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NEW BOUNDS AND NEW ESTIMATES FOR HH INTEGRAL INEQUALITY VIA CONCAVE FUNCTIONS

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Abstract. In this study, new bounds are obtained for the right and left sides of Hermite-Hadamard integral inequality using the concave functions. The reason why we chose concave functions in the study is the ease of use of the methods applied to these functions (as Jensen integral inequality, as Favard inequality). To do this, we used the new lemmas.

1. Short Historical Background and Introduction

One of the three basic properties of real numbers is the order relation. The order relation contributes to mathematical analysis in terms of practice, theory and aesthetics. For this purpose, on November 22, 1881, Hermite (1822-1901) sent a letter to the Journal Mathesis. This letter was published in Mathesis 3 (1883, p.82) and in this letter an inequality is presented that is well-known in the literature as Hermite- Hadamard integral inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I of a real numbers and $a, b \in I$ with $a < b$. If the function f is concave, the inequality in (1.1) is reversed. That is

$$f\left(\frac{a+b}{2}\right) \geq \frac{1}{b-a}\int_a^b f(x)\,dx \geq \frac{f(a)+f(b)}{2}.$$

For recent results and generalizations concerning concave functions, see ([6, 7]). Set et al. wrote some inequalities of the Hadamard type for several types of convex functions, see([8]).

To understand the behavior of a function on an interval, the upward or downward concavity of the curve representing the function is important. The information indicating the direction of the curve relates to the derivatives of the function in question. The importance of the subject in optimization theory cannot be denied. Optimization is the set of studies

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carried out to obtain maximum output from a system within certain constraints. In short, achieving the best or most appropriate situations under constraints is among the main goals. Undoubtedly, the smallest upper bound and the largest lower bound is very important here. For example, maximizing profit at minimum cost.

In [4], Özdemir gave the following results for differantiable mappings on $I^* \subset R$.

Lemma 1.1. *Suppose that $f : I^* \subset R \to R$ be a differentiable mapping on I^* (I^* is interior of I) with $a < b$. If $f' \in L[a, b]$, and $t \in [0, 1]$, then we have*

$$\int_a^b f(x)\,dx - \frac{(b-a)}{2}\left[f(a) + f(b)\right] \tag{1.2}$$
$$= \frac{(b-a)^2}{2}\left\{\int_0^{\frac{1}{2}} (1-2t) f'(tb + (1-t)a)\,dt + \int_{\frac{1}{2}}^1 (1-2t) f'(tb + (1-t)a)\,dt\right\}.$$

Lemma 1.2. *Suppose that $f : I^* \subset R \to R$ be a differentiable mapping on I^* (I^* is interior of I) with $a < b$. If $f'' \in L[a, b]$, and $t \in [0, 1]$, then we have*

$$f'\left(\frac{a+b}{2}\right) - \frac{2}{b-a} f(b) + \frac{2}{(b-a)^2}\int_a^b f(x)\,dx \tag{1.3}$$
$$= (b-a)\left[\int_0^{\frac{1}{2}} t^2 f''(tb + (1-t)a)\,dt + \int_{\frac{1}{2}}^1 \left(t^2 - 1\right) f''(tb + (1-t)a)\,dt\right].$$

Lemma 1.3. *Suppose that $f : I^* \subset R \to R$ be a differentiable mapping on I^* (I^* is interior of I) with $a < b$. If $f'' \in L[a, b]$, and $t \in [0, 1]$, then we have*

$$f'(a) - f'\left(\frac{a+b}{2}\right) - \frac{2}{(b-a)} f\left(\frac{a+b}{2}\right) + \frac{2}{(b-a)^2}\int_a^b f(x)\,dx \tag{1.4}$$
$$= (b-a)\left[\int_0^{\frac{1}{2}} \left(t^2 - 1\right) f''(tb + (1-t)a)\,dt + \int_{\frac{1}{2}}^1 (1-t)^2 f''(tb + (1-t)a)\,dt\right].$$

To prove each of the above lemmas, as many partial integration methods are applied to their right-hand sides as necessary.

For example, when $f(x) = x$, $f(x) = x^2$, $f(x) = x^3$ and $[0, 2]$ are taken respectively in Lemma 1.1-Lemma 1.3 above, it is seen that the equations are satisfied in each lemma.

