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LIMIT PROPERTIES AND INEQUALITIES INVOLVING K-POLYGAMMA FUNCTIONS

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Abstract. Let $\psi_k^{(a)}(z)$ be the k-polygamma function of order $a \in \mathbb{N}$. In this paper, we establish some limit properties and inequalities involving $\psi_k^{(a)}(z)$. In some instances, the inequalities provide bounds for certain ratios involving $\psi_k^{(a)}(z)$. In the other instances, they provide bounds (in terms of Hurwitz zeta function) for the harmonic, arithmetic and geometric means involving the functions $\psi_k^{(a)}(z)$ and $\psi_k^{(a)}(1/z)$. The established results serve as generalization and extension of some recent results. Largely, the techniques employed in proving our results depend on monotonicity properties of certain functions involving the k-polygamma functions.

1. Introduction

The Euler's integral of second kind, which is commonly referred to as the gamma function, is defined as

$$\begin{aligned}\Gamma(z) &= \int_0^\infty s^{z-1}e^{-s}ds,\\ &= \lim_{r\to\infty}\frac{r!r^z}{z(z+1)(z+2)\dots(z+r)},\end{aligned}$$

for $z > 0$. The logarithmic derivative of the gamma function which is called the digamma function is defined as

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \int_0^\infty \frac{e^{-s}-e^{-zs}}{1-e^{-s}}ds, \tag{1.1}$$

$$= \int_0^\infty \left(\frac{e^{-s}}{s} - \frac{e^{-zs}}{1-e^{-s}}\right)ds, \tag{1.2}$$

$$= -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\frac{z}{n(n+z)}, \tag{1.3}$$

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where γ is the Euler-Mascheroni constant. Derivatives of the digamma function are referred to polygamma functions and they are defined as

$$\psi^{(a)}(z) = \frac{d^a}{dz^a}\psi(z) = (-1)^{a+1}\int_0^\infty \frac{s^a e^{-zs}}{1-e^{-s}}ds, \tag{1.4}$$

$$= (-1)^{a+1}\sum_{n=0}^{\infty}\frac{a!}{(n+z)^{a+1}}, \tag{1.5}$$

$$= (-1)^{a+1}a!\zeta(a+1,z), \tag{1.6}$$

for $a \in \mathbb{N}$, where $\zeta(r,z)$ is the Hurwitz zeta function. Specifically, $\zeta(r,1) = \zeta(r)$ where $\zeta(r)$ is the Riemann zeta function. Also, in [12], it has been established that

$$\psi^{(a)}(z) = (-1)^{a+1}\left[\frac{(a-1)!}{z^a} + \frac{a!}{2z^{a+1}} + \int_0^\infty \left(\frac{s}{e^s-1} - 1 + \frac{s}{2}\right)s^{a-1}e^{-zs}ds\right]. \tag{1.7}$$

The k-gamma function (also known as the k-analogue of the gamma function) was introduced by Díaz and Pariguan [14] and it is defined as

$$\Gamma_k(z) = \int_0^\infty s^{z-1}e^{-\frac{s^k}{k}}ds, \tag{1.8}$$

$$= \lim_{r\to\infty}\frac{r!k^r(rk)^{\frac{z}{k}-1}}{(z)_{r,k}}, \tag{1.9}$$

for $k > 0$ and $z \in \mathbb{C}\backslash k\mathbb{Z}$, where $(z)_{r,k} = z(z+k)(z+2k)\dots(z+(r-1)k)$ is the Pochhammer k-symbol. The k-gamma function satisfies the basic identities

$$\Gamma_k(z+k) = z\Gamma_k(z),$$
$$\Gamma_k(k) = 1,$$
$$\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma(\frac{z}{k}),$$
$$\Gamma_k(ak) = k^{a-1}(a-1)!, \quad a \in \mathbb{N}.$$

