [4] :

$$^{P}D_{\alpha}f(x)=K_1(\alpha,t)f(t)+K_0(\alpha,t)f'(t),$$

where K_1 and K_0 are the functions with respect to $\alpha\in[0,1]$ and $t\in\mathbb{R}$ subject to certain conditions and also, the function f is differentiable with respect to $t\in\mathbb{R}$. In robotics and control systems, this mathematical operator is often utilized. The significance of research on the Caputo derivative and the proportionate derivative has significantly increased in recent years [15], [16], [20].

Baleanu et al. provided the following definition in [6], combining the ideas of proportional derivative and Caputo derivative in a novel way to produce a hybrid fractional operator that can be expressed as a linear combination of Riemann-Liouville fractional integral and Caputo fractional derivative.

Definition 1.2. Let $f:I\subset\mathbb{R}^+\to\mathbb{R}$ be a differentiable function on I° and f,f' are locally $L_1(I)$. Then, the proportional Caputo-hybrid operator may be defined as follows:

$$^{C}D_{a^+}^{\alpha}\ f(t)=\frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}\left[K_1(\alpha,\tau)f(\tau)+K_0(\alpha,\tau)f'(\tau)\right](t-\tau)^{-\alpha}d\tau$$

where $\alpha\in[0,1]$ and K_1 and K_0 are functions which satisfy the following conditions:

$$\begin{aligned}\lim_{\alpha\to0^+}K_0(\alpha,\tau)&=0;\quad \lim_{\alpha\to1}K_0(\alpha,\tau)=1;\quad K_0(\alpha,\tau)\neq0,\quad \alpha\in(0,1];\\ \lim_{\alpha\to0}K_1(\alpha,\tau)&=0;\quad \lim_{\alpha\to1^-}K_1(\alpha,\tau)=1;\quad K_1(\alpha,\tau)\neq0,\quad \alpha\in[0,1).\end{aligned}$$

Next, Sarıkaya [24] proposed a new definition by applying distinct K_1 and K_0 functions based on Definition 1.2. Also, Sarıkaya [24] derived the Hermite-Hadamard inequality utilizing his own new definition as presented below:

Definition 1.3. Let $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I° and $f,\ f' \in L_1(I)$. The left-sided and right-sided proportional Caputo-hybrid operator of order α are defined respectively as follows:

$$ {}_{a^+}^{PC}D_b^\alpha\ f(b) = \frac{1}{\Gamma(1-\alpha)} \int_a^b \left[K_1(\alpha, b-\tau) f(\tau) + K_0(\alpha, b-\tau) f'(\tau)\right] (b-\tau)^{-\alpha} d\tau $$

and

$$ {}_{b^-}^{PC}D_a^\alpha\ f(a) = \frac{1}{\Gamma(1-\alpha)} \int_a^b \left[K_1(\alpha, \tau-a) f(\tau) + K_0(\alpha, \tau-a) f'(\tau)\right] (\tau-a)^{-\alpha} d\tau, $$

where $\alpha \in [0,1]$ and $K_0(\alpha,\tau) = (1-\alpha)^2 \tau^{1-\alpha}$ and $K_1(\alpha,\tau) = \alpha^2 \tau^\alpha$.

Theorem 1.1. *Let $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I°, the interior of the interval I, where $a, b \in I^\circ$ with $a < b$ and let f, f' be the convex functions on I. Then, the following inequalities hold:*

$$ \begin{aligned} & \alpha^2 (b-a)^\alpha f\left(\frac{a+b}{2}\right) + \frac{1}{2}(1-\alpha)(b-a)^{1-\alpha} f'\left(\frac{a+b}{2}\right) \\ \leq\ & \frac{\Gamma(1-\alpha)}{2(b-a)^{1-\alpha}} \left[{}_{a^+}^{PC}D_b^\alpha f(b) + {}_{b^-}^{PC}D_a^\alpha f(a)\right] \\ \leq\ & \alpha^2 (b-a)^\alpha \left[\frac{f(a)+f(b)}{2}\right] + (1-\alpha))(b-a)^{1-\alpha} \left[\frac{f'(a)+f'(b)}{4}\right]. \end{aligned} $$

Additionally, Sarıkaya provided the following Simpson's type inequality in [25] by employing his interpretation of the proportional Caputo operator:

