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A COMPREHENSIVE STUDY OF (s, P)-FUNCTIONS AND THEIR APPLICATIONS TO MIDPOINT OF HERMITE-HADAMARD-FEJÉR TYPE INEQUALITIES

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Abstract. Hermite-Hadamard-Fejér type inequalities are a general version of Hermite-Hadamard inequalities, obtained under specific conditions with a certain type of weight function. These inequalities are a fundamental part of analysis, rooted in convex functions and existing inequalities. Moreover, (s, P)-functions represent a more generalized form of convexity, inspired by s-convexity and P-functions. This study is designed to derive midpoint-type Hermite-Hadamard-Fejér inequalities for (s, P)-functions. The newly established inequalities not only reduce to midpoint-type Hermite-Hadamard inequalities for (s, P)-functions, but also, in the special case where $s = 1$, the inequalities for P-functions are obtained. Additionally, applications supported by graphical illustrations are provided for general versions and special cases of derived inequalities.

1. Introduction

Convex functions are crucial in many fields, such as physics, economics, mathematics, statistics, and medicine. They are regarded as one of the most important subjects in modern research, as evidenced by the extensive literature dedicated to their study. Because of their significance, many researchers have identified and investigated different types of convex functions, leading to an ever-expanding body of work. A notable class is that of h-convex functions, which has paved the way for the development of various new types of convexity under specific circumstances. For example, Bombardelli and Varošanec [2] explored the properties of h-convex functions in the context of Hermite-Hadamard-Fejér inequalities, laying a foundation for further research. Similarly, Dragomir and Pearce [6] provided a comprehensive treatment of Hermite-Hadamard inequalities and their applications, which has inspired subsequent studies in the field. In addition, several refinements and extensions have been proposed: Zabandan [20] introduced a new refinement of the classical Hermite-Hadamard inequality for convex functions; Demir [4] developed

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new inequalities for functions whose second derivatives exhibit exponential trigonometric convexity; Turhan [18] offered novel results based on the generalization of integral inequalities for trigonometrically p-convex functions; and further contributions by Demir, Maden, İşcan, and Kadakal [3] as well as by Baloch and İşcan [1] have advanced the understanding of Simpson-type inequalities and Hermite-Hadamard type integral inequalities for harmonically convex functions. Fundamentally, convex functions are based on inequality conditions, which makes them invaluable for establishing new bounds and facilitating optimization tasks.

The Hermite-Hadamard (H-H) inequality is well-established in the literature for a continuous function $\xi : I \to \mathbb{R}$ and for all $k, l \in I \subset \mathbb{R}$ where $k < l$:

$$\xi\left(\frac{k+l}{2}\right) \leq \frac{1}{l-k}\int_{k}^{l} \xi(x)dx \leq \frac{\xi(k)+\xi(l)}{2}.$$

If ξ is concave, the inequality is reversed [8]. This inequality has been applied to various classes of convex functions, and through numerous lemma, theorems related to trapezoidal and midpoint inequalities have been developed and presented.

In 1906, Fejér introduced a weighted version of the Hermite-Hadamard (H-H) inequality, marking a significant development in the study of convex functions. This advancement, known today as the H-H Fejér type inequality, extended the classical result by incorporating weight functions, thereby enhancing the mathematical framework for analyzing convex functions. Building on this foundation, several researchers have provided refinements and generalizations that tailor these inequalities to specific weight function conditions. For example, Iscan, Numan, and Bekar [9] established both Hermite-Hadamard and Simpson-type inequalities for differentiable functions exhibiting harmonically P-convex behavior, which broadened the scope of the classical inequality. Numan and İşcan [14] introduced the concept of exponential type P-functions, offering further generalizations in this area. Moreover, Kunt and İşcan [12] provided a comprehensive treatment of Hermite–Hadamard–Fejér type inequalities for p-convex functions, while Latif, Dragomir, and Momoniat [13] developed new Fejér type inequalities for harmonically convex functions with applications to special means. In addition, refinements by Tseng, Hwang, and Dragomir [17] have sharpened the existing inequalities, and Khan et al. [10] contributed by deriving novel inequalities for s-convex functions. Finally, Dragomir and Fitzpatrick [5] focused on the Hadamard inequalities for s-convex functions in the second sense, further illuminating the intricate relationship between weight functions and the properties of convex function inequalities. Collectively, these developments have significantly expanded our understanding and application of convex function inequalities.

Theorem 1.1. [7] *Assume $\xi : [k, l] \to \mathbb{R}$ is a convex mapping. Then, the following inequality is satisfied:*

$$\xi\left(\frac{k+l}{2}\right)\int_{k}^{l} g(x)\,dx \leq \frac{1}{l-k}\int_{k}^{l} \xi(x)g(x)\,dx \leq \frac{\xi(k)+\xi(l)}{2}\int_{k}^{l} g(x)\,dx \tag{1.2}$$

where the function $g : [k, l] \to \mathbb{R}$ is nonnegative, integrable, and exhibits symmetry about $x = \frac{k+l}{2}$.