The k-digamma function is defined as the logarithmic derivative of the k-gamma function. It is given by any of the following forms (see [18, 19, 22, 23, 28, 32])

$$\psi_k(z) = \frac{\Gamma_k'(z)}{\Gamma_k(z)} = \frac{\ln k - \gamma}{k} - \frac{1}{z} + \sum_{r=1}^{\infty}\frac{z}{rk(rk+z)}, \tag{1.10}$$

$$= \frac{\ln k - \gamma}{k} + \sum_{r=0}^{\infty}\left(\frac{1}{rk+k} - \frac{1}{rk+z}\right), \tag{1.11}$$

$$= \frac{\ln k - \gamma}{k} + \int_0^\infty \frac{e^{-ks}-e^{-zs}}{1-e^{-ks}}ds, \tag{1.12}$$

$$= \frac{\ln k - \gamma}{k} + \int_0^1 \frac{s^{k-1}-s^{z-1}}{1-s^k}ds, \tag{1.13}$$

and satifies the identities

$$\psi_k(z+k) = \frac{1}{z} + \psi_k(z), \tag{1.14}$$

$$\psi_k(k) = \frac{\ln k - \gamma}{k}, \tag{1.15}$$

$$\psi_k(z) = \frac{\ln k}{k} + \frac{1}{k}\psi(\frac{z}{k}). \tag{1.16}$$

The k-polygamma functions are defined as [19, 28]

$$\psi_k^{(a)}(z) = \frac{d^a}{dz^a}\psi_k(z) = (-1)^{a+1}\sum_{r=0}^{\infty}\frac{a!}{(rk+z)^{a+1}}, \tag{1.17}$$

$$= (-1)^{a+1}\int_0^{\infty}\frac{s^a e^{-zs}}{1-e^{-ks}}ds, \tag{1.18}$$

$$= -\int_0^1 \frac{(\ln s)^a s^{z-1}}{1-s^k}ds, \tag{1.19}$$

where $a \in \mathbb{N}$ and satisfies the identities (see also [33])

$$\psi_k^{(a)}(z+k) = \frac{(-1)^a a!}{z^{a+1}} + \psi_k^{(a)}(z), \tag{1.20}$$

$$\psi_k^{(a)}(z) = \frac{1}{k^{a+1}}\psi^{(a)}(\frac{z}{k}), \tag{1.21}$$

$$= \frac{(-1)^{a+1}a!}{k^{a+1}}\zeta(a+1, \frac{z}{k}). \tag{1.22}$$

The k-polygamma functions play a crucial role in the theory of special functions and mathematical analysis. Due to their strong connections with other special functions, as well as their relationship with some famous mathematical constants, they found applications in other areas such as statistics and mathematical physics. In recent times, the k-polygamma functions have been studied along different paths. For instance, see [31, 32, 35, 37, 39].

Let $\mathcal{H}(u,v)$, $\mathcal{G}(u,v)$ and $\mathcal{A}(u,v)$ respectively be the harmonic mean, geometric mean and arithmetic mean of u and v, which are defined as

$$\mathcal{H}(u,v) = \frac{2uv}{u+v}, \quad \mathcal{G}(u,v) = \sqrt{uv} \quad \text{and} \quad \mathcal{A}(u,v) = \frac{u+v}{2}.$$

Gautschi [15] proved that the harmonic mean inequality

$$\mathcal{H}\left(\Gamma(z), \Gamma(1/z)\right) \geq 1 \tag{1.23}$$

holds for $z > 0$. By virtue of (1.23), the inequalities

$$\mathcal{A}\left(\Gamma(z), \Gamma(1/z)\right) \geq 1, \tag{1.24}$$

$$\mathcal{G}\left(\Gamma(z), \Gamma(1/z)\right) \geq 1, \tag{1.25}$$

are obtained for $z > 0$. Other researchers have considered some refinements and generalizations of the inequalities (1.23), (1.24) and (1.25) due to their usefulness in mathematical analysis [1, 2, 4–7, 16, 17].

Alzer and Jameson [8] established striking companions of (1.23), (1.24) and (1.25) by proving that the inequalities

$$\mathcal{H}\left(\psi(z), \psi(1/z)\right) \geq -\gamma, \tag{1.26}$$

$$\mathcal{A}\left(\psi(z), \psi(1/z)\right) < -\gamma, \quad z \neq 1, \tag{1.27}$$

$$\mathcal{G}\left(\psi(z), \psi(1/z)\right) < \gamma, \quad z \neq 1, \tag{1.28}$$

hold for $z > 0$.