We consider the following useful inequality:

For all continuous concave functions $f : [a; b] \to R^+$ and all parameters $q > 1$:

$$\left(\frac{1}{b-a}\int_a^b f^q(x)\,dx\right)^{\frac{1}{q}} \le \frac{2}{(q+1)^{\frac{1}{q}}}\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right). \tag{1.5}$$

This inequality is well known in the literature as Favard inequality see ([2]).

We will also use the well-known Hölder inequality [3] in the literature:

Let be $p > 1$ and $p^{-1} + q^{-1} = 1$, If f and g reel functions on $[a, b]$ such that $|f|^p$ *and* $|f|^q$ are integrable on $[a, b]$. Then

$$\int_a^b |f(x) g(x)|\,dx \le \left(\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b |g(x)|^q\,dx\right)^{\frac{1}{q}}. \tag{1.6}$$

S.S. Dragomir and R.P. Agarwal in [1] developed inequalities for differentiable mapping and give applications to special means of real numbers, and trapezoidal formula.

Theorem 1.1. *Let $f : I^0 \subseteq R \to R$ be a differentiable mapping on I^0, $a, b \in I^0$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \leq \frac{(b-a)^2}{8} \left(|f'(a)| + |f'(b)| \right). \tag{1.7}$$

This study is a continuation of our study in [5].

2. The Results

Theorem 2.1. *$f :\ I^* \subset R \to R$ be a differerantiable mapping on I^*, $a, b \in I^*$ with $a < b$, let $q \geq 1$.If $|f'|^q$ is concave on $[a, b]$ and $|f'|$ is a linear transformation, then the following inequality holds:*

$$\begin{aligned} \left| \int_a^b f(x)\,dx - \frac{(b-a)}{2} [f(a) + f(b)] \right| &\leq \frac{(b-a)^2}{8} \left\{ \left| f'\left(\frac{5a+b}{6} \right) \right| + \left| f'\left(\frac{5b+a}{6} \right) \right| \right\} \\ &= \frac{(b-a)^2}{8} \left| f'(a+b) \right|. \end{aligned} \tag{2.1}$$

Proof. Using the concavity of $|f'|^q$ and the power-mean inequality, we obtain

$$|f'(tb + (1-t)a)|^q \geq t|f'(b)|^q + (1-t)|f'(a)|^q \geq [t|f'(b)| + (1-t)|f'(a)|]^q$$

$$f'|tb + (1-t)a| \geq t|f'(b)| + (1-t)|f'(a)|$$

So $|f'|$ is also concave.

From equality (1.2), by properties of modulus, Jensen integral inequality we have

$$\begin{aligned} & \left| \int_a^b f(x)\,dx - \frac{(b-a)}{2} [f(a) + f(b)] \right| \\ = & \left| \frac{(b-a)^2}{2} \left\{ \int_0^{\frac{1}{2}} (1-2t) f'(tb + (1-t)a)\,dt + \int_{\frac{1}{2}}^1 (1-2t) f'(tb + (1-t)a)\,dt \right\} \right| \\ \leq & \frac{(b-a)^2}{2} \left\{ \int_0^{\frac{1}{2}} |1-2t|\,dt \left| f'\left(\frac{\int_0^{\frac{1}{2}} |1-2t| \ (tb + (1-t)a)\,dt}{\int_0^{\frac{1}{2}} |1-2t| \ dt} \right) \right| \right. \\ & \left. + \int_{\frac{1}{2}}^1 |1-2t| \ dt \left| f'\left(\frac{\int_{\frac{1}{2}}^1 |1-2t| \ (tb + (1-t)a) \ dt}{\int_{\frac{1}{2}}^1 |1-2t| \ dt} \right) \right| \right\} \\ = & \frac{(b-a)^2}{8} \left\{ \left| f'\left(\frac{5a+b}{6} \right) \right| + \left| f'\left(\frac{5b+a}{6} \right) \right| \right\} = \frac{(b-a)^2}{8} |f'(a+b)| \end{aligned}$$

which gives the inequality (2.1).