The domain of convex analysis has been substantially extended with the introduction of h-convexity, as outlined by Varosanec. This more generalized convexity class is based on a modulating function h, which is non-negative and distinct from zero, providing a broader scope than classical convexity.

Definition 1.1. [19] Let G, I be intervals and $h \neq 0$, where $h : G \to \mathbb{R}$ is a non-negative function. The function $\xi : I \to \mathbb{R}$ is called h-convex if for every $k, l \in I$ and $\omega \in (0, 1)$, the following inequality holds:

$$\xi(\omega k + (1-\omega)l) \leq h(\omega)\xi(k) + h(1-\omega)\xi(l).$$

If ξ is concave, the inequality is reversed. Functions in this convexity class are denoted by $SX(h, I)$.

This advanced convexity class has led to the emergence of numerous new convexity types. Among these are the (s, P)-functions introduced by İ. İşcan and S. Numan, along with the associated Hermite-Hadamard inequalities and theorems:

Definition 1.2. [15] Let $\xi : I \to \mathbb{R}$ be a non-negative function, where $k, l \in I$ and $\omega \in [0, 1]$. If the following inequality is satisfied, the function ξ is called a (s, P)-functions:

$$\xi(\omega k + (1-\omega)l) \leq [\omega^s + (1-\omega)^s](\xi(k) + \xi(l)).$$

We will denote by $P_s(I)$ the class of all (s, P)-functions on interval I. Clearly, the definition of $(1, P)$-function is coincide with the definition of P-function.

Theorem 1.2. [15] *Let $s \in (0, 1]$ and $\xi : [k, l] \to \mathbb{R}$ be a (s, P)-function. If $k < l$ and $\xi \in L[k, l]$, then the following Hermite-Hadamard type inequalities hold:*

$$2^{s-2}\xi\left(\frac{k+l}{2}\right) \leq \frac{1}{k-l}\int_k^l \xi(x)\,dx \leq \frac{2}{s+1}\left[\xi(k) + \xi(l)\right].$$

Hermite-Hadamard-Fejér-type inequalities occupy a significant position in the literature, offering a broad scope that includes the derivation of new inequalities and results for various weight functions as well as inequalities obtained for certain fractional integrals. Additionally, the (s, P)-convex function, a newly introduced class of convexity, has been observed to attract considerable interest among researchers. By leveraging the lemma provided by M.Z. Sarıkaya, significant new theorems have been established for midpoint-type Hermite-Hadamard-Fejér inequalities. These results not only highlight the versatility of (s, P)-functions in producing generalized inequalities but also offer new insights and deepen the understanding of the underlying mathematical structures. Through this research, various (s, P)-functions were generated, and their behaviors with respect to different weight functions were illustrated through graphical representations.

Lemma 1.1. [16] *Let $\xi : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, and $k, l \in I^\circ$, where $k < l$, and let $g : [k, l] \to [0, \infty)$ be a differentiable function. If $\xi' \in L[k, l]$, then the* *following equality holds:*

$$\frac{1}{l-k}\int\limits_k^l \xi(x)g(x)dx - \frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int\limits_k^l g(x)dx = (l-k)\int\limits_0^1 m(\omega)\xi'(\omega k+(1-\omega)l)d\omega$$

for each $\omega \in [0,1]$, *where*

$$m(\omega) = \begin{cases} \int\limits_0^\omega g(ks+(1-s)l)ds, & \omega\in[0,\frac{1}{2}] \\ -\int\limits_\omega^1 g(ks+(1-s)l)ds, & \omega\in[\frac{1}{2},1]. \end{cases}$$

Each theorem derived from this study provides novel upper bounds, offering deeper analytical insights into the behaviour of different functions. These theorems yield results that extend beyond the existing bounds of the literature, providing a broader spectrum of applications. In addition, various examples have been illustrated through graphical representations, showcasing the practical applications of these results. These graphs not only demonstrate the accuracy of the derived inequalities but also serve as a visual tool for understanding the behaviour of different function classes under the obtained theorems.