Yildirim [30] established generalizations of (1.26), (1.27) and (1.28) by proving that

$$\mathcal{H}\left(\psi_k(z),\psi_k(1/z)\right) \geq \psi_k(1), \tag{1.29}$$

$$\mathcal{A}\left(\psi_k(z),\psi_k(1/z)\right) < \psi_k(1), \quad z \neq 1, \tag{1.30}$$

$$\mathcal{G}\left(\psi_k(z),\psi_k(1/z)\right) \leq \psi_k(1), \tag{1.31}$$

hold for $z > 0$. The results (1.29), (1.30) and (1.31) are improvements of the results of Yin et al. [36] concerning the same subject.

Nantomah et al. [27] extended these results to the trigamma function by proving that the inequalities

$$\mathcal{H}\left(\psi'(z),\psi'(1/z)\right) \leq \frac{\pi^2}{6}, \tag{1.32}$$

$$\mathcal{A}\left(\psi'(z),\psi'(1/z)\right) \geq \frac{\pi^2}{6}, \tag{1.33}$$

$$\mathcal{G}\left(\psi'(z),\psi'(1/z)\right) \geq \frac{\pi^2}{6}, \tag{1.34}$$

hold for $z > 0$.

Das and Swaminathan [13] considered a more general results involving the polygamma functions. Specifically, they proved that, for $m \in \mathbb{N}$, the inequalities

$$(-1)^m\mathcal{H}\left(\psi^{(m)}(z),\psi^{(m)}(1/z)\right) \geq (-1)^m\psi^{(m)}(1), \tag{1.35}$$

$$(-1)^m\mathcal{A}\left(\psi^{(m)}(z),\psi^{(m)}(1/z)\right) < (-1)^m\psi^{(m)}(1), \quad z \neq 1, \tag{1.36}$$

$$\mathcal{G}\left(\psi^{(m)}(z),\psi^{(m)}(1/z)\right) > \psi^{(m)}(1), \quad z \neq 1, \tag{1.37}$$

hold for $z > 0$. From the work of Das and Swaminathan, inequalities (1.32), (1.33) and (1.34) are recovered as particular cases.

In a recent work, Nantomah [26] established results which are equivalent to (1.35), (1.36) and (1.37) by using different techniques.

Motivated by [26] and [30], the goal of this current work is to extend and generalize the results of [26] and [30] respectively to the the k-polygamma functions. For analogous results relating to other special functions, interested readers may refer to the works [11,20,21,24, 25,30,36].

2. Results and Discussion

Theorem 2.1. *The following limits statements hold for all $a \in \mathbb{N}$.*

$$\lim_{z\to 0} z^{a+1}\psi_k^{(a)}(z) = (-1)^{a+1}a!, \tag{2.1}$$

$$\lim_{z\to \infty} z^{a}\psi_k^{(a)}(z) = (-1)^{a+1}\frac{(a-1)!}{k}, \tag{2.2}$$

$$\lim_{z\to 0} z\frac{\psi_k^{(a+1)}(z)}{\psi_k^{(a)}(z)} = -(a+1), \tag{2.3}$$

$$\lim_{z\to \infty} z\frac{\psi_k^{(a+1)}(z)}{\psi_k^{(a)}(z)} = -a, \tag{2.4}$$

$$\lim_{z\to 0}\frac{\left(\psi_k^{(a+1)}(z)\right)^2}{\psi_k^{(a)}(z)\psi_k^{(a+2)}(z)}=\frac{a+1}{a+2}, \tag{2.5}$$

$$\lim_{z\to \infty}\frac{\left(\psi_k^{(a+1)}(z)\right)^2}{\psi_k^{(a)}(z)\psi_k^{(a+2)}(z)}=\frac{a}{a+1}. \tag{2.6}$$

Proof. From (1.20), we have

$$\lim_{z\to 0} z^{a+1}\psi_k^{(a)}(z)=\lim_{z\to 0}\left[z^{a+1}\psi_k^{(a)}(z+k)-(-1)^a a!\right]=(-1)^{a+1}a!$$

which proves (2.1). Next, by making use of (1.7) and (1.21), we have

$$\begin{aligned}\psi_k^{(a)}(z)&=\frac{1}{k^{a+1}}\psi^{(a)}(\frac{z}{k})\\ &=\frac{(-1)^{a+1}}{k^{a+1}}\left[k^a\frac{(a-1)!}{z^a}+k^{a+1}\frac{a!}{2z^{a+1}}+\int_0^\infty\left(\frac{s}{e^s-1}-1+\frac{s}{2}\right)s^{a-1}e^{-\frac{zs}{k}}ds\right]\\ &=(-1)^{a+1}\left[\frac{1}{k}\frac{(a-1)!}{z^a}+\frac{a!}{2z^{a+1}}+\frac{1}{k^{a+1}}\int_0^\infty\left(\frac{s}{e^s-1}-1+\frac{s}{2}\right)s^{a-1}e^{-\frac{zs}{k}}ds\right].\end{aligned}$$

