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FRACTIONAL INEQUALITIES OF MILNE-TYPE FOR TWICE DIFFERENTIABLE STRONGLY CONVEX FUNCTIONS

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Abstract. This paper aims to prove several novel Milne-type integral inequalities for differentiable strongly convex functions. To prove the main findings, an integral identity, Riemann-Liouville fractional integrals, differentiable strongly convex functions and well-known inequalities such as Hölder and Young have been used. Several special cases of main findings have been considered.

1. Introduction

Numerical integration, a fundamental component of mathematical computation, becomes particularly essential when the analytical integration of complex functions is infeasible. By employing numerical techniques, mathematicians and researchers strive to enhance the accuracy of computed integrals and to more precisely determine upper bounds on the associated errors. Research in this area has critically evaluated the performance of various numerical methods and conducted in-depth analyses to better understand the error behavior of each approach.

Investigating error bounds in numerical integration necessitates a detailed examination of mathematical inequalities for different classes of functions, such as convex, bounded, and Lipschitz-continuous functions. These inequalities offer valuable insights into how error estimations can be optimized by leveraging the structural properties of the underlying functions.

In particular, the study of functions whose first or second derivatives satisfy convexity conditions enables more accurate estimation of error bounds in numerical integration. Such functions exhibit distinct mathematical and geometric properties that play a significant role in error analysis. These insights contribute to the ongoing development of methods aimed at improving the theoretical and practical efficiency of numerical integration.

Key words and phrases. Strongly convex, Milne-type inequalities, fractional integrals.

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Both classical and modern techniques establish different error bounds, leading to varied conclusions about their accuracy and efficiency. In this context, a deeper understanding of numerical integration methods and their associated error estimates may reveal potential improvements that enhance the reliability of the integration process.

This paper aims to provide a comprehensive review of numerical integration techniques and their corresponding error bounds. We now turn to a detailed discussion of the fundamental principles of these methods and the general characteristics of their associated error bounds.

1. The expression below is known as Simpson's $\frac{1}{3}$ rule, which is a widely used formula for numerical integration.

$$\int_{\rho_1}^{\rho_2} \chi(\epsilon)d\epsilon \approx \frac{\rho_2-\rho_1}{6}[\chi(\rho_1)+4\chi(\frac{\rho_1+\rho_2}{2})+\chi(\rho_2)]. \tag{1.1}$$

2. Simpson's second formula, also referred to as the Newton-Cotes quadratic formula or more commonly as Simpson's $\frac{3}{8}$ rule (see [6]), is defined as follows:

$$\int_{\rho_1}^{\rho_2} \chi(\epsilon)d\epsilon \approx \frac{\rho_2-\rho_1}{8}[\chi(\rho_1)+3\chi(\frac{2\rho_1+\rho_2}{3})+3\chi(\frac{\rho_1+2\rho_2}{3})+\chi(\rho_2)]. \tag{1.2}$$

Equations (1.1) and (1.2) hold for any function χ whose fourth derivative exists continuously on the interval $[\rho_1,\rho_2]$:

The general expression for Simpson's inequality is presented in the following standard form:

Theorem 1.1. *When considerning $\chi:[\rho_1,\rho_2]\to\mathbb{R}$, a function with four continuous derivatives within the interval (ρ_1,ρ_2), and $\left\|\chi^{(4)}\right\|_\infty=\sup_{\epsilon\in(\rho_1,\rho_2)}|\chi^{(4)}(\epsilon)|<\infty$, the subsequent inequality holds:*

$$\left|\frac{1}{6}\left[\chi(\rho_1)+4\chi(\frac{\rho_1+\rho_2}{2})+\chi(\rho_2)\right]-\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\chi(\epsilon)d\epsilon\right|\le\frac{1}{2880}\left\|\chi^{(4)}\right\|_\infty(\rho_2-\rho_1)^4.$$

The Simpson-type inequality was first proven by Sarıkaya et al. in [27] for convex functions. In the context of Riemann-Liouville fractional integrals, there exist three distinct types of Simpson-type inequalities, classified based on their fractional integral representations. These inequalities have been further developed and refined in [5, 16, 20], contributing to the extension of Simpson's inequality within fractional analysis and demonstrating its applicability to various types of fractional integrals.

Moreover, special attention has been given to Simpson-type inequalities for twice differentiable functions in [23, 26, 32]. These studies provide an in-depth analysis of the application of Simpson's inequality to such functions and present specific cases of the inequality tailored for this function class. As a result, the scope of Simpson-type inequalities has been broadened, enabling more precise and specialized results for certain classes of functions.