2. Hermite Hadamard Fejer type Inequality for (s,P)- Functions

Theorem 2.1. *Let* $\xi : I^\circ \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* I°, *where* $k,l\in I^\circ$ *with* $k<l$, *and let* $g:[k,l]\to[0,\infty)$ *be a differentiable function that is symmetric about* $\frac{k+l}{2}$. *Given that* ξ' *is* (s,P)*-functions over the interval* $[k,l]$, *the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{l-k}\int_k^l \xi(x)g(x)dx - \frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l g(x)dx\right| \\ \leq\ & \frac{2}{(l-k)^{s+1}(s+1)}\left(\int\limits_{\frac{k+l}{2}}^l g(x)\left[(x-k)^{s+1}-(l-x)^{s+1}\right]dx\right)\left[|\xi'(k)|+|\xi'(l)|\right]. \quad (2.1)\end{aligned}$$

Proof. Considering Lemma 1.1, the absolute value of both sides of the expression is first taken, and then the property that $|\xi'|$ is an (s,P)-function is applied:

$$\begin{aligned}&\left|\frac{1}{l-k}\int_k^l \xi(x)g(x)dx - \frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l g(x)dx\right| \\ \leq\ & (l-k)\int\limits_0^1 |m(\omega)|\left|\xi'(\omega k+(1-\omega)l)\right|d\omega \\ \leq\ & (l-k)\int\limits_0^1 |m(\omega)|\left[w^s+(1-w)^s\right]\left[|\xi'(k)|+|\xi'(l)|\right]d\omega\end{aligned}$$

is obtained. Using the definition of $m(\omega)$ as given in Lemma 1.1, the following inequality is derived:

$$\left|\frac{1}{l-k}\int_k^l \xi(x)g(x)dx - \frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l g(x)dx\right|$$

$$
\begin{aligned}
&\leq (l-k)\left\{\left|\int_0^{\frac{1}{2}}\left(\int_0^{\omega} g(kn+(1-n)l)dn\right)-\int_{\frac{1}{2}}^{1}\left(\int_{\omega}^{1} g(kn+(1-n)l)dn\right)\right|\right. \\
&\quad \left.\times\left[w^s+(1-w)^s\right]\left[\left|\xi'(k)\right|+\left|\xi'(l)\right|\right]d\omega\right\} \\
&\leq (l-k)\left\{\int_0^{\frac{1}{2}}\left(\int_0^{\omega} g(kn+(1-n)l)dn\right)\left[w^s+(1-w)^s\right]\left[\left|\xi'(k)\right|+\left|\xi'(l)\right|\right]d\omega\right. \\
&\quad \left.+\int_{\frac{1}{2}}^{1}\left(\int_{\omega}^{1} g(kn+(1-n)l)dn\right)\left[w^s+(1-w)^s\right]\left[\left|\xi'(k)\right|+\left|\xi'(l)\right|\right]d\omega\right\}. \qquad (2.2)
\end{aligned}
$$

At this point, by changing the order of integration on the right side of the last inequality, the following integrals are obtained:

$$
\begin{aligned}
T_1 &= \int_0^{\frac{1}{2}}\left(\int_0^{\omega} g(kn+(1-n)l)dn\right)\left[w^s+(1-w)^s\right]d\omega \\
&= \int_0^{\frac{1}{2}}\int_0^{\omega} g(kn+(1-n)l)\left[w^s+(1-w)^s\right]dn\,d\omega \\
&= \int_0^{\frac{1}{2}}\int_n^{\frac{1}{2}} g(kn+(1-n)l)\left[w^s+(1-w)^s\right]d\omega\,dn \\
&= \int_0^{\frac{1}{2}} g(kn+(1-n)l)\left[\frac{(1-n)^s-n^s}{s+1}\right]dn \\
&= \int_0^{\frac{1}{2}} g(kn+(1-n)l)\left[\frac{(1-n)^{s+1}-n^{s+1}}{s+1}\right]dn \\
&= \frac{1}{(l-k)^{s+2}(s+1)}\int_{\frac{k+l}{2}}^{l} g(x)\left[(x-k)^{s+1}-(l-x)^{s+1}\right]dx \qquad (2.3)
\end{aligned}
$$

and

$$
\begin{aligned}
T_2 &= \int_{\frac{1}{2}}^{1}\left(\int_{\omega}^{1} g(kn+(1-n)l)dn\right)\left[w^s+(1-w)^s\right]d\omega \\
&= \int_{\frac{1}{2}}^{1}\int_{\omega}^{1} g(kn+(1-n)l)\left[w^s+(1-w)^s\right]dn\,d\omega
\end{aligned}
$$

$$= \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{n} g(kn + (1-n)l) \left[w^s + (1-w)^s\right] d\omega \, dn$$

$$= \int_{\frac{1}{2}}^{1} g(kn + (1-n)l) \left[\frac{n^{s+1} - (1-n)^{s+1}}{s+1}\right] dn$$

$$= \int_{0}^{\frac{1}{2}} g(kn + (1-n)l) \left[\frac{(1-n)^s - n^s}{s+1}\right] dn$$

$$= \frac{1}{(l-k)^{s+2}(s+1)} \int_{\frac{k+l}{2}}^{l} g(x) \left[(l-x)^{s+1} - (x-k)^{s+1}\right] dx. \tag{2.4}$$

Since $g(x)$ is symmetric with respect to $x = \frac{k+l}{2}$ by hypothesis with $g(x) = g(k+l-x)$ for all $x \in [k, l]$, it follows that

$$T_1 = T_2.$$

Substituting the expressions for T_1 and T_2 from (2.3) and (2.4) into inequality (2.2), the proof is completed. □