Theorem 1.2. *Let $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be differantiable function on I°, the interior of the interval I, where $a, b \in I^\circ$ with $a < b$, and $f', f'' \in L[a,b]$. Then, the following identity holds:*

$$ \begin{aligned} & S(a,b;\alpha) \\ =\ & \frac{\alpha^2 (b-a)^\alpha}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \\ & + \frac{(1-\alpha)(b-a)^{2-\alpha}}{12} \left[f'(a) + 4f'\left(\frac{a+b}{2}\right) + f'(b)\right] \\ & - \frac{\Gamma(1-\alpha)}{2(b-a)^{1-\alpha}} \left[{}_{a^+}^{PC}D_b^\alpha f(b) + {}_{b^-}^{PC}D_a^\alpha f(a)\right] \end{aligned} $$

where

$$S(a,b;\alpha) = \frac{\alpha^2(b-a)^{1+\alpha}}{2}\int_0^1 P(t)[f'(ta+(1-t)b)+f'(tb+(1-t)a)]\,dt$$

$$+\frac{(1-\alpha)(b-a)^{2-\alpha}}{4}\int_0^1 Q(t)[f''(ta+(1-t)b)+f''(tb+(1-t)a)]\,dt,$$

$$P(t)=\begin{cases}\frac{1}{6}-t, & 0\le t<\frac{1}{2}\\ \frac{5}{6}-t, & \frac{1}{2}\le t\le 1,\end{cases}$$

and

$$Q(t)=\begin{cases}\frac{1}{6}-t^{2-2\alpha}, & 0\le t<\frac{1}{2}\\ \frac{5}{6}-t^{2-2\alpha}, & \frac{1}{2}\le t\le 1.\end{cases}$$

Besides, Demir [12] offered a novel integral identity related to the Milne-type integral inequalities using twice-differentiable convex mappings for the proportional Caputo-hybrid operator . This identity is essential for demonstrating our other main results:

Lemma 1.1. *Let $f: I\subset\mathbb{R}^+\to\mathbb{R}$ be a twice differentiable function on I^o, the interior of the interval I, where $a,b\in I^o$ satisfying $a<b$ and let $f,f',f''\in L_1[a,b]$. Then, the following identity is satisfied:*

$$\alpha^2(b-a)^{\alpha+1}2^{-\alpha-1}\int_0^1\left(\frac{t}{2}-\frac{2}{3}\right)\left[f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)-f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right]dt$$

$$+(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-3}\int_0^1\left(\frac{t^{2-2\alpha}}{2}-\frac{2}{3}\right)\left[f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)-f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right]dt$$

$$=\frac{\alpha^2(b-a)^{\alpha}2^{-\alpha}}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)$$

$$+\frac{(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}}{3}\left(2f'(a)-f'\left(\frac{a+b}{2}\right)+2f'(b)\right)$$

$$-\frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}}\left[{}^{PC}_{\left(\frac{a+b}{2}\right)^+}D^{\alpha}_{b}f(b)+{}^{PC}_{\left(\frac{a+b}{2}\right)^-}D^{\alpha}_{a}f(a)\right].$$

In the present research, with the help of the identity provided by Demir [12], we prove several integral inequalities connected to the Milne-type integral inequalities for proportional Caputo-hybrid operator. In the beginning, we give a set of Milne-type inequalities for bounded mappings using proportionate Caputo-hybrid operator. Additionally, we use the properties of the Lipschitz condition and bounded variation with proportional Caputo-hybrid operator to establish a number of Milne-type inequalities. We note that the obtained results enhance and expand certain of the earlier findings in the field of integral inequalities.

2. Main Results

First, by using a proportional Caputo-hybrid operator, we intend to derive several Milne-type inequalities for bounded mappings.

Theorem 2.1. *Suppose that the conditions of Lemma 1.1 hold. If there exist $m, n, M, N \in \mathbb{R}$ such that $m \leq f'(t) \leq M$ and $n \leq f''(t) \leq N$ for all $t \in [a,b]$, then we have*

$$\begin{aligned}
&\left| \frac{\alpha^2 (b-a)^{\alpha} 2^{-\alpha}}{3} \left(2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b) \right) \right. \\
&+ \frac{(1-\alpha)(b-a)^{1-\alpha} 2^{\alpha-2}}{3} \left(2f'(a) - f'\left(\frac{a+b}{2}\right) + 2f'(b) \right) \\
&\left. - \frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}} \left[{}^{PC}_{\left(\frac{a+b}{2}\right)^+} D_b^{\alpha} f(b) + {}^{PC}_{\left(\frac{a+b}{2}\right)^-} D_a^{\alpha} f(a) \right] \right| \\
\leq\ & \frac{5\alpha^2 (b-a)^{\alpha+1} 2^{-\alpha-1}}{12} (M-m) \\
&+ (1-\alpha)(b-a)^{2-\alpha} 2^{\alpha-3} \left(\frac{2}{3} - \frac{1}{2(3-2\alpha)} \right) (N-n).
\end{aligned}$$