Here we used the facts that,

$$\int_0^{\frac{1}{2}} |1-2t|\,dt = \frac{1}{4},$$

$$\int_0^{\frac{1}{2}} |1-2t| \ (tb+(1-t)\,a)\,dt = \int_0^{\frac{1}{2}} (1-2t) \ (tb+(1-t)\,a)\,dt = \frac{5a+b}{24},$$

$$\int_{\frac{1}{2}}^1 |1-2t| \ dt = \int_{\frac{1}{2}}^1 (2t-1)\,dt = \frac{1}{4},$$

$$\int_{\frac{1}{2}}^1 |1-2t| \ (tb+(1-t)\,a) \ dt = \int_{\frac{1}{2}}^1 (2t-1) \ (tb+(1-t)\,a)\,dt = \frac{5b+a}{24},$$

and

$$\begin{aligned} & \left|f'\left(\frac{5a+b}{6}\right)\right| + \left|f'\left(\frac{5b+a}{6}\right)\right| \\ = \ & \frac{5}{6}\left|f'(a)\right| + \frac{1}{6}\left|f'(a)\right| + \frac{1}{6}\left|f'b\right| + \frac{5}{6}\left|f'(b)\right| \\ = \ & \left|f'(a)\right| + \left|f'(b)\right| \\ = \ & \left|f'(a+b)\right|. \end{aligned}$$

Finally, we used the linearity of $|f'|$. □

Corollary 2.1. *As can be seen in Theorem 2.1, when $|f'|^q$ and therefore $|f'|$ are concave and $|f'|$ is linear, we obtain the inequality (1.7).*

Theorem 2.2. *$f: I^* \subset R \to R$ be a differantiable mapping on I^*, $a, b \in I^*$ with $a < b$, let $q \geq 1$. If $|f''|^q$ is concave on $[a,b]$ and $|f''|$ is a linear map, then the following inequality holds:*

$$\begin{aligned} & \left|f'\left(\frac{a+b}{2}\right) - \frac{2}{b-a} f(b) + \frac{2}{(b-a)^2}\int_a^b f(x)\,dx\right| \\ \leq \ & \frac{b-a}{24}\left\{\left|f''\left(\frac{3b+5a}{8}\right)\right| + 5\left|f''\left(\frac{13a+27b}{40}\right)\right|\right\} \\ = \ & (b-a)\left(0,15625\left|f''(b)\right| + (0,09375)\left|f''(a)\right|\right). \end{aligned} \tag{2.2}$$

Proof. Using the concavity of $|f''|^q$ and the power-mean inequality, we obtain

$$\left|f''(tb+(1-t)\,b)\right|^q \geq t\left|f''(b)\right|^q + (1-t)\left|f''(a)\right|^q \geq \left[t\left|f''(b)\right| + (1-t)\left|f''(a)\right|\right]^q,$$

$$f''\left|tb+(1-t)\,a\right| \geq t\left|f''(b)\right| + (1-t)\left|f''(a)\right|.$$

So $|f''|$ is also concave. From inequality (1.3), by properties of modulus and the Jensen integral inequality for concave functions, we have

$$\begin{aligned}
& \left| f'\left(\frac{a+b}{2}\right) - \frac{2}{b-a} f(b) + \frac{2}{(b-a)^2} \int_a^b f(x)\,dx \right| \\
\leq\ & (b-a) \left| \int_0^{\frac{1}{2}} t^2 f''(tb+(1-t)a)\,dt + \int_{\frac{1}{2}}^1 \left(t^2-1\right) f''(tb+(1-t)a)\,dt \right| \\
=\ & (b-a) \left[\int_0^{\frac{1}{2}} \left|t^2\right| \left|f''(tb+(1-t)a)\right| dt + \int_{\frac{1}{2}}^1 \left|\left(t^2-1\right)\right| \left|f''(tb+(1-t)a)\right| dt \right] \\
\leq\ & (b-a) \left(\int_0^{\frac{1}{2}} t^2 dt \right) \left| f''\left(\frac{\int_0^{\frac{1}{2}} t^2 (tb+(1-t)a)\,dt}{\int_0^{\frac{1}{2}} t^2 dt} \right) \right| \\
& + (b-a) \left(\int_{\frac{1}{2}}^1 \left|t^2-1\right| dt \right) \left| f''\left(\frac{\int_{\frac{1}{2}}^1 \left|t^2-1\right| (tb+(1-t)a)\,dt}{\int_{\frac{1}{2}}^1 \left|t^2-1\right| dt} \right) \right| .
\end{aligned}$$

Considering the linearity of $|f''|$, if we calculate the following integrals and replace them, we obtain the inequality (2.2).

