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NEW APPROACH TO NESTED FUNCTIONS, SOME INEQUALITIES AND NEW RESULTS

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ABSTRACT. This work focuses on H and T nested functions T_{pj} and H_{pj}. We define particular nested functions based on the nested functions T_{pj} and H_{pj} and study peculiar properties of them. We propose some identities and prove important trigonometric and hyperbolic inequalities relating to these nested functions. In order to highlight the results, we give several examples.

1. Introduction

Inequalities are one of the most important topics that have received significant attention in mathematics, and much research has been conducted in this field (see [3–8]). In this work, as the special case of Mittag-Leffler functions, we consider H and T nested functions $T_{p\ j}$ and $H_{p\ j}$ and we will investigate several identities about these nested functions. Also, we define new nested functions based on $T_{p\ j}$ and $H_{p\ j}$. We give inequalities and identities related to these new nested functions. Also, we give some counter examples to one of the important theorems of paper [6].

For more information about hyperbolic functions, nested functions and some kinds of generalizations of these functions, we refer to [5] and [7–12].

We start this section by definitions of $T_{p\ j}$ and $H_{p\ j}$ and give some properties of these special functions.

Definition 1.1. [1,2] H and T nested functions $T_{p\ j}, H_{p\ j} : \mathbb{R} \to \mathbb{R},\ j = 0, 1, 2, \cdots, p-1, p \in \mathbb{N}$, are defined respectively, as follows:

$$T_{p\ j}(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{pn+j}}{(pn+j)!}, \quad H_{p\ j}(t) = \sum_{n=0}^{\infty} \frac{t^{pn+j}}{(pn+j)!}.$$

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Theorem 1.1. [1] *For each $t \in \mathbb{R}$, following identities are valid:*

$$\begin{aligned} T'_{p\ 0}(t) &= -T_{p\ p-1}(t), & H'_{p\ 0}(t) &= H_{p\ p-1}(t), \\ T'_{p\ 1}(t) &= T_{p\ 0}(t), & H'_{p\ 1}(t) &= H_{p\ 0}(t), \\ &\vdots & &\vdots \\ T'_{p\ p-1}(t) &= T_{p\ p-2}(t), & H'_{p\ p-1}(t) &= H_{p\ p-2}(t). \end{aligned}$$

Remark 1.1. [2] Let $\lambda^p = 1, \lambda \neq 1, w^p = -1, w \neq -1$, we have

$$T_{p\ 0}(t) = \frac{\sum_{j=0}^{p-1} e^{w^j t}}{p}, j = 0, 1, ..., p-1,$$
$$T_{p\ 1}(t) = \frac{\sum_{j=0}^{p-1} w^{p-j} e^{w^j t}}{p}, j = 0, 1, ..., p-1,$$
$$\cdots$$
$$T_{p\ p-1}(t) = \frac{\sum_{j=0}^{p-1} w^j e^{w^j t}}{p}, j = 0, 1, ..., p-1,$$

$$H_{p\ 0}(t) = \frac{\sum_{j=0}^{p-1} e^{\lambda^j t}}{p}, j = 0, 1, ..., p-1,$$
$$H_{p\ 1}(t) = \frac{\sum_{j=0}^{p-1} \lambda^{p-j} e^{\lambda^j t}}{p}, j = 0, 1, ..., p-1,$$
$$H_{p\ 2}(t) = \frac{\sum_{j=0}^{p-1} \lambda^{p-2j} e^{\lambda^j t}}{p}, j = 0, 1, ..., p-1,$$
$$\cdots$$
$$H_{p\ p-1}(t) = \frac{\sum_{j=0}^{p-1} \lambda^j e^{\lambda^j t}}{p}, j = 0, 1, ..., p-1.$$

For example, for $p = 3$, we find that

$$H_{3\ 0}(t) = \frac{e^t + e^{\lambda t} + e^{\lambda^2 t}}{3},$$
$$H_{3\ 1}(t) = \frac{e^t + \lambda^2 e^{\lambda t} + \lambda e^{\lambda^2 t}}{3},$$
$$H_{3\ 2}(t) = \frac{e^t + \lambda e^{\lambda t} + \lambda^2 e^{\lambda^2 t}}{3}.$$