The classical Newton inequality holds a fundamental place in mathematical analysis due to its broad applicability in various fields, including algebra, combinatorics, and optimization. This important inequality provides valuable insights into the relationships between symmetric polynomials and has numerous implications in theoretical and applied mathematics. It is formally stated as follows:

Theorem 1.2 (See [6]). *If $\chi : [\rho_1,\rho_2] \to \mathbb{R}$ represents a function with a continuous fourth derivative defined over (ρ_1, ρ_2), and $\left\|\chi^{(4)}\right\|_\infty = \sup_{\epsilon\in(\rho_1,\rho_2)} |\chi^{(4)}(\epsilon)| < \infty$, then the inequality presented below is valid:*

$$\begin{aligned}&\left|\frac{1}{8}\left[\chi(\rho_1) + 3\chi(\frac{2\rho_1+\rho_2}{3}) + 3\chi(\frac{\rho_1+2\rho_2}{3}) + \chi(\rho_2)\right] - \frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\chi(\epsilon)d\epsilon\right| \\ \leq\ & \frac{1}{6480}\left\|\chi^{(4)}\right\|_\infty (\rho_2-\rho_1)^4.\end{aligned}$$

In [18,19,21], the authors establish Newton-type inequalities by utilizing convex functions within the framework of local fractional integrals. These works not only extend the classical Newton inequality to the realm of fractional analysis but also offer new insights into its applications, thereby enriching the theoretical foundations of the field. By adapting Newton's inequality to the context of fractional calculus, these contributions highlight the interplay between convexity and integral inequalities in non-integer order settings.

A significant milestone in this direction was achieved in [31], where the first rigorous proofs of Newton-type inequalities for Riemann-Liouville fractional integrals were presented. This study laid the groundwork for further exploration in fractional calculus, providing fundamental results that have since become a key reference for researchers in the field. Following this foundational work, numerous studies have been conducted on Riemann-Liouville fractional integrals, with particular emphasis on deriving Newton-type inequalities and thoroughly examining their validity under different conditions [14,33]. These contributions have played a crucial role in advancing the theory of fractional integrals, addressing gaps in the literature, and fostering the development of new mathematical techniques within fractional analysis.

In [12], Djenaoui and Meftah made a significant contribution to the study of Milne-type inequalities by incorporating the concept of convexity for the first time. Their work represents a crucial step in analyzing Milne inequalities within the framework of convex analysis, offering a broader and more generalized perspective on these inequalities. By leveraging convexity, this research provided a refined mathematical structure, facilitating a deeper understanding of the underlying properties of Milne-type inequalities.

Building upon this foundation, Budak and his colleagues expanded the applicability of these inequalities in [4], extending their scope to the domain of Riemann-Liouville fractional integrals. This study played a key role in strengthening the theoretical framework of Milne-type inequalities within fractional analysis. By adapting these inequalities to fractional calculus, the authors introduced a fresh perspective on their application in different analytical settings, thereby bridging classical and modern mathematical approaches. In [17], the authors have proved several new Milne-type inequalities by using tempered fractional integral operators.

More recently, significant advancements have been made in [1,3], where novel fractional variations of Milne-type inequalities were introduced and analyzed through the lens of separable convex functions. These studies further extended the applicability of Milne-type inequalities by examining their validity across different function classes, including bounded functions, Lipschitz functions, and functions of bounded variation. Through these detailed investigations, the scope of Milne-type inequalities has been considerably broadened, enhancing their utility in various mathematical contexts.

For a more comprehensive understanding of Milne-type inequalities and to explore the latest developments in this area, references [11, 13, 30] provide valuable insights. These works offer an in-depth analysis of various generalizations of Milne-type inequalities, their theoretical underpinnings, and their applications in different branches of mathematical analysis.
We will proceed by giving the following definition which is very important for fractional calculus.

Definition 1.1. (See [22]) Let $\Psi \in L_1[\epsilon_1, \epsilon_2]$. The RL-integrals $J^{\alpha}_{\epsilon_1^+}\Psi$ and $J^{\alpha}_{\epsilon_2^-}\Psi$ of order $\alpha > 0$ with $\epsilon_1 \geq 0$ are defined by

$$J^{\alpha}_{\epsilon_1^+}\Psi(u_1) = \frac{1}{\Gamma(\alpha)}\int_{\epsilon_1}^{u_1}(u_1-\zeta)^{\alpha-1}\Psi(\zeta)d\zeta, \qquad u_1 > \epsilon_1$$

and

$$J^{\alpha}_{\epsilon_2^-}\Psi(u_1) = \frac{1}{\Gamma(\alpha)}\int_{u_1}^{\epsilon_2}(\zeta-u_1)^{\alpha-1}\Psi(\zeta)d\zeta, \qquad u_1 < \epsilon_2,$$

where $\Gamma(\alpha) = \int_0^{\infty} e^{-\zeta}u^{\alpha-1}du$, here is $J^{0}_{\epsilon_1^+}\Psi(u_1) = J^{0}_{\epsilon_2^-}\Psi(u_1) = \Psi(u_1)$.