Proof. Lemma 1.1 helps us arrive at

$$\begin{aligned}
&\frac{\alpha^2 (b-a)^{\alpha} 2^{-\alpha}}{3} \left(2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b) \right) \\
&+ \frac{(1-\alpha)(b-a)^{1-\alpha} 2^{\alpha-2}}{3} \left(2f'(a) - f'\left(\frac{a+b}{2}\right) + 2f'(b) \right) \\
&- \frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}} \left[{}^{PC}_{\left(\frac{a+b}{2}\right)^+} D_b^{\alpha} f(b) + {}^{PC}_{\left(\frac{a+b}{2}\right)^-} D_a^{\alpha} f(a) \right] \\
=\ & \alpha^2 (b-a)^{\alpha+1} 2^{-\alpha-1} \left\{ \int_0^1 \left(\frac{t}{2} - \frac{2}{3} \right) \left[f'\left(\frac{2-t}{2}a + \frac{t}{2}b \right) - \frac{m+M}{2} \right] dt \right. \\
&\left. + \int_0^1 \left(\frac{t}{2} - \frac{2}{3} \right) \left[\frac{m+M}{2} - f'\left(\frac{t}{2}a + \frac{2-t}{2}b \right) \right] dt \right\} \\
&+ (1-\alpha)(b-a)^{2-\alpha} 2^{\alpha-3} \left\{ \int_0^1 \left(\frac{t^{2-2\alpha}}{2} - \frac{2}{3} \right) \left[f''\left(\frac{2-t}{2}a + \frac{t}{2}b \right) - \frac{n+N}{2} \right] dt \right. \\
&\left. + \int_0^1 \left(\frac{t^{2-2\alpha}}{2} - \frac{2}{3} \right) \left[\frac{n+N}{2} - f''\left(\frac{t}{2}a + \frac{2-t}{2}b \right) \right] dt \right\}.
\end{aligned} \tag{2.1}$$

Upon calculating the absolute value of (2.1), we obtain

$$\begin{aligned}
&\Bigg|\frac{\alpha^2(b-a)^{\alpha}2^{-\alpha}}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)\\
&+\frac{(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}}{3}\left(2f'(a)-f'\left(\frac{a+b}{2}\right)+2f'(b)\right)\\
&-\frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}}\left[{}^{PC}_{\left(\frac{a+b}{2}\right)^{+}}D^{\alpha}_{b}f(b)+{}^{PC}_{\left(\frac{a+b}{2}\right)^{-}}D^{\alpha}_{a}f(a)\right]\Bigg|\\
\leq\;& \alpha^2(b-a)^{\alpha+1}2^{-\alpha-1}\left\{\int_0^1\left(\frac{2}{3}-\frac{t}{2}\right)\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)-\frac{m+M}{2}\right|dt\right.\\
&\left.+\int_0^1\left(\frac{2}{3}-\frac{t}{2}\right)\left|\frac{m+M}{2}-f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt\right\}\\
&+(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-3}\left\{\int_0^1\left(\frac{2}{3}-\frac{t^{2-2\alpha}}{2}\right)\left|f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)-\frac{n+N}{2}\right|dt\right.\\
&\left.+\int_0^1\left(\frac{2}{3}-\frac{t^{2-2\alpha}}{2}\right)\left|\frac{n+N}{2}-f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt\right\}.
\end{aligned}$$

Because $m\leq f'(t)\leq M$ and $n\leq f''(t)\leq N$ for all $t\in[a,b]$, we have

$$\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)-\frac{m+M}{2}\right|\leq\frac{M-m}{2},\qquad\left|\frac{m+M}{2}-f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|\leq\frac{M-m}{2}$$

and

$$\left|f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)-\frac{n+N}{2}\right|\leq\frac{N-n}{2},\qquad\left|\frac{n+N}{2}-f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|\leq\frac{N-n}{2}.$$