Remark 1.2. [2] Also, for $p = 3$, we have, $w^p = -1, w \neq -1$, and the following identities are valid:

$$\begin{aligned} T_{3\ 0}(t) &= \frac{e^{-t} + e^{wt} + e^{-w^2 t}}{3}, \\ T_{3\ 1}(t) &= \frac{-e^{-t} - w^2 e^{wt} + w e^{-w^2 t}}{3}, \\ T_{3\ 2}(t) &= \frac{e^{-t} - w e^{wt} + w^2 e^{-w^2 t}}{3}. \end{aligned}$$

Definition 1.2. The functions ${}_p\tan_{ij},\ {}_p\tanh_{ij} : \mathbb{R} \to \mathbb{R},\ i, j = 0, 1, 2, \cdots, p-1,\ p \in \mathbb{N}, i \neq j$ are defined respectively, as follows:

$$ {}_p\tan_{i\ j}(t) = \frac{T_{p\ i}(t)}{T_{p\ j}(t)}, \quad {}_p\tanh_{i\ j}(t) = \frac{H_{p\ i}(t)}{H_{p\ j}(t)}. $$

For example for $i = 1$ and $j = 0$, we get

$$ {}_p\tan_{1\ 0}(t) = \frac{T_{p\ 1}(t)}{T_{p\ 0}(t)}, \quad {}_p\tanh_{1\ 0}(t) = \frac{H_{p\ 1}(t)}{H_{p\ 0}(t)}. $$

Remark 1.3. By the use of Remark (1.1), for $p = 3$, we obtain the following identities:

$$ \begin{aligned} {}_3\tan_{1\ 0}(t) &= \frac{H_{3\ 1}(t)}{H_{3\ 0}(t)} = \frac{e^t + \lambda^2 e^{\lambda t} + \lambda e^{\lambda^2 t}}{e^t + e^{\lambda t} + e^{\lambda^2 t}}, \\ {}_3\tanh_{2\ 1}(t) &= \frac{H_{3\ 2}(t)}{H_{3\ 1}(t)} = \frac{e^t + \lambda e^{\lambda t} + \lambda^2 e^{\lambda^2 t}}{e^t + \lambda^2 e^{\lambda t} + \lambda e^{\lambda^2 t}}, \\ {}_3\tanh_{0\ 2}(t) &= \frac{H_{3\ 0}(t)}{H_{3\ 2}(t)} = \frac{e^t + e^{\lambda t} + e^{\lambda^2 t}}{e^t + \lambda e^{\lambda t} + \lambda^2 e^{\lambda^2 t}}. \end{aligned} $$

Lemma 1.1. *For each $x \neq 0$, the following inequalities hold:*

$$ \frac{{}_p\tanh_{1\ 0}(x)}{x} < 1 \quad \text{and} \quad \frac{H_{p\ 1}(x)}{x} > 1, $$

$$ \frac{{}_p\tanh_{i\ i-1}(x)}{x} < 1 \quad , \ i = 0, 1, 2, \cdots, p-1. $$

According to the above mentioned results, for $x > 0$ we have,

$$ (1 + p\ H_{p\ 0}(x)) \sum_{n=1}^{\infty} \frac{x^{pn}}{(pn+1)!} + 1 - H_{p\ 0}(x) > 0. $$

That is,

$$ (1 + p\ H_{p\ 0}(x))(H_{p\ 1}(x) - x) > x(H_{p\ 0}(x) - 1). $$

So, we get

$$ (1 + p\ H_{p\ 0}(x))H_{p\ 1}(x) > (1 + p)x\ H_{p\ 0}(x). $$

The following lemmas are essential for the next sections.

Lemma 1.2. [10] *Let $-\infty \leq u < v \leq \infty$ and p and q be continuous functions that are differentiable on (u, v), with $p(u+) = q(u+) = 0$ or $p(v-) = q(v-) = 0$. Suppose that $q(z)$ and $q\prime(z)$ are nonzero for all $z \in (u, v)$. If $p\prime(z)/q\prime(z)$ is increasing (or decreasing) on (u, v), then $p(x)/q(x)$ is also increasing (or decreasing) on (u, v).*

Definition 1.3. [6] The function f is superadditive on an interval I, if for every $x, y \in I$,

$$ f(x + y) \geq f(x) + f(y). $$

The function f is subadditive on an interval I, if for every $x, y \in I$,

$$ f(x + y) \leq f(x) + f(y). $$

Lemma 1.3. [6]*If a function $\frac{p(x)}{x}$ is increasing or decreasing on an interval I, then $p(x)$ is supperadditive or subadditive on I respectively.*

2. Some Properties of the Hi Integral

In this section, we study some properties of the nested function Hi by using an integral method. We prove that, in some conditions, Hi is monotonic and supperadditive. Also, we establish some inequalities related to this particular nested function and give various examples. We start this section with the definition of the function $Hc_{p\ i}(z)$, which generalizes the cardinal hyperbolic sine function as defined in [6].