In the above definition, if we set $\alpha = 1$, the definition overlaps with the classical integral. Features of the fractional integral operator can be found in the references [7–10, 15, 22, 28]. Let us continue our study by giving the definition of convexity and s-convexity in the second sense.

Definition 1.2. [24] Let $\mathbb{A}$ be on interval in $\mathbb{R}$. Then $\chi : \mathbb{A} \to \mathbb{R}$ is said to be convex, if

$$\chi\left(\epsilon\rho_1 + (1-\epsilon)\rho_2\right) \leq \epsilon\chi\left(\rho_1\right) + (1-\epsilon)\chi\left(\rho_2\right)$$

holds for all $\rho_1, \rho_2 \in \mathbb{A}$ and $\epsilon \in [0,1]$.

In [25], Polyak B.T defines the strongly convex function as follows

Definition 1.3. A function such that $\chi : \mathbb{A} \to \mathbb{R}$, is called strongly convex function with moduls $\eta > 0$, if

$$\chi\left(\epsilon\rho_1 + (1-\epsilon)\rho_2\right) \leq \epsilon\chi\left(\rho_1\right) + (1-\epsilon)\chi\left(\rho_2\right) - \eta\epsilon(1-\epsilon)(\rho_1-\rho_2)^2$$

for all $\rho_1, \rho_2 \in \mathbb{A}$, and $\epsilon \in [0,1]$.

Lemma 1.1. [2] *If $\chi : [\rho_1, \rho_2] \longrightarrow R$ is absolutely continuous over (ρ_1, ρ_2) and $\chi'' \in L_1([\rho_1, \rho_2])$, then the following equality holds:*

$$\frac{\Gamma(\zeta+1)}{2(\rho_2-\rho_1)^{\zeta}}[\Im^{\zeta}_{\rho_1^+}\chi(\rho_2) + \Im^{\zeta}_{\rho_2^-}\chi(\rho_1)] - \frac{1}{3}[2\chi(\rho_1) - \chi(\frac{\rho_1+\rho_2}{2}) + 2\chi(\rho_2)] = \frac{(\rho_2-\rho_1)^2}{2(\zeta+1)}\sum_{k=1}^{4} I_k,$$

where

$$I_1 = \int_0^{\frac{1}{2}} (\epsilon^{\zeta+1} - \frac{\zeta+4}{3}\epsilon)\chi''(\epsilon\rho_2 + (1-\epsilon)\rho_1)d\epsilon,$$
$$I_2 = \int_0^{\frac{1}{2}} (\epsilon^{\zeta+1} - \frac{\zeta+4}{3}\epsilon)\chi''(\epsilon\rho_1 + (1-\epsilon)\rho_2)d\epsilon,$$
$$I_3 = \int_{\frac{1}{2}}^1 (\epsilon^{\zeta+1} - \epsilon)\chi''(\epsilon\rho_2 + (1-\epsilon)\rho_1)d\epsilon,$$
$$I_4 = \int_{\frac{1}{2}}^1 (\epsilon^{\zeta+1} - \epsilon)\chi''(\epsilon\rho_1 + (1-\epsilon)\rho_2)d\epsilon.$$

The primary objective of this paper is to establish new fractional Milne-type inequalities applicable to functions whose second derivatives exhibit convexity. To accomplish this, we first lay a foundational framework by introducing key mathematical concepts, including Riemann-Liouville fractional integrals, convexity, and strongly convex functions. These fundamental principles provide the necessary groundwork for the subsequent development of novel fractional Milne-type inequalities under convexity constraints.

Building upon this theoretical foundation, we present a refined formulation of fractional Milne inequalities that incorporate convexity conditions, thereby extending their applicability within the realm of fractional analysis. By integrating convexity properties into the fractional setting, this study offers deeper insights into the behavior of these inequalities and their potential applications. Ultimately, this work aims to enhance the understanding of Milne-type inequalities in fractional calculus, contributing to the ongoing development of mathematical inequalities and their applications in various analytical contexts.