Here we used the facts that,

$$\int_0^{\frac{1}{2}} t^2 dt = \frac{1}{24}, \qquad \left| f''\left(\frac{\int_0^{\frac{1}{2}} t^2 (tb+(1-t)a)\,dt}{\int_0^{\frac{1}{2}} t^2 dt} \right) \right| = \left| f''\left(\frac{3b+5a}{8}\right) \right|,$$

and

$$\int_{\frac{1}{2}}^1 \left|t^2-1\right| dt = \frac{5}{24}, \qquad \left| f''\left(\frac{\int_{\frac{1}{2}}^1 \left|t^2-1\right| (tb+(1-t)a)\,dt}{\int_{\frac{1}{2}}^1 \left|t^2-1\right| dt} \right) \right| = \left| f''\left(\frac{13a+27b}{40}\right) \right|.$$

□

Theorem 2.3. *Suppose that $f : I^* \subset R \to R$ be a differentiable mapping on I^* (I^* is interior of I) with $a < b$. If $f'' \in L[a,b]$, If $|f''|^q$ is concave on $[a,b]$ and $t \in [0,1]$. $|f''|$ is a linear map, then we have*

$$\begin{aligned}
& \left| \frac{f'(a) - f'\left(\frac{a+b}{2}\right)}{b-a} - \frac{2}{(b-a)^2} f\left(\frac{a+b}{2}\right) + \frac{2}{(b-a)^3} \int_a^b f(x)\,dx \right| \\
\leq\ & \frac{5}{24} \left| f''\left(\frac{21b+67a}{40}\right) \right| + \frac{1}{24} \left| f''\left(\frac{5b+3a}{8}\right) \right| \\
=\ & (0,1054166667)\ \left|f''(b)\right| + (0,3645833333)\ \left|f''(a)\right| .
\end{aligned} \tag{2.3}$$

Proof. First of all, If $|f''|^q$ is concave on $[a,b]$ $|f''|$ is also concave on $[a,b]$. From (1.4), by the property of modulus and the Jensen's integral inequality for concave functions, we have

$$\begin{aligned}
& \left|\frac{f'(a)-f'\left(\frac{a+b}{2}\right)}{b-a}-\frac{2}{(b-a)^2}f\left(\frac{a+b}{2}\right)+\frac{2}{(b-a)^3}\int_a^b f(x)\,dx\right| \\
= & \left|\int_0^{\frac{1}{2}}\left(t^2-1\right)f''(tb+(1-t)a)\,dt+\int_{\frac{1}{2}}^1(1-t)^2 f''(tb+(1-t)a)\,dt\right| \\
\leq & \left[\int_0^{\frac{1}{2}}\left|\left(t^2-1\right)\right|\ \left|f''(tb+(1-t)a)\right|dt+\int_{\frac{1}{2}}^1\left|(1-t)^2\right|\ \left|f''(tb+(1-t)a)\right|dt\right] \\
= & \left[\int_0^{\frac{1}{2}}\left(1-t^2\right)\ \left|f''(tb+(1-t)a)\right|dt+\int_{\frac{1}{2}}^1(1-t)^2\ \left|f''(tb+(1-t)a)\right|dt\right] \\
\leq & \int_0^{\frac{1}{2}}\left(1-t^2\right)dt\,\left|f''\left(\frac{\int_0^{\frac{1}{2}}(1-t^2)(tb+(1-t)a)\,dt}{\int_0^{\frac{1}{2}}(1-t^2)\,dt}\right)\right| \\
& +\int_{\frac{1}{2}}^1(1-t)^2\,dt\,\left|f''\left(\frac{\int_{\frac{1}{2}}^1(1-t)^2(tb+(1-t)a)\,dt}{\int_{\frac{1}{2}}^1(1-t)^2\,dt}\right)\right| \\
= & \frac{5}{24}\left|f''\left(\frac{21b+67a}{40}\right)\right|+\frac{1}{24}\left|f''\left(\frac{5b+3a}{8}\right)\right|.
\end{aligned}$$

On the other hand, since $|f''|$ is linear we get (2.3).