Definition 2.1. The function $Hc_{p\ i}$ is defined as follows:

$$Hc_{p\ i}(z) = \begin{cases} \frac{H_{p\ i}(z)}{z^i}, & z \neq 0 \\ 1, & z = 0. \end{cases}$$

According to the following definition, we introduce a generalization of the hyperbolic sine integral function as described in [6].

Definition 2.2. The function $Hi_{p\ i}$ is defined as follows:

$$Hi_{p\ i}(z) = \int_0^1 \frac{H_{p\ i}(zt)}{t^i} dt, i = 1, 2, ..., p-1, (z > 0). \tag{2.1}$$

So, we get

$$Hi_{p\ i}(z) = \int_0^1 \frac{\sum_{n=0}^{\infty} \frac{z^{pn+i} t^{pn+i}}{(pn+i)!}}{t^i} dt = \int_0^z \sum_{n=0}^{\infty} \frac{z^{pn+i} t^{pn}}{(pn+i)!} dt = \sum_{n=0}^{\infty} \frac{z^{pn+i}}{(pn+1)(pn+i)!}. \tag{2.2}$$

Example 2.1. According to the Relation (2.2), for $p = 3$ we have the following identities:

$$\begin{aligned} Hi_{3\ 1}(z) &= \int_0^1 \frac{H_{3\ 1}(zt)}{t} dt = \sum_{n=0}^{\infty} \frac{z^{pn+1}}{(pn+1)(pn+1)!}, \\ Hi_{3\ 2}(z) &= \int_0^1 \frac{H_{3\ 1}(zt)}{t} dt = \sum_{n=0}^{\infty} \frac{z^{pn+2}}{(pn+1)(pn+2)!}. \end{aligned}$$

Based on the relation (2.1), the derivatives of $Hi_{p\ i}(z)$ are obtained by the following corollary.

Corollary 2.1. *The following identities are valid:*

$$Hi_{p\ i}^{(j)}(z) = \int_0^1 \frac{H_{p\ i}^{(j)}(zt)}{t^i} dt = \int_0^1 t^{j-i} H_{p\ i-j(i-j \equiv^p 0)}(zt) dt,$$

$$Hi_{p\ i}^{(i)}(z) = \int_0^1 H_{p\ 0}(zt) dt,$$

$$\cdots$$

$$Hi_{p\ i}^{(p)}(z) = \int_0^1 t^{p-i} H_{p\ 0}(zt) dt,$$

$$Hi_{p\ i}^{(p+1)}(z) = \int_0^1 t^{p+1-i} H_{p\ p-1}(zt) dt,$$

$$Hi_{p\ i}^{(p+2)}(z) = \int_0^1 t^{p+2-i} H_{p\ p-2}(zt) dt,$$

$$Hi_{p\ \ i}^{(p+3)}(z)=\int_0^1 t^{p+3-i}H_{p\ \ p-3}(zt)dt.$$

Remark 2.1. From Corollary 2.1, we obtain the following identity.

$$Hi_{p\ \ i}^{(i)}(z)=\int_0^1 H_{p\ \ 0}(zt)dt=\frac{H_{p\ \ 1}(z)}{z}.$$

Example 2.2. The following identities are valid:[6]

$$\begin{aligned}
Hi_{2\ \ 1}^{(1)}(z) &= \int_0^1 H_{2\ \ 0}(zt)dt=\frac{H_{2\ \ 1}(z)}{z},\\
Hi_{2\ \ 1}^{(2)}(z) &= \int_0^1 t\ H_{2\ \ 1}(zt)dt=\frac{H_{2\ \ 0}(z)}{z}-\frac{1}{z}\int_0^1 H_{2\ \ 0}(zt)dt\\
&= \frac{H_{2\ \ 0}(z)}{z}-\frac{H_{2\ \ 1}(z)}{z^2}.
\end{aligned}$$