2. Main results

Theorem 2.1. *Assume the conditions of Lemma 1.1 are satisfied. Furthermore, if* $\left|\chi''\right|$ *exhibits strongly convex over* $[\rho_1, \rho_2]$, *then:*

$$\begin{aligned}
&\left|\frac{\Gamma(\zeta+1)}{2(\rho_2-\rho_1)^{\zeta}}[\Im^{\zeta}_{\rho_1^+}\chi(\rho_2) + \Im^{\zeta}_{\rho_2^-}\chi(\rho_1)] - \frac{1}{3}[2\chi(\rho_1) - \chi(\frac{\rho_1+\rho_2}{2}) + 2\chi(\rho_2)]\right| \\
\leq\ & \frac{(\rho_2-\rho_1)^2}{2(\zeta+1)} \times \left[\frac{(\zeta^2+15\zeta+2)}{24(\zeta+2)}\left(\left|\chi''(\rho_1)\right| + \left|\chi''(\rho_2)\right|\right)\right. \\
& \left. -\eta(\rho_1-\rho_2)^2\left[\frac{(5\zeta+53)(\zeta+3)(\zeta+4)-576}{288(\zeta+3)(\zeta+4)}\right]\right],
\end{aligned}$$

where $\eta > 0$.

Proof. On applying the modulus operation to Lemma 1.1, we obtain:

$$\begin{aligned}
&\left|\frac{\Gamma(\zeta+1)}{2(\rho_2-\rho_1)^{\zeta}}[\Im^{\zeta}_{\rho_1^+}\chi(\rho_2) + \Im^{\zeta}_{\rho_2^-}\chi(\rho_1)] - \frac{1}{3}[2\chi(\rho_1) - \chi(\frac{\rho_1+\rho_2}{2}) + 2\chi(\rho_2)]\right| \\
\leq\ & \frac{(\rho_2-\rho_1)^2}{2(\zeta+1)}\left[\int_0^{\frac{1}{2}}\left|\epsilon^{\zeta+1} - \frac{\zeta+4}{3}\epsilon\right|\left|\chi''(\epsilon\rho_2 + (1-\epsilon)\rho_1)\right|d\epsilon\right.
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{\frac{1}{2}}\left|\epsilon^{\zeta+1}-\frac{\zeta+4}{3}\epsilon\right|\left|\chi''(\epsilon\rho_1+(1-\epsilon)\rho_2)\right|d\epsilon\\
&+\int_{\frac{1}{2}}^1\left|\epsilon^{\zeta+1}-\epsilon\right|\left|\chi''(\epsilon\rho_2+(1-\epsilon)\rho_1)\right|d\epsilon\\
&+\left.\int_{\frac{1}{2}}^1\left|\epsilon^{\zeta+1}-\epsilon\right|\left|\chi''(\epsilon\rho_1+(1-\epsilon)\rho_2)\right|d\epsilon\right]. \qquad (2.1)
\end{aligned}
$$

Utilizing the strong convexity property for $\left|\chi''\right|$, we derive

$$
\begin{aligned}
&\left|\frac{\Gamma(\zeta+1)}{2(\rho_2-\rho_1)^\zeta}[\Im^\zeta_{\rho_1^+}\chi(\rho_2)+\Im^\zeta_{\rho_2^-}\chi(\rho_1)]-\frac{1}{3}[2\chi(\rho_1)-\chi(\frac{\rho_1+\rho_2}{2})+2\chi(\rho_2)]\right|\\
\leq\ &\frac{(\rho_2-\rho_1)^2}{2(\zeta+1)}\left[\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\epsilon-\epsilon^{\zeta+1}\right)\left[\epsilon\left|\chi''(\rho_2)\right|+(1-\epsilon)\left|\chi''(\rho_1)\right|-\eta\epsilon(1-\epsilon)(\rho_1-\rho_2)^2\right]d\epsilon\right.\\
&+\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\epsilon-\epsilon^{\zeta+1}\right)\left[\epsilon\left|\chi''(\rho_1)\right|+(1-\epsilon)\left|\chi''(\rho_2)\right|-\eta\epsilon(1-\epsilon)(\rho_1-\rho_2)^2\right]d\epsilon\\
&+\int_{\frac{1}{2}}^1\left(\epsilon-\epsilon^{\zeta+1}\right)\left[\epsilon\left|\chi''(\rho_2)\right|+(1-\epsilon)\left|\chi''(\rho_1)\right|-\eta\epsilon(1-\epsilon)(\rho_1-\rho_2)^2\right]d\epsilon\\
&+\left.\int_{\frac{1}{2}}^1\left(\epsilon-\epsilon^{\zeta+1}\right)\left[\epsilon\left|\chi''(\rho_1)\right|+(1-\epsilon)\left|\chi''(\rho_2)\right|-\eta\epsilon(1-\epsilon)(\rho_1-\rho_2)^2\right]d\epsilon\right].\\
&=\frac{(\rho_2-\rho_1)^2}{2(\zeta+1)}\times\left[\frac{(\zeta^2+15\zeta+2)}{24(\zeta+2)}\left(\left|\chi''(\rho_1)\right|+\left|\chi''(\rho_2)\right|\right)\right.\\
&\left.-\eta(\rho_1-\rho_2)^2\left[\frac{(5\zeta+53)(\zeta+3)(\zeta+4)-576}{288(\zeta+3)(\zeta+4)}\right]\right].
\end{aligned}
$$