Thus, we obtain

$$\begin{aligned}
&\Bigg|\frac{\alpha^2(b-a)^{\alpha}2^{-\alpha}}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)\\
&+\frac{(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}}{3}\left(2f'(a)-f'\left(\frac{a+b}{2}\right)+2f'(b)\right)\\
&-\frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}}\left[{}^{PC}_{\left(\frac{a+b}{2}\right)^{+}}D^{\alpha}_{b}f(b)+{}^{PC}_{\left(\frac{a+b}{2}\right)^{-}}D^{\alpha}_{a}f(a)\right]\Bigg|\\
\leq\;&\frac{5\alpha^2(b-a)^{\alpha+1}2^{-\alpha-3}}{3}(M-m)\\
&+(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-3}\left(\frac{2}{3}-\frac{1}{2(3-2\alpha)}\right)(N-n).
\end{aligned}$$

□

Corollary 2.1. *The following particular situation arises from Theorem 2.1 as α goes to* 0*:*

$$\left|\frac{1}{3}\left(2f'(a)-f'\left(\frac{a+b}{2}\right)+2f'(b)\right)-\frac{4}{(b-a)^2}\left(\int_{\frac{a+b}{2}}^{b}f(x)dx-\int_{a}^{\frac{a+b}{2}}f(x)dx\right)\right|$$
$$\leq \frac{b-a}{4}(N-n).$$

Corollary 2.2. *Under the assumptions of Theorem 2.1, if there exist $M,N\in\mathbb{R}^+$ such that $|f'(t)|\leq M$ and $|f''(t)|\leq N$ for all $t\in[a,b]$, then we have*

$$\left|\frac{\alpha^2(b-a)^{\alpha}2^{-\alpha}}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)\right.$$
$$+\frac{(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}}{3}\left(2f'(a)-f'\left(\frac{a+b}{2}\right)+2f'(b)\right)$$
$$\left.-\frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}}\left[{}^{PC}_{\left(\frac{a+b}{2}\right)^+}D^{\alpha}_{b}f(b)+{}^{PC}_{\left(\frac{a+b}{2}\right)^-}D^{\alpha}_{a}f(a)\right]\right|$$
$$\leq \frac{5\alpha^2(b-a)^{\alpha+1}2^{-\alpha-2}}{3}M$$
$$+(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-2}\left(\frac{2}{3}-\frac{1}{2(3-2\alpha)}\right)N.$$

Remark 2.1. According to Theorem 2.1, under particular case when α tends to 1, we obtain

$$\left|\frac{1}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|$$
$$\leq \frac{5(b-a)}{24}(M-m),$$

which was demonstrated by Budak et al. in [10].

Remark 2.2. In Corollary 2.2, for the specific circumstance when α approaches to 1, we get

$$\left|\frac{1}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|$$
$$\leq \frac{5(b-a)}{12}M,$$

which was shown by Alomari and Liu in [3].

Now, we obtain several Milne-type inequalities for Lipschitz functions with the help of a proportional Caputo-hybrid operator.

Theorem 2.2. *Suppose that the conditions of Lemma 1.1 hold. If f' and f'' are L_1-Lipschitz function and L_2-Lipschitz function on $[a,b]$, respectively, then we get the following inequality:*

$$\begin{aligned}
&\left|\frac{\alpha^2(b-a)^\alpha 2^{-\alpha}}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)\right.\\
&+\frac{(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}}{3}\left(2f'(a)-f'\left(\frac{a+b}{2}\right)+2f'(b)\right)\\
&\left.-\frac{\Gamma(1-\alpha)}{2^\alpha(b-a)^{-\alpha+1}}\left[{}^{PC}_{\left(\frac{a+b}{2}\right)^+}D^\alpha_b f(b)+{}^{PC}_{\left(\frac{a+b}{2}\right)^-}D^\alpha_a f(a)\right]\right|\\
\leq\ & \alpha^2(b-a)^{\alpha+2}2^{-\alpha-3}L_1\\
&+(1-\alpha)(b-a)^{3-\alpha}2^{\alpha-3}\left(\frac{1}{3}-\frac{1}{2(3-2\alpha)}+\frac{1}{2(4-2\alpha)}\right)L_2,
\end{aligned}$$

where L_1, L_2 are positive real constants.