Example 2.3. Based on the relation (2.2), for $p=3$, we get the following identities:

$$\begin{aligned}
Hi_{3\ \ 1}(z) &= \int_0^1 \frac{H_{3\ \ 1}(zt)}{t}dt,\\
Hi_{3\ \ 1}^{(1)}(z) &= \int_0^1 H_{3\ \ 0}(zt)dt,\\
Hi_{3\ \ 1}^{(2)}(z) &= \int_0^1 t\ H_{3\ \ 2}(zt)dt,\\
Hi_{3\ \ 1}^{(3)}(z) &= \int_0^1 t^2\ H_{3\ \ 1}(zt)dt.
\end{aligned}$$

Example 2.4. Derivative of $Hi_{3\ \ 1}$ for $j=1,2,3$ are given by:

$$\begin{aligned}
Hi_{3\ \ 1}^{(1)}(z) &= \int_0^1 H_{3\ \ 0}(zt)dt=\frac{H_{3\ \ 1}(z)}{z},\\
Hi_{3\ \ 1}^{(2)}(z) &= \int_0^1 t\ H_{3\ \ 2}(zt)dt=\frac{H_{3\ \ 0}(z)}{z}-\frac{1}{z}\int_0^1 H_{3\ \ 0}(zt)dt\\
&= \frac{H_{3\ \ 0}(z)}{z}-\frac{H_{3\ \ 1}(z)}{z^2},\\
Hi_{3\ \ 1}^{(3)}(z) &= \int_0^1 t^2\ H_{3\ \ 1}(zt)dt\\
&= \frac{H_{3\ \ 2}(z)}{z}-\frac{2\ H_{3\ \ 0}(z)}{z^2}+\frac{2\ H_{3\ \ 1}(z)}{z^3}.
\end{aligned}$$

Example 2.5. Derivative of $Hi_{3\ \ 2}$ for $j=1,2,3$ are given by:

$$\begin{aligned}
Hi_{3\ \ 2}(z) &= \int_0^1 \frac{H_{3\ \ 2}(zt)}{t^2}dt,\\
Hi_{3\ \ 2}^{(1)}(z) &= \int_0^1 \frac{H_{3\ \ 1}(zt)}{t}dt,\\
Hi_{3\ \ 2}^{(2)}(z) &= \int_0^1 H_{3\ \ 0}(zt)dt,\\
Hi_{3\ \ 2}^{(3)}(z) &= \int_0^1 t\ H_{3\ \ 2}(zt)dt.
\end{aligned}$$

Example 2.6. The following identities are valid:

$$\begin{aligned}
Hi_{3\ \ 2}^{(2)}(z) &= \int_0^1 H_{3\ \ 0}(zt)dt = \frac{H_{3\ \ 1}(z)}{z},\\
Hi_{3\ \ 2}^{(3)}(z) &= \int_0^1 t\ H_{3\ \ 2}(zt)dt = \frac{H_{3\ \ 0}(z)}{z} - \frac{1}{z}\int_0^1 H_{3\ \ 0}(zt)dt\\
&= \frac{H_{3\ \ 0}(z)}{z} - \frac{H_{3\ \ 1}(z)}{z^2},\\
Hi_{3\ \ 2}^{(4)}(z) &= \int_0^1 t^2\ H_{3\ \ 1}(zt)dt\\
&= \frac{H_{3\ \ 2}(z)}{z} - \frac{2\ H_{3\ \ 0}(z)}{z^2} + \frac{2\ H_{3\ \ 1}(z)}{z^3}.
\end{aligned}$$

Remark 2.2. According to the definition of $Hi_{3\ \ 2}(z)$, for $z > 0$, we have

$$Hi_{3\ \ 2}^{(4)}(z) > 0.$$

Thus, by Example 2.6, we have the following inequality,

$$z > 0 \qquad \Longrightarrow \qquad \frac{H_{3\ \ 2}(z)}{z} - \frac{2\ H_{3\ \ 0}(z)}{z^2} + \frac{2\ H_{3\ \ 1}(z)}{z^3} > 0.$$

Theorem 2.1. *The function $Hi_{3\ \ 1}(z)$ is supperadditive on $(0,\infty)$. That is, the inequality*

$$Hi_{3\ \ 1}(u+v) > Hi_{3\ \ 1}(u) + Hi_{3\ \ 1}(v),$$

holds for $u > 0$ and $v > 0$.