Corollary 2.1. *If $g(x) = 1$ is taken in inequality (2.1), the following result is obtained:*

$$\left| \frac{1}{l-k} \int_{k}^{l} \xi(x)dx - \xi\left(\frac{k+l}{2}\right) \right| \leq \frac{(l-k)(2^{s+1}-1)}{2^{s+1}(s+1)(s+2)} \left[|\xi'(k)| + |\xi'(l)|\right].$$

In particular, if $s = 1$ is also chosen in the obtained inequality, then we get the following inequality [11, Theorem 2.2]*:*

$$\left| \frac{1}{l-k} \int_{k}^{l} \xi(x)dx - \xi\left(\frac{k+l}{2}\right) \right| \leq \frac{l-k}{8} \left[|\xi'(k)| + |\xi'(l)|\right]. \tag{2.5}$$

Theorem 2.2. *Let $\xi : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a function differentiable on I°, where $k, l \in I^{\circ}, k < l$, and let $g : [k, l] \to [0, \infty)$ be a differentiable function that is symmetric with respect to $\frac{k+l}{2}$. Under the condition that $q \geq 1, \frac{1}{p} + \frac{1}{q} = 1$, and given that the function $|\xi'|^q$ is (s, P)-functions on the interval $[k, l]$, it follows that:*

$$\left| \frac{1}{l-k} \int_k^l \xi(x)g(x)dx - \frac{1}{l-k}\xi\left(\frac{k+l}{2}\right) \int_k^l g(x)dx \right| \tag{2.6}$$

$$\leq 2\,(l-k)^{1-\frac{2}{p}} \left(\int_{\frac{k+l}{2}}^{l} g^p(x) \left(x - \frac{k+l}{2}\right) dx \right)^{\frac{1}{p}} \left(\frac{2^{s+1}-1}{2^{s+1}(s+1)(s+2)} \right)^{\frac{1}{q}} \left[|\xi'(k)|^q + |\xi'(l)|^q\right]^{\frac{1}{q}}.$$

Proof. Taking Lemma 1.1 into consideration, by applying the absolute value and changing the order of integration, the following inequality is derived:

$$\begin{aligned}
&\left|\frac{1}{l-k}\int_k^l \xi(x)g(x)dx - \frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l g(x)dx\right| \\
&= (l-k)\left|\int_0^1 m(\omega)\xi'(\omega k+(1-\omega)l)\right| d\omega \\
&= (l-k)\left|\int_0^{\frac{1}{2}}\left(\int_0^{\omega} g(nk+(1-n)l)dn\right)\xi'(k\omega+(1-\omega)l)d\omega\right. \\
&\left.-\int_{\frac{1}{2}}^{1}\left(\int_{\omega}^{1} g(nk+(1-n)l)dn\right)\xi'(k\omega+(1-\omega)l)d\omega\right| \\
&\leq (l-k)\int_0^{\frac{1}{2}}\int_0^{\omega} g(nk+(1-n)l)\left|\xi'(k\omega+(1-\omega)l)\right| dnd\omega \\
&+\int_{\frac{1}{2}}^{1}\int_{\omega}^{1} g(nk+(1-n)l)\left|\xi'(k\omega+(1-\omega)l)\right| dnd\omega \\
&= (l-k)\int_0^{\frac{1}{2}}\int_n^{\frac{1}{2}} g(nk+(1-n)l)\left|\xi'(k\omega+(1-\omega)l)\right| d\omega dn \\
&+\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{n} g(nk+(1-n)l)\left|\xi'(k\omega+(1-\omega)l)\right| d\omega dn.
\end{aligned}$$

On the right-hand side of the obtained inequality, applying Hölder's inequality and utilizing the fact that the function $|f'|^q$ is an (s, P)-function, the following inequality is derived:

$$\begin{aligned}
&\left|\frac{1}{l-k}\int_k^l \xi(x)g(x)dx - \frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l g(x)dx\right| \\
\leq\ & (l-k)\left\{\left(\int_0^{\frac{1}{2}}\int_n^{\frac{1}{2}} g^p(nk+(1-n)l)d\omega dn\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\int_n^{\frac{1}{2}}\left|\xi'(nk+(1-n)l)\right|^q d\omega dn\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{n} g^p(nk+(1-n)l)d\omega dn\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{n}\left|\xi'(nk+(1-n)l)\right|^q d\omega dn\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