Proof. With help of Lemma 1.1 and because of the property of Lipschitz functions on $[a,b]$, we have

$$\begin{aligned}
&\left|\frac{\alpha^2(b-a)^\alpha 2^{-\alpha}}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)\right.\\
&+\frac{(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}}{3}\left(2f'(a)-f'\left(\frac{a+b}{2}\right)+2f'(b)\right)\\
&\left.-\frac{\Gamma(1-\alpha)}{2^\alpha(b-a)^{-\alpha+1}}\left[{}^{PC}_{\left(\frac{a+b}{2}\right)^+}D^\alpha_b f(b)+{}^{PC}_{\left(\frac{a+b}{2}\right)^-}D^\alpha_a f(a)\right]\right|\\
\leq\ & \alpha^2(b-a)^{\alpha+1}2^{-\alpha-1}\int_0^1\left(\frac{2}{3}-\frac{t}{2}\right)\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)-f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt\\
&+(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-3}\int_0^1\left(\frac{2}{3}-\frac{t^{2-2\alpha}}{2}\right)\left|f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)-f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt\\
\leq\ & \alpha^2(b-a)^{\alpha+1}2^{-\alpha-1}\int_0^1\left(\frac{2}{3}-\frac{t}{2}\right)L_1(1-t)(b-a)dt\\
&+(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-3}\int_0^1\left(\frac{2}{3}-\frac{t^{2-2\alpha}}{2}\right)L_2(1-t)(b-a)dt.
\end{aligned}$$

Evaluating the integrals in the inequality mentioned above, we get

$$\int_0^1 \left(\frac{2}{3}-\frac{t}{2}\right)(1-t)\,dt=\frac{1}{4} \quad \text{and} \quad \int_0^1 \left(\frac{2}{3}-\frac{t^{2-2\alpha}}{2}\right)(1-t)\,dt=\frac{1}{3}-\frac{1}{2(3-2\alpha)}+\frac{1}{2(4-2\alpha)}.$$

Therefore, the proof is completed. □

Remark 2.3. Theorem 2.2 states that in the specific condition when α approaches to 1, we obtain

$$\begin{aligned}&\left|\frac{1}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq\ & \frac{(b-a)^2}{8}L_1,\end{aligned}$$

which was shown by Budak et al. in [10].

Corollary 2.3. *As α gets closer to 0, the following particular case happens, based on Theorem 2.2:*

$$\begin{aligned}&\left|\frac{1}{3}\left(2f'(a)-f'\left(\frac{a+b}{2}\right)+2f'(b)\right)-\frac{4}{(b-a)^2}\left(\int_{\frac{a+b}{2}}^{b} f(x)dx-\int_a^{\frac{a+b}{2}} f(x)dx\right)\right|\\ \leq\ & \frac{7(b-a)^2}{48}L_2.\end{aligned}$$

Following that, we provide a few Milne-type inequalities for proportional Caputo-hybrid operators by the use of bounded variation mappings.

Theorem 2.3. *Let $f,f':[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then, we have*

$$\begin{aligned}&\left|\frac{\alpha^2(b-a)^{\alpha}2^{-\alpha}}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)\right.\\ &+\frac{(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}}{3}\left(2f'(a)-f'\left(\frac{a+b}{2}\right)+2f'(b)\right)\\ &\left.-\frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}}\left[{}^{PC}_{\left(\frac{a+b}{2}\right)^+}D^{\alpha}_{b}f(b)+{}^{PC}_{\left(\frac{a+b}{2}\right)^-}D^{\alpha}_{a}f(a)\right]\right|\\ \leq\ & \frac{2^{-\alpha+1}\alpha^2}{3(b-a)^{-\alpha}}\bigvee_a^b(f)+\frac{2^{-1+\alpha}(1-\alpha)}{3(b-a)^{-1-\alpha}}\bigvee_a^b(f').\end{aligned}$$

where $\bigvee_a^b(f)$ and $\bigvee_a^b(f')$ denote the total variations of f and f' on $[a,b]$, respectively.