The proof is completed. □

Corollary 2.1. *If we choose $\zeta=1$ in Theorem 2.1, the following inequality is obtained.*

$$
\begin{aligned}
&\left|\frac{1}{(\rho_2-\rho_1)}\int_{\rho_1}^{\rho_2}\chi(\epsilon)d\epsilon-\frac{1}{3}[2\chi(\rho_1)-\chi(\frac{\rho_1+\rho_2}{2})+2\chi(\rho_2)]\right|\\
&\leq\frac{(\rho_2-\rho_1)^2}{4}\left[\frac{\left[\left|\chi''(\rho_1)\right|+\left|\chi''(\rho_2)\right|\right]}{4}-\frac{73\eta(\rho_2-\rho_1)^2}{720}\right].
\end{aligned}
$$

Theorem 2.2. *Assume that the conditions of Lemma 1.1 are valid and, additionally, if $\left|\chi''\right|^q$, where $q>1$, exhibitis strongly convex over the interval $[\rho_1,\rho_2]$, then:*

$$
\begin{aligned}
&\left|\frac{\Gamma(\zeta+1)}{2(\rho_2-\rho_1)^\zeta}[\Im^\zeta_{\rho_1^+}\chi(\rho_2)+\Im^\zeta_{\rho_2^-}\chi(\rho_1)]-\frac{1}{3}[2\chi(\rho_1)-\chi(\frac{\rho_1+\rho_2}{2})+2\chi(\rho_2)]\right|\\
\leq\ &\frac{(\rho_2-\rho_1)^2}{2(\zeta+1)}\left[\left(\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\epsilon-\epsilon^{\zeta+1}\right)^p d\epsilon\right)^{\frac{1}{p}}+\left(\frac{1}{\zeta}\mathcal{B}\left(p+1,\frac{p+1}{\zeta},1-\left(\frac{1}{2}\right)^\zeta\right)\right)^{\frac{1}{p}}\right]\\
&\times\left[\left(\frac{9\left|\chi''(\rho_2)\right|^q+3\left|\chi''(\rho_1)\right|^q-2\eta(\rho_1-\rho_2)^2}{24}\right)^{\frac{1}{q}}\right.
\end{aligned}
$$

$$+\left(\frac{9\left|\chi''(\rho_1)\right|^q+3\left|\chi''(\rho_2)\right|^q-2\eta(\rho_1-\rho_2)^2}{24}\right)^{\frac{1}{q}}\Bigg],$$

where $\frac{1}{p}+\frac{1}{q}=1, \eta>0$ *and* $\mathcal{B}$ *represents the incomplete Beta function, defined as:*

$$\mathcal{B}(k,y,r)=\int_0^r \epsilon^{k-1}(1-\epsilon)^{y-1}d\epsilon.$$

Proof. When we apply Hölder's inequality to the inequality (2.1), we obtain the following result:

$$\begin{aligned}
&\left|\frac{\Gamma(\zeta+1)}{2(\rho_2-\rho_1)^\zeta}[\Im^\zeta_{\rho_1^+}\chi(\rho_2)+\Im^\zeta_{\rho_2^-}\chi(\rho_1)]-\frac{1}{3}[2\chi(\rho_1)-\chi(\frac{\rho_1+\rho_2}{2})+2\chi(\rho_2)]\right|\\
\leq\ &\frac{(\rho_2-\rho_1)^2}{2(\zeta+1)}\left[\left(\int_0^{\frac{1}{2}}\left|\epsilon^{\zeta+1}-\frac{\zeta+4}{3}\epsilon\right|^p d\epsilon\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\chi''(\epsilon\rho_2+(1-\epsilon)\rho_1)\right|^q d\epsilon\right)^{\frac{1}{q}}\right.\\
&+\left(\int_0^{\frac{1}{2}}\left|\epsilon^{\zeta+1}-\frac{\zeta+4}{3}\epsilon\right|^p d\epsilon\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\chi''(\epsilon\rho_1+(1-\epsilon)\rho_2)\right|^q d\epsilon\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^1\left|\epsilon^{\zeta+1}-\epsilon\right|^p d\epsilon\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|\chi''(\epsilon\rho_2+(1-\epsilon)\rho_1)\right|^q d\epsilon\right)^{\frac{1}{q}}\\
&\left.+\left(\int_{\frac{1}{2}}^1\left|\epsilon^{\zeta+1}-\epsilon\right|^p d\epsilon\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|\chi''(\epsilon\rho_1+(1-\epsilon)\rho_2)\right|^q d\epsilon\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Utilizing the strongly convexity of the function $\left|\chi''\right|^q$, we obtain

$$\begin{aligned}
&\left|\frac{\Gamma(\zeta+1)}{2(\rho_2-\rho_1)^\zeta}[\Im^\zeta_{\rho_1^+}\chi(\rho_2)+\Im^\zeta_{\rho_2^-}\chi(\rho_1)]-\frac{1}{3}[2\chi(\rho_1)-\chi(\frac{\rho_1+\rho_2}{2})+2\chi(\rho_2)]\right|\\
\leq\ &\frac{(\rho_2-\rho_1)^2}{2(\zeta+1)}\times\\
&\left[\left(\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\epsilon-\epsilon^{\zeta+1}\right)^p d\epsilon\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\epsilon\left|\chi''(\rho_2)\right|^q+(1-\epsilon)\left|\chi''(\rho_1)\right|^q-\eta\epsilon(1-\epsilon)(\rho_1-\rho_2)^2d\epsilon\right)^{\frac{1}{q}}\right.\\
&+\left(\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\epsilon-\epsilon^{\zeta+1}\right)^p d\epsilon\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\epsilon\left|\chi''(\rho_1)\right|^q+(1-\epsilon)\left|\chi''(\rho_2)\right|^q-\eta\epsilon(1-\epsilon)(\rho_1-\rho_2)^2d\epsilon\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^1\left(\epsilon-\epsilon^{\zeta+1}\right)^p d\epsilon\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\epsilon\left|\chi''(\rho_2)\right|^q+(1-\epsilon)\left|\chi''(\rho_1)\right|^q-\eta\epsilon(1-\epsilon)(\rho_1-\rho_2)^2d\epsilon\right)^{\frac{1}{q}}\\
&\left.+\left(\int_{\frac{1}{2}}^1\left(\epsilon-\epsilon^{\zeta+1}\right)^p d\epsilon\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\epsilon\left|\chi''(\rho_1)\right|^q+(1-\epsilon)\left|\chi''(\rho_2)\right|^q-\eta\epsilon(1-\epsilon)(\rho_1-\rho_2)^2d\epsilon\right)^{\frac{1}{q}}\right]
\end{aligned}$$

$$= \frac{(\rho_2-\rho_1)^2}{2(\zeta+1)}\left[\left(\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\epsilon-\epsilon^{\zeta+1}\right)^p d\epsilon\right)^{\frac{1}{p}}+\left(\frac{1}{\zeta}\mathcal{B}\left(p+1,\frac{p+1}{\zeta},1-\left(\frac{1}{2}\right)^{\zeta}\right)\right)^{\frac{1}{p}}\right]$$
$$\times\left[\left(\frac{9\left|\chi''(\rho_2)\right|^q+3\left|\chi''(\rho_1)\right|^q-2\eta(\rho_1-\rho_2)^2}{24}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{9\left|\chi''(\rho_1)\right|^q+3\left|\chi''(\rho_2)\right|^q-2\eta(\rho_1-\rho_2)^2}{24}\right)^{\frac{1}{q}}\right].$$