Another upper bound for the right hand of the Hermite-Hadamard inequality (in short HH) is given by the following theorem. ☐

Theorem 2.4. *$f: I\subset R\to R$, $I\subset[0,\infty)$ be a differantiable mapping on I^* such that $f'\in L\ [a,b]$, where $a,b\in I\subset\ I^*$ with $a<b$, let $q>1$. If $|f'|^q$ is concave on $[a,b]$, then we have the following inequality*

$$\begin{aligned}
& \left|\int_a^b f(x)\,dx-\frac{(b-a)}{2}[f(a)+f(b)]\right| \qquad (2.4)\\
\leq & \frac{(b-a)^2}{4}\frac{1}{(2q-1)^{\frac{q-1}{q}}}\left[\left|f'\left(\frac{3a+b}{16}\right)\right|+\left|f'\left(\frac{3b+a}{16}\right)\right|\right].
\end{aligned}$$

Proof. From Lemma 1.1, we have

$$\begin{aligned}
& \left|\int_a^b f(x)\,dx-\frac{(b-a)}{2}[f(a)+f(b)]\right| \qquad (2.5)\\
= & \frac{(b-a)^2}{2}\left\{\int_0^{\frac{1}{2}}(1-2t)\left|f'(tb+(1-t)a)\right|dt+\int_{\frac{1}{2}}^1|(1-2t)|\ \left|f'(tb+(1-t)a)\right|dt\right\} \\
= & \frac{(b-a)^2}{2}\left\{\int_0^{\frac{1}{2}}(1-2t)\left|f'(tb+(1-t)a)\right|dt+\int_{\frac{1}{2}}^1(2t-1)\left|f'(tb+(1-t)a)\right|dt\right\}.
\end{aligned}$$

By using Hölder's inequality (see (1.6)) for $q > 1$ and $p = \frac{q}{q-1}$, we have

$$\int_0^{\frac{1}{2}} (1-2t) \left| f'(tb + (1-t)a) \right| dt \le \left(\int_0^{\frac{1}{2}} (1-2t)^{\frac{q-1}{q}} dt \right)^{\frac{q-1}{q}} \left(\int_0^{\frac{1}{2}} \left| f'(tb+(1-t)a) \right|^q dt \right)^{\frac{1}{q}}$$

and

$$\int_{\frac{1}{2}}^1 (2t-1) \left| f'(tb + (1-t)a) \right| dt \le \left(\int_{\frac{1}{2}}^1 (2t-1)^{\frac{q-1}{q}} \right)^{\frac{q-1}{q}} \left(\int_{\frac{1}{2}}^1 \left| f'(tb+(1-t)a) \right|^q dt \right)^{\frac{1}{q}}.$$

It can be checked that,

$$\int_0^{\frac{1}{2}} (1-2t)^{\frac{q-1}{q}} dt = \frac{1}{2(2q-1)} = \int_{\frac{1}{2}}^1 (2t-1)^{\frac{q-1}{q}}.$$

Since

$$f' \left| tb + (1-t)a \right|^q \ge t \left| f'(b) \right|^q + (1-t) \left| f'(a) \right|^q \ge \left[t \left| f'(b) \right| + (1-t) \left| f'(a) \right| \right]^q,$$

$$f' \left| tb + (1-t)a \right| \ge t \left| f'(b) \right| + (1-t) \left| f'(a) \right|$$

we can use the the Jensen integral inequality for concave functions to obtain

$$\begin{aligned} \int_0^{\frac{1}{2}} \left| f'(tb+(1-t)a) \right|^q dt &= \int_0^{\frac{1}{2}} t^0 \left| f'(tb+(1-t)a) \right|^q dt \\ &\le \left(\int_0^{\frac{1}{2}} t^0 dt \right) \left| f' \left(\frac{\int_0^{\frac{1}{2}} (tb+(1-t)a)\, dt}{\int_0^{\frac{1}{2}} t^0 dt} \right) \right|^q \\ &= \frac{1}{2} \left| f' \left(\frac{3a+b}{16} \right) \right|^q \end{aligned}$$

and analogously

$$\int_{\frac{1}{2}}^1 \left| f'(tb+(1-t)a) \right|^q dt \le \left(\int_{\frac{1}{2}}^1 t^0 dt \right) \left| f' \left(\frac{\int_{\frac{1}{2}}^1 (tb+(1-t)a)\, dt}{\int_0^{\frac{1}{2}} t^0 dt} \right) \right|^q = \frac{1}{2} \left| f' \left(\frac{3b+a}{16} \right) \right|^q.$$