Hence

$$\lim_{z\to\infty} z^a\psi_k^{(a)}(z)=(-1)^{a+1}\frac{(a-1)!}{k},$$

which proves (2.2). Next, by applying (2.1), we obtain

$$\lim_{z\to 0} z\frac{\psi_k^{(a+1)}(z)}{\psi_k^{(a)}(z)}=\lim_{z\to 0}\frac{z^{a+2}\psi_k^{(a+1)}(z)}{z^{a+1}\psi_k^{(a)}(z)}=\frac{(-1)^{a+2}(a+1)!}{(-1)^{a+1}a!}=-(a+1)$$

which proves (2.3). Next, by applying (2.2), we obtain

$$\lim_{z\to\infty} z\frac{\psi_k^{(a+1)}(z)}{\psi_k^{(a)}(z)}=\lim_{z\to\infty}\frac{z^{a+1}\psi_k^{(a+1)}(z)}{z^{a}\psi_k^{(a)}(z)}=\frac{(-1)^{a+2}\frac{a!}{k}}{(-1)^{a+1}\frac{(a-1)!}{k}}=-a$$

which proves (2.4). Next, by applying (2.1), we obtain

$$\begin{aligned}\lim_{z\to 0}\frac{\left(\psi_k^{(a+1)}(z)\right)^2}{\psi_k^{(a)}(z)\psi_k^{(a+2)}(z)}&=\lim_{z\to 0}\frac{z^{a+2}\psi_k^{(a+1)}(z).z^{a+2}\psi_k^{(a+1)}(z)}{z^{a+1}\psi_k^{(a)}(z).z^{a+3}\psi_k^{(a+2)}(z)}\\ &=\frac{\lim_{z\to 0}z^{a+2}\psi_k^{(a+1)}(z).\lim_{z\to 0}z^{a+2}\psi_k^{(a+1)}(z)}{\lim_{z\to 0}z^{a+1}\psi_k^{(a)}(z).\lim_{z\to 0}z^{a+3}\psi_k^{(a+2)}(z)}\\ &=\frac{(-1)^{a+2}(a+1)!(-1)^{a+2}(a+1)!}{(-1)^{a+1}a!(-1)^{a+3}(a+2)!}\\ &=\frac{(a+1)!(a+1)!}{a!(a+2)!}\\ &=\frac{a+1}{a+2}\end{aligned}$$

which proves (2.5). Finally, by applying (2.2), we obtain

$$\begin{aligned}
\lim_{z\to\infty} \frac{\left(\psi_k^{(a+1)}(z)\right)^2}{\psi_k^{(a)}(z)\psi_k^{(a+2)}(z)} &= \lim_{z\to\infty} \frac{z^{a+1}\psi_k^{(a+1)}(z).z^{a+1}\psi_k^{(a+1)}(z)}{z^a\psi_k^{(a)}(z).z^{a+2}\psi_k^{(a+2)}(z)} \\
&= \frac{\lim_{z\to\infty} z^{a+1}\psi_k^{(a+1)}(z). \lim_{z\to\infty} z^{a+1}\psi_k^{(a+1)}(z)}{\lim_{z\to\infty} z^a\psi_k^{(a)}(z). \lim_{z\to\infty} z^{a+2}\psi_k^{(a+2)}(z)} \\
&= \frac{(-1)^{a+2}\frac{a!}{k}.(-1)^{a+2}\frac{a!}{k}}{(-1)^{a+1}\frac{(a-1)!}{k}.(-1)^{a+3}\frac{(a+1)!}{k}} \\
&= \frac{a!a!}{(a-1)!(a+1)!} \\
&= \frac{a}{a+1}
\end{aligned}$$

which proves (2.6). □

Remark 2.1. The limit (2.2) and other limits equivalent to (2.5) and (2.6) earlier appeared in [38]. Also, some more general results concerning these limits have been established in [34] and [35]. Here, we provided a very simple way of obtaining the limits.