$$\begin{aligned}
&\leq (l-k)\left\{\left(\int_0^{\frac{1}{2}} g^p(nk+(1-n)l)\left(\frac{1}{2}-n\right)dn\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\int_n^{\frac{1}{2}}\left[\omega^s+(1-\omega)^s\right]\left[|\xi'(k)|^q+|\xi'(l)|^q\right]d\omega dn\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_{\frac{1}{2}}^{1} g^p(nk+(1-n)l)\left(n-\frac{1}{2}\right)dn\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{n}\left[\omega^s+(1-\omega)^s\right]\left[|\xi'(k)|^q+|\xi'(l)|^q\right]d\omega dn\right)^{\frac{1}{q}}\right\}\\
&\leq (l-k)\left\{\left(\int_0^{\frac{1}{2}} g^p(x)\left(\frac{1}{2}-n\right)dn\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\int_n^{\frac{1}{2}}\left[\omega^s+(1-\omega)^s\right]\left[|\xi'(k)|^q+|\xi'(l)|^q\right]d\omega dn\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_{\frac{1}{2}}^{1} g^p(nk+(1-n)l)\left(n-\frac{1}{2}\right)dn\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{n}\left[\omega^s+(1-\omega)^s\right]\left[|\xi'(k)|^q+|\xi'(l)|^q\right]d\omega dn\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Utilizing the assumption that the function $g(x)$ is symmetric with respect to $x = \frac{k+l}{2}$ and acknowledging that $|\xi'|^q$ represents an (s,P)-function, the inequality is further refined to:

$$\begin{aligned}
&\left|\frac{1}{l-k}\int_k^l \xi(x)g(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l g(x)dx\right|\\
\leq\ & (l-k)\left(\int_{\frac{k+l}{2}}^{l} g^p(x)\left(\frac{2x-l-k}{2(l-k)^2}\right)dx\right)^{\frac{1}{p}}\left[|\xi'(k)|^q+|\xi'(l)|^q\right]^{\frac{1}{q}}\\
&\times\left[\left(\int_0^{\frac{1}{2}}\int_n^{\frac{1}{2}}(\omega^s+(1-\omega)^s)d\omega dn\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{n}(\omega^s+(1-\omega)^s)d\omega dn\right)^{\frac{1}{q}}\right]\\
=\ & (l-k)\left(\int_{\frac{k+l}{2}}^{l} g^p(x)\left(\frac{2x-l-k}{2(l-k)^2}\right)dx\right)^{\frac{1}{p}}\left[|\xi'(k)|^q+|\xi'(l)|^q\right]^{\frac{1}{q}}\\
&\times\left[\left(\int_0^{\frac{1}{2}}\left(\frac{(1-n)^{s+1}-n^{s+1}}{s+1}\right)dn\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^{1}\left(\frac{n^{s+1}-(1-n)^{s+1}}{s+1}\right)dn\right)^{\frac{1}{q}}\right].
\end{aligned}$$

The integrals obtained on the right side of the final expression are as follows:

$$\int_0^{\frac{1}{2}}\left(\frac{(1-n)^{s+1}-n^{s+1}}{s+1}\right)dn=\int_{\frac{1}{2}}^{1}\left(\frac{n^{s+1}-(1-n)^{s+1}}{s+1}\right)dn=\frac{2^{s+1}-1}{2^{s+1}(s+1)(s+2)}.$$

Thus, the proof is completed. □

Corollary 2.2. *If $g(x) = 1$ is taken in Theorem 2.2, the following inequality is obtained:*

$$\left|\frac{1}{l-k}\int\limits_{k}^{l}\xi(x)dx-\xi\left(\frac{k+l}{2}\right)\right|\leq\frac{l-k}{2^{\frac{3}{p}-1}}\left(\frac{2^{s+1}-1}{2^{s+1}(s+1)(s+2)}\right)^{\frac{1}{q}}\left[|\xi'(k)|^{q}+|\xi'(l)|^{q}\right]^{\frac{1}{q}}. \quad (2.7)$$

By setting $s = 1$ in (2.7), the following inequality is obtained :

$$\left|\frac{1}{l-k}\int\limits_{k}^{l}\xi(x)dx-\xi\left(\frac{k+l}{2}\right)\right|\leq\frac{l-k}{4}\left[|\xi'(k)|^{q}+|\xi'(l)|^{q}\right]^{\frac{1}{q}}. \quad (2.8)$$

The goal should be to achieve the optimal upper bound. Building on this, the obtained theorem and results are derived below by employing the Power Mean inequality, which is a consequence of Hölder's inequality:

Theorem 2.3. *Let $\xi : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a function differentiable on I°, where $k, l \in I^{\circ}, k < l$, and let $g : [k, l] \to [0, \infty)$ be a differentiable function that is symmetric to $\frac{k+l}{2}$. Under the condition that $q > 1$, and given that the function $|\xi'|^{q}$ are (s, P)-functions on the interval $[k, l]$, it follows that:*

$$\begin{aligned}&\left|\frac{1}{l-k}\int\limits_{k}^{l}\xi(x)g(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_{k}^{l}g(x)dx\right|\\ \leq\ &\frac{2}{(l-k)^{1+\frac{s}{q}}}\left[\int\limits_{\frac{k+l}{2}}^{l}\left(x-\frac{k+l}{2}\right)g(x)dx\right]^{1-\frac{1}{q}}\\ &\times\left(\int\limits_{\frac{k+l}{2}}^{l}g(x)\left[(x-k)^{s+1}-(l-x)^{s+1}\right]dx\right)^{\frac{1}{q}}\left[|\xi'(k)|^{q}+|\xi'(l)|^{q}\right]^{\frac{1}{q}}. \quad (2.9)\end{aligned}$$