Proof. Let $L(z) = \frac{Hi_{3\ \ 1}(z)}{z}$ for $z > 0$. Then, by Definition 2.2, we have,

$$\begin{aligned}
z^2 L^{(1)}(z) &= z\ Hi_{3\ \ 1}^{(1)}(z) - Hi_{3\ \ 1}(z)\\
&= H_{3\ \ 1}(z) - \sum_{n=0}^{\infty} \frac{z^{pn+1}}{(pn+1)(pn+1)!}\\
&= \sum_{n=0}^{\infty} \frac{z^{pn+1}}{(pn+1)!} - \sum_{n=0}^{\infty} \frac{z^{pn+1}}{(pn+1)(pn+1)!} > 0.
\end{aligned}$$

Hence, $L(z)$ is increasing. Now, by applying Lemma 1.3, we get the result. □

Theorem 2.2. *The function $Hi_{3\ \ 2}(z)$ is supperadditive on $(0,\infty)$. That is, the inequality*

$$Hi_{3\ \ 2}(u+v) > Hi_{3\ \ 2}(u) + Hi_{3\ \ 2}(v),$$

holds for $u > 0$ and $v > 0$.

Proof. Let $L(z) = \frac{Hi_{3\ 2}(z)}{z}$ for $z > 0$. Then by Definition 2.2, we have,

$$\begin{aligned}
z^2 L^{(1)}(z) &= z\ Hi_{3\ 2}^{(1)}(z) - Hi_{3\ 2}(z) \\
&= z\int_0^1 \frac{H_{3\ 1}(zt)}{t}dt - \sum_{n=0}^{\infty} \frac{z^{pn+1}}{(pn+1)(pn+2)!} \\
&= z\sum_{n=0}^{\infty} \frac{z^{pn+1}}{(pn+1)(pn+1)!} - \sum_{n=0}^{\infty} \frac{z^{pn+2}}{(pn+1)(pn+1)!} \\
&= \sum_{n=0}^{\infty} \frac{z^{pn+2}}{(pn+1)(pn+1)!} - \sum_{n=0}^{\infty} \frac{z^{pn+2}}{(pn+1)(pn+2)!} > 0.
\end{aligned}$$

Hence, $L(z)$ is increasing. Now, by using Lemma 1.3, we get the result. □

Theorem 2.3. *The inequality*

$$Hi_{3\ 1}(u) + Hi_{3\ 1}(v) > u + v,$$

holds for $u, v > 0$ *and the inequality*

$$\frac{Hi_{3\ 1}(u)}{Hi_{3\ 1}(v)} < \frac{u}{v},$$

holds for $0 < u \leq v$.

Proof. The first term of the series $S = \sum_{n=0}^{\infty} \frac{z^{3n+1}}{(3n+1)(3n+1)!}$ (when $n = 0$) is

$$\frac{z^1}{1 \cdot 1!} = z.$$

Therefore, we can express the series S as follows:

$$S = z + \sum_{n=1}^{\infty} \frac{z^{3n+1}}{(3n+1)(3n+1)!}.$$

Note that, for all $n \geq 1$ and $z > 0$, we have

$$z^{3n+1} > 0, \quad (3n+1) > 0, \quad (3n+1)! > 0.$$

Therefore, we get

$$\frac{z^{3n+1}}{(3n+1)(3n+1)!} > 0,$$

implying that

$$\sum_{n=1}^{\infty} \frac{z^{3n+1}}{(3n+1)(3n+1)!} > 0.$$

Hence, we deduce that

$$S = z + \underbrace{\sum_{n=1}^{\infty} \frac{z^{3n+1}}{(3n+1)(3n+1)!}}_{>0} > z + 0 = z.$$

Hence, by Definition 2.2, we have

$$Hi_{3\ 1}(z) = \int_0^1 \frac{H_{3\ 1}(zt)}{t} dt = \sum_{n=0}^{\infty} \frac{z^{3n+1}}{(3n+1)(3n+1)!} > z.$$

So, we get

$$Hi_{3\ 1}(u) + Hi_{3\ 1}(v) > u + v.$$

The monotonicity property of the function $\frac{Hi_{3\ 1}(z)}{z}$, implies that

$$\frac{Hi_{3\ 1}(u)}{u} < \frac{Hi_{3\ 1}(v)}{v}.$$

So, we obtain

$$\frac{Hi_{3\ 1}(u)}{Hi_{3\ 1}(v)} < \frac{u}{v}.$$

□

Theorem 2.4. *The inequality*

$$Hi_{3\ 2}(u) + Hi_{3\ 2}(v) > u^2 + v^2,$$

holds for $u, v > 0$ *and the inequality*

$$\frac{Hi_{3\ 2}(u)}{Hi_{3\ 2}(v)} < \frac{u}{v},$$

holds for $0 < u \leq v$.