If we write all the inequalities in their place in (1.2), the desired inequality (2.4) is found. □

Corollary 2.2. *$f : I \subset R \to R$, $I \subset [0, \infty)$ be a differantiable mapping on I^*such that $f' \in L[a,b]$, where $a, b \in I \subset I^*$ with $a < b$, let $q > 1$. If $|f'|^q$ is concave on $[a,b]$. Then we have the following inequality*

$$\left| \int_a^b f(x)\, dx - \frac{(b-a)}{2} [f(a) + f(b)] \right| \le \frac{(b-a)^2}{4} \left[\left| f' \left(\frac{3a+b}{16} \right) \right| + \left| f' \left(\frac{3b+a}{16} \right) \right| \right]. \quad (2.6)$$

Proof. With power mean inequality, since $|f'|^q$ is concave $|f'|$ is also concave see ([6]). Thus

$$\begin{aligned} & \frac{\left|f'\left(\frac{3a+b}{16}\right)\right|+\left|f'\left(\frac{3b+a}{16}\right)\right|}{4} \\ \geq & \frac{\frac{3}{16}\left|f'(a)\right|+\frac{1}{16}\left|f'(b)\right|+\frac{3}{16}\left|f'(b)\right|+\frac{1}{16}\left|f'(a)\right|}{4} \\ = & \frac{\left|f'(a)\right|+\left|f'(b)\right|}{16} \end{aligned}$$

and

$$\frac{\left|f'(a)\right|+\left|f'(b)\right|}{16} \leq \frac{\left|f'\left(\frac{3a+b}{16}\right)\right|+\left|f'\left(\frac{3b+a}{16}\right)\right|}{4}$$

$\frac{1}{(2q-1)^{\frac{q-1}{q}}} \to 0$ for $q \to \infty$ and $\frac{1}{(2q-1)^{\frac{q-1}{q}}} \to 1$ for q$\to 1^{+}$. So we obtain

$$0 < \frac{1}{(2q-1)^{\frac{q-1}{q}}} < 1$$

Thus we can not write the term $\frac{|f'(a)|+|f'(b)|}{16}$ instead of $\frac{\left|f'\left(\frac{3a+b}{16}\right)\right|+\left|f'\left(\frac{3b+a}{16}\right)\right|}{4}$. But inequality (2.6) becomes better as q increas for $q \in (1,\infty)$. That is, the inequality (2.6) is a simple consequence of the inequality (2.4). □

Özdemir and his colleagues (see [5]), Theorem 2, (6)) made a new prediction regarding the left side of the HH inequality. In the following theorem, we made a new prediction about the right side of HH with the help of Hölder and Jensen integral inequalities for concave functions.

Theorem 2.5. *$f : I \subset R \to R$, $I \subset [0,\infty)$ be a differantiable mapping on I^*such that $f' \in L\ [a,b]$, where $a,b \in I \subset I^*$ with $a<b$, let $q>1$. If $|f'|^q$ is concave on $[a,b]$. Then we have the following inequality*

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{2}\left[f(a)+f(b)\right]\right| \leq \frac{b-a}{(2q-1)^{\frac{q-1}{q}}(1+q)^{\frac{1}{q}}}\int_a^b \left|f'(x)\right|dx. \tag{2.7}$$

Proof. In the proof, we will use Hölder's and Favard's inequality for concave functions, respectively, for $q>1$ and $p=\frac{q}{q-1}$

$$\begin{aligned} & \int_0^{\frac{1}{2}}(1-2t)\left|f'(tb+(1-t)a)\right|dt \\ \leq & \left(\int_0^{\frac{1}{2}}(1-2t)^{\frac{q-1}{q}}dt\right)^{\frac{q-1}{q}}\left(\int_0^{\frac{1}{2}}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} & \int_{\frac{1}{2}}^1(2t-1)\left|f'(tb+(1-t)a)\right|dt \\ \leq & \left(\int_{\frac{1}{2}}^1(2t-1)^{\frac{q-1}{q}}\right)^{\frac{q-1}{q}}\left(\int_{\frac{1}{2}}^1\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}. \end{aligned} \tag{2.9}$$