□

Corollary 2.2. *If we choose $\zeta=1$ in Theorem 2.2, the following inequality is obtained.*

$$\left|\frac{1}{(\rho_2-\rho_1)}\int_{\rho_1}^{\rho_2}\chi(\epsilon)d\epsilon-\frac{1}{3}[2\chi(\rho_1)-\chi(\frac{\rho_1+\rho_2}{2})+2\chi(\rho_2)]\right|$$
$$\leq \frac{(\rho_2-\rho_1)^2}{4}\left[\left(\int_0^{\frac{1}{2}}\left(\frac{5}{3}\epsilon-\epsilon^2\right)^p d\epsilon\right)^{\frac{1}{p}}+\left(\mathcal{B}\left(p+1,p+1,\frac{1}{2}\right)\right)^{\frac{1}{p}}\right]$$
$$\times\left[\left(\frac{9\left|\chi''(\rho_2)\right|^q+3\left|\chi''(\rho_1)\right|^q-2\eta(\rho_1-\rho_2)^2}{24}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{9\left|\chi''(\rho_1)\right|^q+3\left|\chi''(\rho_2)\right|^q-2\eta(\rho_1-\rho_2)^2}{24}\right)^{\frac{1}{q}}\right],$$

where $\frac{1}{p}+\frac{1}{q}=1,\eta>0$ and $\mathcal{B}$ represents the incomplete Beta function, defined as in Theorem 2.2.

Theorem 2.3. *Assume that the conditions of Lemma 1.1 are satisfied, if $\left|\chi''\right|^q$ is strongly convex function for $q>1$, over the interval $[\rho_1,\rho_2]$, then:*

$$\left|\frac{\Gamma(\zeta+1)}{2(\rho_2-\rho_1)^{\zeta}}[\Im^{\zeta}_{\rho_1^+}\chi(\rho_2)+\Im^{\zeta}_{\rho_2^-}\chi(\rho_1)]-\frac{1}{3}[2\chi(\rho_1)-\chi(\frac{\rho_1+\rho_2}{2})+2\chi(\rho_2)]\right|$$
$$\leq \frac{(\rho_2-\rho_1)^2}{(\zeta+1)}\left[\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\epsilon-\epsilon^{\zeta+1}\right)^p d\epsilon\right)\right.$$
$$\left.+\frac{1}{q}\left(\frac{1}{\zeta}\mathcal{B}\left(p+1,\frac{p+1}{\zeta},1-\left(\frac{1}{2}\right)^{\zeta}\right)\right)+\frac{3\left|\chi''(\rho_2)\right|^q+3\left|\chi''(\rho_1)\right|^q-2\eta(\rho_1-\rho_2)^2}{6q}\right],$$

where $\frac{1}{p}+\frac{1}{q}=1,\eta>0$ and $\mathcal{B}$ represents the incomplete Beta function defined as in Theorem 2.2.

Proof. When we apply Young inequality to the inequality (2.1), we obtain the following result:

$$\begin{aligned}
&\left|\frac{\Gamma(\zeta+1)}{2(\rho_2-\rho_1)^{\zeta}}[\Im^{\zeta}_{\rho_1^+}\chi(\rho_2)+\Im^{\zeta}_{\rho_2^-}\chi(\rho_1)]-\frac{1}{3}[2\chi(\rho_1)-\chi(\frac{\rho_1+\rho_2}{2})+2\chi(\rho_2)]\right| \\
\leq\ & \frac{(\rho_2-\rho_1)^2}{2(\zeta+1)}\left[\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left|\epsilon^{\zeta+1}-\frac{\zeta+4}{3}\epsilon\right|^p d\epsilon\right)+\frac{1}{q}\left(\int_0^{\frac{1}{2}}\left|\chi''(\epsilon\rho_2+(1-\epsilon)\rho_1)\right|^q d\epsilon\right)\right. \\
&+\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left|\epsilon^{\zeta+1}-\frac{\zeta+4}{3}\epsilon\right|^p d\epsilon\right)+\frac{1}{q}\left(\int_0^{\frac{1}{2}}\left|\chi''(\epsilon\rho_1+(1-\epsilon)\rho_2)\right|^q d\epsilon\right) \\
&+\frac{1}{p}\left(\int_{\frac{1}{2}}^1\left|\epsilon^{\zeta+1}-\epsilon\right|^p d\epsilon\right)+\frac{1}{q}\left(\int_{\frac{1}{2}}^1\left|\chi''(\epsilon\rho_2+(1-\epsilon)\rho_1)\right|^q d\epsilon\right) \\
&\left.+\frac{1}{p}\left(\int_{\frac{1}{2}}^1\left|\epsilon^{\zeta+1}-\epsilon\right|^p d\epsilon\right)+\frac{1}{q}\left(\int_{\frac{1}{2}}^1\left|\chi''(\epsilon\rho_1+(1-\epsilon)\rho_2)\right|^q d\epsilon\right)\right].
\end{aligned}$$