Proof. Take the mappings $K, L_\alpha : [a,b] \to \mathbb{R}$ defined by

$$K(t) = \begin{cases} (t-a) - \frac{2(b-a)}{3}, & a \le t \le \frac{a+b}{2} \\ \frac{2(b-a)}{3} - (b-t), & \frac{a+b}{2} \le t \le b, \end{cases}$$

and

$$L_\alpha(t) = \begin{cases} (t-a)^{2-2\alpha} - \frac{2(b-a)^{2-2\alpha}}{3.2^{1-2\alpha}}, & a \le t \le \frac{a+b}{2} \\ \frac{2(b-a)^{2-2\alpha}}{3.2^{1-2\alpha}} - (b-t)^{2-2\alpha}, & \frac{a+b}{2} \le t \le b, \end{cases}$$

respectively. From integration by parts it follows that

$$\int_a^b K(t)df(t) = \int_a^{\frac{a+b}{2}} \left((t-a) - \frac{2(b-a)}{3}\right) df(t) \tag{2.2}$$
$$+ \int_{\frac{a+b}{2}}^b \left(\frac{2(b-a)}{3} - (b-t)\right) df(t)$$
$$= -\frac{(b-a)}{6} f\left(\frac{a+b}{2}\right) + \frac{2(b-a)}{3} f(a) - \int_a^{\frac{a+b}{2}} f(t)dt$$
$$+ \frac{2(b-a)}{3} f(b) - \frac{(b-a)}{6} f\left(\frac{a+b}{2}\right) - \int_{\frac{a+b}{2}}^b f(t)\,dt,$$

and

$$\int_a^b L_\alpha(t)df'(t) = \int_a^{\frac{a+b}{2}} \left((t-a)^{2-2\alpha} - \frac{2(b-a)^{2-2\alpha}}{3.2^{1-2\alpha}}\right) df'(t) \tag{2.3}$$
$$+ \int_{\frac{a+b}{2}}^b \left(\frac{2(b-a)^{2-2\alpha}}{3.2^{1-2\alpha}} - (b-t)^{2-2\alpha}\right) df'(t)$$
$$= -\frac{(b-a)^{2-2\alpha}}{3.2^{2-2\alpha}} f'\left(\frac{a+b}{2}\right) + \frac{4(b-a)^{2-2\alpha}}{3.2^{2-2\alpha}} f'(a) - (2-2\alpha) \int_a^{\frac{a+b}{2}} (t-a)^{1-2\alpha} f'(t)dt$$
$$+ \frac{4(b-a)^{2-2\alpha}}{3.2^{2-2\alpha}} f'(b) - \frac{(b-a)^{2-2\alpha}}{3.2^{2-2\alpha}} f'\left(\frac{a+b}{2}\right) - (2-2\alpha) \int_{\frac{a+b}{2}}^b (b-t)^{1-2\alpha} f'(t)dt.$$

We obtain the following outcome by multiplying (2.2) by $2^{-\alpha}\alpha^2(b-a)^{-1+\alpha}$ and (2.3) by $2^{-1-\alpha}(1-\alpha)(b-a)^{-1+\alpha}$ and adding them side by side:

$$\begin{aligned}
&\frac{2^{-\alpha}\alpha^2}{(b-a)^{1-\alpha}}\int_a^b K(t)df(t)+\frac{2^{-1-\alpha}(1-\alpha)}{(b-a)^{1-\alpha}}\int_a^b L_\alpha(t)df'(t)\\
=\;& \frac{\alpha^2(b-a)^{\alpha}2^{-\alpha}}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)\\
&+\frac{(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}}{3}\left(2f'(a)-f'\left(\frac{a+b}{2}\right)+2f'(b)\right)\\
&-\frac{2^{-\alpha}\Gamma(1-\alpha)}{(b-a)^{-\alpha+1}}\left[{}^{PC}_{\left(\frac{a+b}{2}\right)^+}D^{\alpha}_{b}f(b)+{}^{PC}_{\left(\frac{a+b}{2}\right)^-}D^{\alpha}_{a}f(a)\right].
\end{aligned}$$

It is commonly known that if $g,h:[a,b]\to\mathbb{R}$ are in the sense that g is continuous on $[a,b]$ and h is of bounded variation on $[a,b]$, then $\int\limits_a^b g(t)dh(t)$ exists and

$$\left|\int_a^b g(t)dh(t)\right|\le \sup_{t\in[a,b]}|g(t)|\bigvee_a^b(h). \tag{2.4}$$