Proof. To determine an upper bound, we first take the absolute value of both sides of the equality given in Lemma 1.1 and apply Fubini's theorem. This yields the following expression:

$$\begin{aligned}&\left|\frac{1}{l-k}\int_{k}^{l}\xi(x)g(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_{k}^{l}g(x)dx\right|\\ =\ &(l-k)\left|\int\limits_{0}^{1}m(\omega)\xi'(\omega k+(1-\omega)l)\right|d\omega\\ =\ &(l-k)\left|\int\limits_{0}^{\frac{1}{2}}\left(\int\limits_{0}^{\omega}g(nk+(1-n)l)dn\right)\xi'(k\omega+(1-\omega)l)d\omega\right.\\ &\left.-\int\limits_{\frac{1}{2}}^{1}\left(\int\limits_{\omega}^{1}g(nk+(1-n)l)dn\right)\xi'(k\omega+(1-\omega)l)d\omega\right|\end{aligned}$$

$$\begin{aligned}
&\leq (l-k)\int_{0}^{\frac{1}{2}}\int_{0}^{\omega} g(nk+(1-n)l)\left|\xi'(k\omega+(1-\omega)l)\right| dnd\omega \\
&\quad+\int_{\frac{1}{2}}^{1}\int_{\omega}^{1} g(nk+(1-n)l)\left|\xi'(k\omega+(1-\omega)l)\right| dnd\omega \\
&= (l-k)\int_{0}^{\frac{1}{2}}\int_{n}^{\frac{1}{2}} g(nk+(1-n)l)\left|\xi'(k\omega+(1-\omega)l)\right| d\omega dn \\
&\quad+\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{n} g(nk+(1-n)l)\left|\xi'(k\omega+(1-\omega)l)\right| d\omega dn.
\end{aligned}$$

If the Power Mean inequality, which is a consequence of Hölder's inequality, is applied to the obtained inequality to achieve the desired upper bound, the following result is obtained:

$$\begin{aligned}
&\left|\frac{1}{l-k}\int_k^l \xi(x)g(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l g(x)dx\right| \\
&\leq (l-k)\left[\left(\int_{0}^{\frac{1}{2}}\int_{n}^{\frac{1}{2}} g(kn+(1-n)l)d\omega dn\right)^{1-\frac{1}{q}}\right. \\
&\quad\times\left(\int_{0}^{\frac{1}{2}}\int_{n}^{\frac{1}{2}} g(kn+(1-n)l)\left|\xi'(k\omega+(1-\omega)l)\right|^q d\omega dn\right)^{\frac{1}{q}} \\
&\quad+\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{n} g(kn+(1-n)l)d\omega dn\right)^{1-\frac{1}{q}} \\
&\quad\left.\times\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{n} g(kn+(1-n)l)\left|\xi'(k\omega+(1-\omega)l)\right|^q d\omega dn\right)^{\frac{1}{q}}\right] \\
&\leq (l-k)\left[\left(\int_{0}^{\frac{1}{2}} g(kn+(1-n)l)\left(\frac{1}{2}-n\right)dn\right)^{1-\frac{1}{q}}\right. \\
&\quad\times\left(\int_{0}^{\frac{1}{2}}\int_{n}^{\frac{1}{2}} g(kn+(1-n)l)\left|\xi'(k\omega+(1-\omega)l)\right|^q d\omega dn\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\int_{\frac{1}{2}}^{1} g(kn+(1-n)l)\left(n-\frac{1}{2}\right)dn\right)^{1-\frac{1}{q}}$$
$$\left.\times\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{n} g(kn+(1-n)l)\left|\xi'(\omega k+(1-\omega)l)\right|^{q} d\omega dn\right)^{\frac{1}{q}}\right].$$

Since the function $|\xi'|^q$ is (s,P)-functions, the following inequality is obtained:

$$\begin{aligned}
&\left|\frac{1}{l-k}\int_k^l \xi(x)g(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l g(x)dx\right|\\
\leq\ &(l-k)\left[\left(\int_0^{\frac{1}{2}} g(kn+(1-n)l)\left(\frac{1}{2}-n\right)dn\right)^{1-\frac{1}{q}}\right.\\
&\times\left(\int_0^{\frac{1}{2}}\int_n^{\frac{1}{2}} g(kn+(1-n)l)\left[\omega^s+(1-\omega)^s\right]\left[|\xi'(k)|^q+|\xi'(l)|^q\right]d\omega dn\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^{1} g(kn+(1-n)l)\left(n-\frac{1}{2}\right)dn\right)^{1-\frac{1}{q}}\\
&\left.\times\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{n} g(kn+(1-n)l)\left[\omega^s+(1-\omega)^s\right]\left[|\xi'(k)|^q+|\xi'(l)|^q\right]d\omega dn\right)^{\frac{1}{q}}\right].
\end{aligned}$$