Theorem 2.2. *For $z > 0$ and $a \in \mathbb{N}$, the function*

$$T(z) = z\frac{\psi_k^{(a+1)}(z)}{\psi_k^{(a)}(z)} \tag{2.7}$$

is strictly increasing and that being so, the inequality

$$-\frac{a+1}{z} < \frac{\psi_k^{(a+1)}(z)}{\psi_k^{(a)}(z)} < -\frac{a}{z} \tag{2.8}$$

holds.

Proof. It has been shown in Lemma 2 of [9] and [3] that, for $u > 0$ and $a \in \mathbb{N}$, the function

$$f(u) = u\frac{\psi^{(a+1)}(u)}{\psi^{(a)}(u)}$$

is strictly increasing. By using the identity (1.21), we have

$$T(z) = z\frac{\psi_k^{(a+1)}(z)}{\psi_k^{(a)}(z)} = \frac{z}{k}\frac{\psi^{(a+1)}(\frac{z}{k})}{\psi^{(a)}(\frac{z}{k})}.$$

Hence $T(z)$ is strictly increasing. The increasing property of $T(z)$ and the limit properties in equations (2.3) and (2.4) imply that

$$-(a+1) = \lim_{z\to 0} T(z) < T(z) < \lim_{z\to\infty} T(z) = -a$$

which yields the inequality (2.8). □

Remark 2.2. It is known in [10] that, a positive function h is said to be GG-convex or geometrically convex if and only if $zh'(z)/h(z)$ is increasing. Theorem 2.2 implies that, for odd a, the function $\psi_k^{(a)}(z)$ is geometrically convex. This is because $\psi_k^{(a)}(z) > 0$ if a is odd and $\psi_k^{(a)}(z) < 0$ if a is even.

Theorem 2.3. *For $z > 0$ and $a \in \mathbb{N}$, the function*

$$Q(z) = \frac{\left(\psi_k^{(a+1)}(z)\right)^2}{\psi_k^{(a)}(z)\psi_k^{(a+2)}(z)} \tag{2.9}$$

is strictly decreasing and as a result, the inequality

$$\frac{a}{a+1} < \frac{\left(\psi_k^{(a+1)}(z)\right)^2}{\psi_k^{(a)}(z)\psi_k^{(a+2)}(z)} < \frac{a+1}{a+2} \tag{2.10}$$

holds.

Proof. It has been proved in Theorem 2 of [29] that, for $u > 0$ and $a \in \mathbb{N}$, the function

$$g(u) = \frac{\left(\psi^{(a+1)}(u)\right)^2}{\psi^{(a)}(u)\psi^{(a+2)}(u)}$$

is strictly decreasing. By using the identity (1.21), we have

$$Q(z) = \frac{\left(\psi_k^{(a+1)}(z)\right)^2}{\psi_k^{(a)}(z)\psi_k^{(a+2)}(z)} = \frac{\left(\psi^{(a+1)}(\frac{z}{k})\right)^2}{\psi^{(a)}(\frac{z}{k})\psi^{(a+2)}(\frac{z}{k})}.$$

Hence $Q(z)$ is strictly decreasing. The decreasing property of $Q(z)$ and the limit properties in equations (2.5) and (2.6) imply that

$$\frac{a}{a+1} = \lim_{z\to\infty} Q(z) < Q(z) < \lim_{z\to 0} Q(z) = \frac{a+1}{a+2}$$

which yields the inequality (2.10). □

Remark 2.3. Generalized forms of Theorem 2.3 have been obtained in [34] and [35]. Also, an inequality equivalent to (2.10) was established in [38] by using a different procedure.

Remark 2.4. The left hand side of (2.10) implies that

$$\psi_k^{(a)}(z)\psi_k^{(a+2)}(z) - 2\left(\psi_k^{(a+1)}(z)\right)^2 < 0. \tag{2.11}$$

Lemma 2.1. *For $z > 0$ and $a \in \mathbb{N}$, the function*

$$A(z) = z\frac{\psi_k^{(a+1)}(z)}{\left(\psi_k^{(a)}(z)\right)^2} \tag{2.12}$$

is strictly decreasing if a is odd and strictly increasing if a is even.