Utilizing the strongly convexity of the function $\left|\chi''\right|^q$, we obtain

$$\begin{aligned}
&\left|\frac{\Gamma(\zeta+1)}{2(\rho_2-\rho_1)^{\zeta}}[\Im^{\zeta}_{\rho_1^+}\chi(\rho_2)+\Im^{\zeta}_{\rho_2^-}\chi(\rho_1)]-\frac{1}{3}[2\chi(\rho_1)-\chi(\frac{\rho_1+\rho_2}{2})+2\chi(\rho_2)]\right| \\
\leq\ & \frac{(\rho_2-\rho_1)^2}{2(\zeta+1)} \\
&\times\left[\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\epsilon-\epsilon^{\zeta+1}\right)^p d\epsilon\right)+\frac{1}{q}\left(\int_0^{\frac{1}{2}}\epsilon\left|\chi''(\rho_2)\right|^q+(1-\epsilon)\left|\chi''(\rho_1)\right|^q-\eta\epsilon(1-\epsilon)(\rho_1-\rho_2)^2 d\epsilon\right)\right. \\
&+\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\epsilon-\epsilon^{\zeta+1}\right)^p d\epsilon\right)+\frac{1}{q}\left(\int_0^{\frac{1}{2}}\epsilon\left|\chi''(\rho_1)\right|^q+(1-\epsilon)\left|\chi''(\rho_2)\right|^q-\eta\epsilon(1-\epsilon)(\rho_1-\rho_2)^2 d\epsilon\right) \\
&+\frac{1}{p}\left(\int_{\frac{1}{2}}^1\left(\epsilon-\epsilon^{\zeta+1}\right)^p d\epsilon\right)+\frac{1}{q}\left(\int_{\frac{1}{2}}^1\epsilon\left|\chi''(\rho_2)\right|^q+(1-\epsilon)\left|\chi''(\rho_1)\right|^q-\eta\epsilon(1-\epsilon)(\rho_1-\rho_2)^2 d\epsilon\right) \\
&\left.+\frac{1}{p}\left(\int_{\frac{1}{2}}^1\left(\epsilon-\epsilon^{\zeta+1}\right)^p d\epsilon\right)+\frac{1}{q}\left(\int_{\frac{1}{2}}^1\epsilon\left|\chi''(\rho_1)\right|^q+(1-\epsilon)\left|\chi''(\rho_2)\right|^q-\eta\epsilon(1-\epsilon)(\rho_1-\rho_2)^2 d\epsilon\right)\right] \\
&=\frac{(\rho_2-\rho_1)^2}{(\zeta+1)}\left[\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\epsilon-\epsilon^{\zeta+1}\right)^p d\epsilon\right)\right. \\
&\left.+\frac{1}{q}\left(\frac{1}{\zeta}\mathcal{B}\left(p+1,\frac{p+1}{\zeta},1-\left(\frac{1}{2}\right)^{\zeta}\right)\right)+\frac{3\left|\chi''(\rho_2)\right|^q+3\left|\chi''(\rho_1)\right|^q-\eta(\rho_1-\rho_2)^2}{6q}\right].
\end{aligned}$$

□

Corollary 2.3. *If we set $\zeta=1$ in Theorem 2.3, the following inequality is obtained.*

$$\begin{aligned}
&\left|\frac{1}{(\rho_2-\rho_1)}\int_{\rho_1}^{\rho_2}\chi(\epsilon)d\epsilon-\frac{1}{3}\left[2\chi(\rho_1)-\chi(\frac{\rho_1+\rho_2}{2})+2\chi(\rho_2)\right]\right| \\
\leq\ & \frac{(\rho_2-\rho_1)^2}{2}\left[\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left(\frac{5}{3}\epsilon-\epsilon^2\right)^p d\epsilon\right)\right.
\end{aligned}$$

$$+\frac{1}{q}\left(\mathcal{B}\left(p+1,p+1,\frac{1}{2}\right)\right)+\frac{3\left|\chi''(\rho_2)\right|^q+3\left|\chi''(\rho_1)\right|^q-\eta(\rho_1-\rho_2)^2}{6q}\Bigg],$$

where $\frac{1}{p}+\frac{1}{q}=1, \eta>0$ *and* $\mathcal{B}$ *represents the incomplete Beta function.*

3. Conclusion

In many studies with Riemann-Liouville integral operators, one of the important concepts of fractional analysis, new integral inequalities have been mentioned. In this study, various new Milne type inequalities for strongly convex functions have been proved using an integral identity obtained with the help of Riemann-Liouville integral operators. It has been observed that the special cases of these inequalities includes new estimations.

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