In the final inequality, after taking the integrals on the right side and applying the variable change $x=kn+(1-n)l$, the following inequality is obtained:

$$\begin{aligned}
&\left|\frac{1}{l-k}\int_k^l \xi(x)g(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l g(x)dx\right|\\
\leq\ &(l-k)\left[|\xi'(k)|^q+|\xi'(l)|^q\right]^{\frac{1}{q}}\\
&\times\left[\left(\int_0^{\frac{1}{2}} g(kn+(1-n)l)\left(\frac{1}{2}-n\right)dn\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}} g(kn+(1-n)l)\left(\frac{(1-n)^{s+1}-n^{s+1}}{s+1}\right)dn\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_{\frac{1}{2}}^{1} g(kn+(1-n)l)\left(n-\frac{1}{2}\right)dn\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1} g(kn+(1-n)l)\left(\frac{n^{s+1}-(1-n)^{s+1}}{s+1}\right)dn\right)^{\frac{1}{q}}\right]
\end{aligned}$$

$$
\begin{aligned}
= \quad & (l-k)\left[|\xi'(k)|^q+|\xi'(l)|^q\right]^{\frac{1}{q}} \\
& \times\left[\left(\int_{\frac{k+l}{2}}^{l} g(x)\left(\frac{1}{2}-\frac{l-x}{l-k}\right)\frac{dx}{l-k}\right)^{1-\frac{1}{q}}\left(\int_{\frac{k+l}{2}}^{l} g(x)\left(\frac{(x-k)^{s+1}-(l-x)^{s+1}}{(l-k)^{s+1}(s+1)}\right)\frac{dx}{l-k}\right)^{\frac{1}{q}}\right. \\
& \left.+\left(\int_{k}^{\frac{k+l}{2}} g(x)\left(\frac{l-x}{l-k}-\frac{1}{2}\right)\frac{dx}{l-k}\right)^{1-\frac{1}{q}}\left(\int_{k}^{\frac{k+l}{2}} g(x)\left(\frac{(l-x)^{s+1}-(x-k)^{s+1}}{(l-k)^{s+1}(s+1)}\right)\frac{dx}{l-k}\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

By using the fact that the function $g(x)$ is symmetric concerning $x=\frac{k+l}{2}$, the proof is completed. □

Corollary 2.3. *If $g(x)=1$ is taken in Theorem 2.3, the following inequality is obtained:*

$$
\left|\frac{1}{l-k}\int_{k}^{l}\xi(x)dx-\xi\left(\frac{k+l}{2}\right)\right|\leq\frac{(l-k)(2^{s+1}-1)^{\frac{1}{q}}}{2^{2-\frac{2}{q}}}\left[\frac{|\xi'(k)|^q+|\xi'(l)|^q}{2^s(s+2)}\right]^{\frac{1}{q}}. \tag{2.10}
$$

Taking $s=1$ in (2.10), the following inequality is obtained [11, Theorem 2.4]*:*

$$
\left|\frac{1}{l-k}\int_{k}^{l}\xi(x)dx-\xi\left(\frac{k+l}{2}\right)\right|\leq\frac{l-k}{2^{2-\frac{2}{q}}}\left[\frac{|\xi'(k)|^q+|\xi'(l)|^q}{2}\right]^{\frac{1}{q}}. \tag{2.11}
$$

3. Application

In the literature, visualizing the theorems or results obtained from studies through graphs across different values and functions plays a significant role in understanding the outcomes. In this section, we present graphical applications of the Hermite-Hadamard-Fejér type inequalities obtained for different weighted functions. These graphs not only verify the accuracy of the derived inequalities but also illustrate the upper bounds achieved by incorporating Hölder's inequality and the Power Mean inequality alongside the weighted functions.

Example 3.1. The function $\xi(x)=\exp(x)$ and the Gaussian function $g(x,k,l,a=0.1)=\exp\left(-a\left(x-\frac{k+l}{2}\right)^2\right)$, which is symmetrically defined with respect to $\frac{k+l}{2}$, are considered. Based on these functions, theorems (2.1), (2.2), and (2.3) are applied to the inequalities (2.1) and (2.6), respectively, within different intervals and for each curve with parameters $s=0.5$, $p=3$, and $q=\frac{2}{3}$. Furthermore, in the inequality (2.9), where the interval remains the same, the case with $s=0.5$ and $q=3$ is considered, and it is demonstrated that the theoretical inequalities hold for different values within the given intervals. Here, while k is kept constant, graphs were obtained by calculating the values of l for 100 different points. These graphs clearly illustrate not only the results provided by the classical inequality but also how the Hölder and Power Mean inequalities behave, as evidenced through the graphical results.