Proof. It has been shown in Lemma 2.3 of [26] that, for $u > 0$ and $a \in \mathbb{N}$, the function

$$h(u) = u\frac{\psi^{(a+1)}(u)}{\left(\psi^{(a)}(u)\right)^2}$$

is strictly decreasing if a is odd and strictly increasing if a is even. Using (1.21), we obtain

$$A(z) = z\frac{\psi_k^{(a+1)}(z)}{\left(\psi_k^{(a)}(z)\right)^2} = k^{a+1}\cdot\frac{z}{k}\frac{\psi^{(a+1)}(\frac{z}{k})}{\left(\psi^{(a)}(\frac{z}{k})\right)^2}.$$

Hence, $A(z)$ is strictly decreasing if a is odd and strictly increasing if a is even. □

Theorem 2.4. *Suppose that $z > 0$ and $a \in \mathbb{N}$. Then*

$$\frac{2\psi_k^{(a)}(z)\psi_k^{(a)}(1/z)}{\psi_k^{(a)}(z)+\psi_k^{(a)}(1/z)} \leq \frac{a!}{k^{a+1}}\zeta\left(a+1,\frac{1}{k}\right) \tag{2.13}$$

is valid if a is odd and

$$\frac{2\psi_k^{(a)}(z)\psi_k^{(a)}(1/z)}{\psi_k^{(a)}(z)+\psi_k^{(a)}(1/z)} \geq -\frac{a!}{k^{a+1}}\zeta\left(a+1,\frac{1}{k}\right) \tag{2.14}$$

is valid if a is even. Under each situation, equality is arrived at when $z = 1$.

Proof. The condition for equality in (2.13) and (2.14) is easy to establish. On grounds of this, we shall only prove the results for $z \in (0,1)\cup(1,\infty)$. Let $\mathcal{K}(z)$ be defined as

$$\mathcal{K}(z) = \frac{2\psi_k^{(a)}(z)\psi_k^{(a)}(1/z)}{\psi_k^{(a)}(z)+\psi_k^{(a)}(1/z)}$$

where $a \in \mathbb{N}$ and $z \in (0,1)\cup(1,\infty)$. Then

$$\frac{\mathcal{K}'(z)}{\mathcal{K}(z)} = \frac{\psi_k^{(a+1)}(z)}{\psi_k^{(a)}(z)} - \frac{1}{z^2}\frac{\psi_k^{(a+1)}(1/z)}{\psi_k^{(a)}(1/z)} - \frac{\psi_k^{(a+1)}(z)-\frac{1}{z^2}\psi_k^{(a+1)}(1/z)}{\psi_k^{(a)}(z)+\psi_k^{(a)}(1/z)}$$

which upon rearrangement gives rise to

$$z\left[\psi_k^{(a)}(z)+\psi_k^{(a)}(1/z)\right]\frac{\mathcal{K}'(z)}{\mathcal{K}(z)} = z\frac{\psi_k^{(a+1)}(z)}{\psi_k^{(a)}(z)}\psi_k^{(a)}(1/z) - \frac{1}{z}\frac{\psi_k^{(a+1)}(1/z)}{\psi_k^{(a)}(1/z)}\psi_k^{(a)}(z).$$

Further rearrangement gives

$$\begin{aligned} z\left[\frac{1}{\psi_k^{(a)}(z)}+\frac{1}{\psi_k^{(a)}(1/z)}\right]\frac{\mathcal{K}'(z)}{\mathcal{K}(z)} &= z\frac{\psi_k^{(a+1)}(z)}{\left(\psi_k^{(a)}(z)\right)^2} - \frac{1}{z}\frac{\psi_k^{(a+1)}(1/z)}{\left(\psi_k^{(a)}(1/z)\right)^2} \\ &:= \mathcal{O}(z). \end{aligned}$$

Now assume that a is odd. Since the function $A(z)$ in Lemma 2.1 is decreasing when a is odd, we conclude that $\mathcal{O}(z) > 0$ if $z \in (0,1)$ and $\mathcal{O}(z) < 0$ if $z \in (1,\infty)$. It is worth noting that if $z \in (0,1)$, then $z < 1/z$ and if $z \in (1,\infty)$, then $z > 1/z$. Thus, $\mathcal{K}'(z) > 0$ if $z \in (0,1)$

and $\mathcal{K}'(z) < 0$ if $z \in (1, \infty)$. Hence, $\mathcal{K}(z)$ is increasing on $(0, 1)$ and decreasing on $(1, \infty)$. Therefore, for $z \in (0, 1) \cup (1, \infty)$, we have