As a result, using (2.4), we discover

$$\begin{aligned}
&\left|\frac{\alpha^2(b-a)^{\alpha}2^{-\alpha}}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)\right.\\
&+\frac{(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}}{3}\left(2f'(a)-f'\left(\frac{a+b}{2}\right)+2f'(b)\right)\\
&\left.-\frac{2^{-\alpha}\Gamma(1-\alpha)}{(b-a)^{-\alpha+1}}\left[{}^{PC}_{\left(\frac{a+b}{2}\right)^+}D^{\alpha}_{b}f(b)+{}^{PC}_{\left(\frac{a+b}{2}\right)^-}D^{\alpha}_{a}f(a)\right]\right|\\
\le\;& \frac{2^{-\alpha}\alpha^2}{(b-a)^{1-\alpha}}\left[\left|\int_a^{\frac{a+b}{2}}\left((t-a)-\frac{2(b-a)}{3}\right)df(t)\right|\right.\\
&\left.+\left|\int_{\frac{a+b}{2}}^b\left(\frac{2(b-a)}{3}-(b-t)\right)df(t)\right|\right]\\
&+\frac{2^{-1-\alpha}(1-\alpha)}{(b-a)^{1-\alpha}}\left[\left|\int_a^{\frac{a+b}{2}}\left((t-a)^{2-2\alpha}-\frac{2(b-a)^{2-2\alpha}}{3.2^{1-2\alpha}}\right)df'(t)\right|\right.
\end{aligned}$$

$$+\left|\int_{\frac{a+b}{2}}^{b}\left(\frac{2\,(b-a)^{2-2\alpha}}{3.2^{1-2\alpha}}-(b-t)^{2-2\alpha}\right)df'(t)\right|\Bigg]$$

$$\leq \frac{2^{-\alpha}\alpha^2}{(b-a)^{1-\alpha}}\left[\sup_{t\in[a,\frac{a+b}{2}]}\left|(t-a)-\frac{2\,(b-a)}{3}\right|\bigvee_{a}^{\frac{a+b}{2}}(f)\right.$$

$$\left.+\sup_{t\in[\frac{a+b}{2},b]}\left|\frac{2\,(b-a)}{3}-(b-t)\right|\bigvee_{\frac{a+b}{2}}^{b}(f)\right]$$

$$+\frac{2^{-1-\alpha}(1-\alpha)}{(b-a)^{1-\alpha}}\left[\sup_{t\in[a,\frac{a+b}{2}]}\left|(t-a)^{2-2\alpha}-\frac{2\,(b-a)^{2-2\alpha}}{3.2^{1-2\alpha}}\right|\bigvee_{a}^{\frac{a+b}{2}}(f')\right.$$

$$\left.+\sup_{t\in[\frac{a+b}{2},b]}\left|\frac{2\,(b-a)^{2-2\alpha}}{3.2^{1-2\alpha}}-(b-t)^{2-2\alpha}\right|\bigvee_{\frac{a+b}{2}}^{b}(f')\right]$$

$$=\frac{2^{-\alpha}\alpha^2}{(b-a)^{1-\alpha}}\left[\frac{2(b-a)}{3}\bigvee_{a}^{\frac{a+b}{2}}(f)+\frac{2(b-a)}{3}\bigvee_{\frac{a+b}{2}}^{b}(f)\right]$$

$$+\frac{2^{-1-\alpha}(1-\alpha)}{(b-a)^{1-\alpha}}\left[\frac{2\,(b-a)^{2-2\alpha}}{3.2^{1-2\alpha}}\bigvee_{a}^{\frac{a+b}{2}}(f')+\frac{2\,(b-a)^{2-2\alpha}}{3.2^{1-2\alpha}}\bigvee_{\frac{a+b}{2}}^{b}(f')\right]$$

$$=\frac{2^{-\alpha+1}\alpha^2}{3(b-a)^{-\alpha}}\bigvee_{a}^{b}(f)+\frac{2^{-1+\alpha}(1-\alpha)}{3(b-a)^{-1+\alpha}}\bigvee_{a}^{b}(f').$$

□

Remark 2.4. In the particular situation when α approaches 1 in Theorem 2.3, we obtain

$$\left|\frac{1}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|$$

$$\leq \frac{2}{3}\bigvee_{a}^{b}(f),$$

which is equal to Theorem 6 proposed by Budak et al. [10], on the basis of the supposition that $\alpha=1$

Corollary 2.4. *In the specific case where α goes to 0 in Theorem 2.3, we get*

$$\left|\frac{1}{3}\left(2f'(a)-f'\left(\frac{a+b}{2}\right)+2f'(b)\right)-\frac{4}{(b-a)^2}\left(\int_{\frac{a+b}{2}}^{b} f(x)dx-\int_{a}^{\frac{a+b}{2}} f(x)dx\right)\right|$$
$$\leq \frac{2}{3}\bigvee_{a}^{b}(f').$$

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