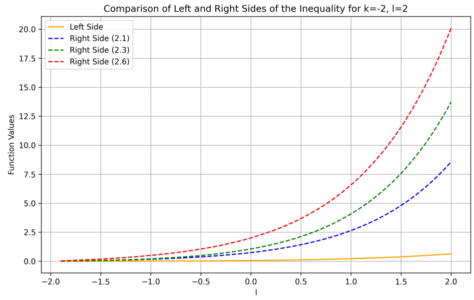

FIGURE 1

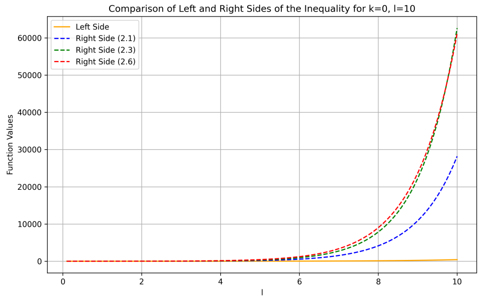

FIGURE 2

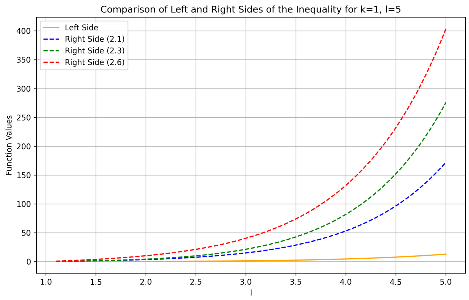

FIGURE 3

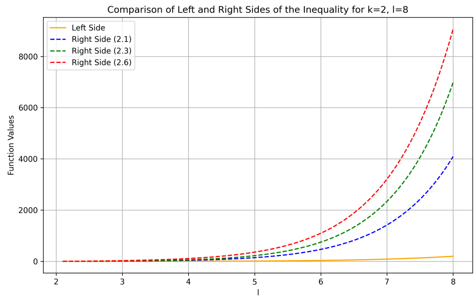

FIGURE 4

Example 3.2. It is well known that when $g(x) = 1$ is taken in the Hermite-Hadamard-Fejér type inequality, the classical Hermite-Hadamard inequalities are obtained. We also present these results in our study. The inequalities we derived include (2.7), (2.8), and (2.10). Using the same data as in Example (3.1), we plotted curves for both the left-hand and right-hand sides of the obtained inequalities to compare the results. These graphs demonstrate that the theoretically valid expressions are also satisfied in practice within the context of Hermite-Hadamard type inequalities.

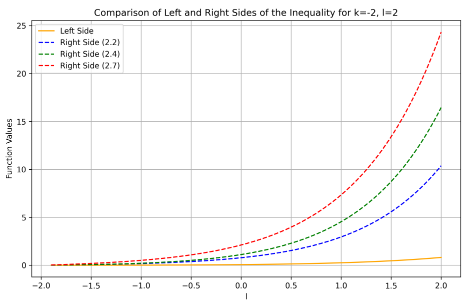

FIGURE 5

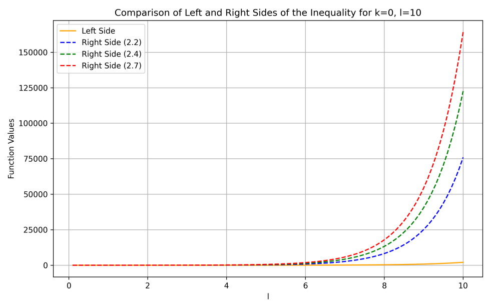

FIGURE 6

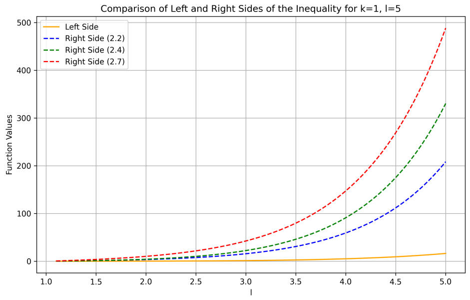

FIGURE 7

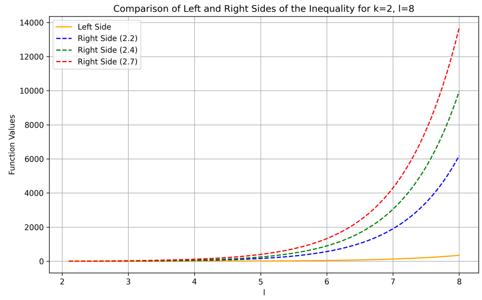

FIGURE 8

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