$$\mathcal{K}(z) < \lim_{z \to 1} \mathcal{K}(z) = \psi_k^{(a)}(1) = \frac{a!}{k^{a+1}} \zeta \left(a + 1, \frac{1}{k}\right)$$

which proves the inequality in (2.13). Next, assume that a is even. Likewise, since the function $A(z)$ in Lemma 2.1 is increasing for even a, we conclude that $\mathcal{O}(z) < 0$ if $z \in (0, 1)$ and $\mathcal{O}(z) > 0$ if $z \in (1, \infty)$. In this way, $\mathcal{K}'(z) < 0$ if $z \in (0, 1)$ and $\mathcal{K}'(z) > 0$ if $z \in (1, \infty)$. Hence, $\mathcal{K}(z)$ is decreasing on $(0, 1)$ and increasing on $(1, \infty)$. Therefore, for $z \in (0, 1) \cup (1, \infty)$, we have

$$\mathcal{K}(z) > \lim_{z \to 1} \mathcal{K}(z) = \psi^{(a)}(1) = -\frac{a!}{k^{a+1}} \zeta \left(a + 1, \frac{1}{k}\right)$$

which proves the inequality in (2.14). This completes the proof of Theorem 2.4. ☐

Theorem 2.5. *Let $z > 0$ and $a \in \mathbb{N}$. Then the inequality*

$$\sqrt{\psi_k^{(a)}(z)\psi_k^{(a)}(1/z)} \geq \frac{a!}{k^{a+1}} \zeta \left(a + 1, \frac{1}{k}\right) \tag{2.15}$$

is valid. Equality is arrived at when $z = 1$

Proof. The condition for equality is easy to establish. For this reason, let $\mathcal{D}(z) = \psi_k^{(a)}(z)\psi_k^{(a)}(1/z)$ for $a \in \mathbb{N}$ and $z \in (0, 1) \cup (1, \infty)$. Then

$$\begin{aligned} z\frac{\mathcal{D}'(z)}{\mathcal{D}(z)} &= z\frac{\psi_k^{(a+1)}(z)}{\psi_k^{(a)}(z)} - \frac{1}{z}\frac{\psi_k^{(a+1)}(1/z)}{\psi_k^{(a)}(1/z)} \\ &:= \Delta(z). \end{aligned}$$

Since $T(z)$ is increasing (see Theorem 2.2), we conclude that $\Delta(z) < 0$ if $z \in (0, 1)$ and $\Delta(z) > 0$ if $z \in (1, \infty)$. Consequently, $\mathcal{D}'(z) < 0$ if $z \in (0, 1)$ and $\mathcal{D}'(z) > 0$ if $z \in (1, \infty)$. That is, $\mathcal{D}(z)$ is decreasing on $(0, 1)$ and increasing on $(1, \infty)$. Therefore, for $z \in (0, 1) \cup (1, \infty)$, we have

$$\mathcal{D}(z) > \lim_{z \to 1} \mathcal{D}(z) = \left(\psi_k^{(a)}(1)\right)^2 = \left[\frac{a!}{k^{a+1}} \zeta \left(a + 1, \frac{1}{k}\right)\right]^2$$

which completes the proof of Theorem 2.5. ☐

Remark 2.5. Results analogous to the results of Theorem 2.5 were obtained in Theorem 3.2 of [38] by using different procedures.

Lemma 2.2. *Let $z > 0$ and $a \in \mathbb{N}$. Then the function*

$$E(z) = z\psi_k^{(a+1)}(z). \tag{2.16}$$

is strictly increasing if a is odd and strictly decreasing if a is even.

Proof. It has been shown in Lemma 2.7 of [26] that, for $u > 0$ and $a \in \mathbb{N}$, the function

$$q(u) = u\psi^{(a+1)}(u)$$

is strictly increasing if a is odd and strictly decreasing if a is even. By applying (1.21), we obtain

$$E(z) = z\psi_k^{(a+1)}(z) = \frac{1}{k^{a+1}}\frac{z}{k}\psi_k^{(a+1)}(\frac{z}{k}).$$

Hence, $E(z)$ is strictly increasing if a is odd and strictly decreasing if a is even. □

Theorem 2.6. *Suppose that $z > 0$ and $a \in \mathbb{N}$. Then*

$$\frac{\psi_k^{(a)}(z) + \psi_k^{(a)}(1/z)}{2} \geq \frac{a!}{k^{a+1}}\zeta\left(a+1, \frac{1}{k}\right) \tag{2.17}$$

is valid if a is odd and

$$\frac{\psi_k^{(a)}(z) + \psi_k^{(a)}(1/z)}{2} \leq -\frac{a!}{k^{a+1}}\zeta\left(a+1, \frac{1}{k}\right) \tag{2.18}$$

is valid if a is even. Under each situation, equality is arrived at when $z = 1$.