Now If we apply the Favard inequality to the right hands of (2.8) and (2.9) we have

$$\begin{aligned}\left(\int_0^{\frac{1}{2}} \left|f'\left(tb+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}} &= \left(\frac{1}{2\left(\frac{a+b}{2}-a\right)}\int_a^{\frac{a+b}{2}} \left|f'(x)\right|^q dx\right)^{\frac{1}{q}} \\ &\leq \frac{2}{2^{\frac{1}{q}}(q+1)^{\frac{1}{q}}}\left(\frac{1}{\frac{a+b}{2}-a}\int_a^{\frac{a+b}{2}} \left|f'(x)\right| dx\right) \\ &\leq \frac{1}{2^{\frac{1-2q}{q}}(q+1)^{\frac{1}{q}}}\left(\frac{1}{b-a}\int_a^{\frac{a+b}{2}} \left|f'(x)\right| dx\right).\end{aligned}$$

Anologously

$$\begin{aligned}\left(\int_{\frac{1}{2}}^{1} \left|f'\left(tb+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}} &= \left(\frac{1}{2\left(b-\frac{a+b}{2}\right)}\int_{\frac{a+b}{2}}^{b} \left|f'(x)\right|^q dx\right)^{\frac{1}{q}} \\ &\leq \frac{2}{2^{\frac{1}{q}}(q+1)^{\frac{1}{q}}}\left(\frac{1}{b-\frac{a+b}{2}}\int_{\frac{a+b}{2}}^{b} \left|f'(x)\right| dx\right) \\ &\leq \frac{1}{2^{\frac{1-2q}{q}}(q+1)^{\frac{1}{q}}}\left(\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b} \left|f'(x)\right| dx\right).\end{aligned}$$

If we adding to the side to side

$$\begin{aligned}&\left(\int_0^{\frac{1}{2}} \left|f'\left(tb+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^{1} \left|f'\left(tb+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}} \\ \leq\ & \frac{1}{2^{\frac{1-2q}{q}}(q+1)^{\frac{1}{q}}(b-a)}\left(\int_a^{\frac{a+b}{2}} \left|f'(x)\right| dx + \int_{\frac{a+b}{2}}^{b} \left|f'(x)\right| dx\right) \\ =\ & \frac{1}{2^{\frac{1-2q}{q}}(q+1)^{\frac{1}{q}}(b-a)}\int_a^b \left|f'(x)\right| dx\end{aligned}$$

and since

$$\int_0^{\frac{1}{2}} (1-2t)^{\frac{q-1}{q}} dt = \frac{1}{2(2q-1)} = \int_{\frac{1}{2}}^{1} (2t-1)^{\frac{q-1}{q}} dt.$$

After the necessary mathematical operations and using (1.2) we obtain the required inequality (2.7). ☐

Corollary 2.3. *$f: I \subset R \to R$, $I \subset [0,\infty)$ be a differantiable mapping on I^* such that $f' \in L\ [a,b]$, where $a,b \in I \subset I^*$ with $a<b$, let $q>1$. If $|f'|^q$ is concave on $[a,b]$. Then we have the following inequality*

$$\left|\int_a^b f(x)\,dx - \frac{(b-a)}{2}\left[f(a)+f(b)\right]\right| \leq \frac{(b-a)}{2}\int_a^b \left|f'(x)\right| dx. \tag{2.10}$$

Proof. Since

$$\frac{b-a}{(2q-1)^{\frac{q-1}{q}}(1+q)^{\frac{1}{q}}} \to 0 \text{ for } q\to\infty \text{ and } \frac{b-a}{(2q-1)^{\frac{q-1}{q}}(1+q)^{\frac{1}{q}}} \to \frac{b-a}{2} \text{ for } q \to 1^+.$$

□

We can write the inequality

$$0 < \frac{b-a}{(2q-1)^{\frac{q-1}{q}}(1+q)^{\frac{1}{q}}} < \frac{b-a}{2}.$$

Thus we obtain inequality (2.10). That is, the inequality (2.10) is a simple consequence of the inequality (2.7).

Conclusion

In this study, using almost the same methods but different lemmas 1.1-1.3, new upper bounds for the right and left sides of HH were obtained and the results were found and to be consistent with some results in the literature. (2.1), (2.4) and (2.7) are connected with the right side of (1.1) and the inequalities (2.2) and (2.3) are also related to the left side of (1.1). Researchers interested in the subject can make comparisons by using the same lemmas and different methods,for the right or left side of the (1.1).

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