Proof. The condition for equality is easy to establish. For this reason, let $\mathcal{U}(z) = \psi_k^{(a)}(z) + \psi_k^{(a)}(1/z)$ for $a \in \mathbb{N}$ and $z \in (0,1) \cup (1,\infty)$. Then

$$\begin{aligned} z\mathcal{U}'(z) &= z\psi_k^{(a+1)}(z) - \frac{1}{z}\psi_k^{(a+1)}(1/z) \\ &:= \theta(z). \end{aligned}$$

Now, assume that a is odd. Since the function $E(z)$ in Lemma 2.2 is increasing when a is odd, we conclude that $\theta(z) < 0$ if $z \in (0,1)$ and $\theta(z) > 0$ if $z \in (1,\infty)$. These imply that, $\mathcal{U}(z)$ is decreasing on $(0,1)$ and increasing on $(1,\infty)$. Therefore, for $z \in (0,1) \cup (1,\infty)$, we obtain

$$\mathcal{U}(z) > \lim_{z\to 1}\mathcal{U}(z) = 2\psi_k^{(a)}(1) = 2\left[\frac{a!}{k^{a+1}}\zeta\left(a+1, \frac{1}{k}\right)\right]$$

which proves (2.17). By the same approache, assume that a is even. Because the function $E(z)$ in Lemma 2.2 is decreasing for even a, we conclude that $\theta(z) > 0$ if $z \in (0,1)$ and $\theta(z) < 0$ if $z \in (1,\infty)$. These imply that, $\mathcal{U}(z)$ is increasing on $(0,1)$ and decreasing on $(1,\infty)$. Therefore, for $z \in (0,1) \cup (1,\infty)$, we obtain

$$\mathcal{U}(z) < \lim_{z\to 1}\mathcal{U}(z) = 2\psi_k^{(a)}(1) = -2\left[\frac{a!}{k^{a+1}}\zeta\left(a+1, \frac{1}{k}\right)\right]$$

whichproves (2.18). This completes the proof of Theorem 2.6. □

Remark 2.6. Results equivalent to the results of Theorem 2.6 were obtained in Lemma 2.5 of [38] by using different procedures.

3. Conclusion

In this paper, we established some limit properties involving the k-polygamma function, $\psi_k^{(a)}(z)$ where $z > 0$ and $a \in \mathbb{N}$. By using the limits, we established bounds for certain ratios involving $\psi_k^{(a)}(z)$. Furthermore, we established bounds (in terms of Hurwitz zeta function) for the harmonic, arithmetic and geometric means involving the functions $\psi_k^{(a)}(z)$ and $\psi_k^{(a)}(1/z)$. Our results serve as generalizations and extensions of some recent results.

The present results could inspire further research on the subject. Lastly, we pose the following open problem.

Open Problem: Among other things, Bouali [11] provided bounds for

$$\mathcal{H}\left(\psi_q(z), \psi_q(1/z)\right),$$

$$\mathcal{A}\left(\psi_q(z), \psi_q(1/z)\right),$$

$$\mathcal{G}\left(\psi_q(z), \psi_q(1/z)\right),$$

where $z > 0$ and $\psi_q(z)$ is the q-digamma function. Taking inspiration from this, find bounds for

$$\mathcal{H}\left(\psi_q^{(r)}(z), \psi_q^{(r)}(1/z)\right),$$

$$\mathcal{A}\left(\psi_q^{(r)}(z), \psi_q^{(r)}(1/z)\right),$$

$$\mathcal{G}\left(\psi_q^{(r)}(z), \psi_q^{(r)}(1/z)\right),$$

where $z > 0$, $r \in \mathbb{N}$ and $\psi_q^{(r)}(z)$ is the q-polygamma function.

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