Proof. The first term of the series $S = \sum_{n=0}^{\infty} \frac{z^{3n+2}}{(3n+1)(3n+1)!}$ (when $n = 0$) is

$$\frac{z^2}{1 \cdot 1!} = z^2.$$

Therefore, we can express the series S as follows:

$$S = z^2 + \sum_{n=1}^{\infty} \frac{z^{3n+2}}{(3n+1)(3n+1)!}.$$

Note that, for all $n \geq 1$ and $z > 0$, we have

$$z^{3n+2} > 0, \quad (3n+1) > 0, \quad (3n+1)! > 0.$$

Therefore, we get

$$\frac{z^{3n+2}}{(3n+1)(3n+1)!} > 0,$$

implying that

$$\sum_{n=1}^{\infty} \frac{z^{3n+2}}{(3n+1)(3n+1)!} > 0.$$

Hence, we deduce that

$$S = z^2 + \underbrace{\sum_{n=1}^{\infty} \frac{z^{3n+2}}{(3n+1)(3n+1)!}}_{>0} > z^2 + 0 = z^2.$$

By Definition 2.2, we have

$$Hi_{3\ \ 2}(z) = \int_0^1 \frac{H_{3\ \ 2}(zt)}{t^2} dt = \sum_{n=0}^{\infty} \frac{z^{3n+2}}{(3n+1)(3n+1)!} > z^2.$$

Thus, we get

$$Hi_{3\ \ 2}(u) + Hi_{3\ \ 2}(v) > u^2 + v^2.$$

The monotonicity property of the function $\frac{Hi_{3\ \ 2}(z)}{z}$ implies that

$$\frac{Hi_{3\ \ 2}(u)}{u} < \frac{Hi_{3\ \ 2}(v)}{v}.$$

Hence, we obtain

$$\frac{Hi_{3\ \ 2}(u)}{Hi_{3\ \ 2}(v)} < \frac{u}{v}.$$

□

Theorem 2.5. *For* $1 \neq z > 0$*, the inequality*

$$Hi_{3\ \ 1}(z) + Hi_{3\ \ 1}(1/z) > 2\int_0^1 \frac{H_{3\ \ 1}(t)}{t} dt \cong 2.02089,$$

holds.

Proof. Let $P(z) = Hi_{3\ \ 1}(z) + Hi_{3\ \ 1}(1/z)$. So, by Example 2.4, we get

$$P'(z) = Hi'_{3\ \ 1}(z) - \frac{1}{z^2} Hi'_{3\ \ 1}(1/z).$$

Thus,

$$\begin{aligned} zP'(z) &= H_{3\ \ 1}(z) - H_{3\ \ 1}(1/z) \\ &\Longrightarrow \\ z &\in (0,1) \Longrightarrow zP'(z) < 0 \\ z &\in (1,\infty) \Longrightarrow zP'(z) > 0. \end{aligned}$$

Consequently, $P(z)$ is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

Therefore, the following inequality is valid:

$$Hi_{3\ \ 1}(z) + Hi_{3\ \ 1}(1/z) = P(z) > P(1) = 2\int_0^1 \frac{H_{3\ \ 1}(t)}{t} dt \cong 2.02089.$$

□

Theorem 2.6. *For* $z > 0$*, the following inequality is valid:*

$$\frac{5z}{6} + \frac{H_{3\ \ 0}(z) - 1}{z^2} < Hi_{3\ \ 1}(z) < 3\left(\frac{H_{3\ \ 0}(z) - 1}{z^2}\right).$$

Proof.

$$\begin{aligned}
& Hi_{3\ 1}(z) - 3(\frac{H_{3\ 0}(z)-1}{z^2}) \\
= & \sum_{n=0}^{\infty} \frac{z^{3n+1}}{(3n+1)(3n+1)!} - 3\sum_{n=1}^{\infty} \frac{z^{3n-2}}{(3n)!} \\
= & \sum_{n=0}^{\infty} \frac{z^{3n+1}}{(3n+1)(3n+1)!} - 3\sum_{n=0}^{\infty} \frac{z^{3n+1}}{(3(n+1))!} \\
= & \sum_{n=0}^{\infty} [\frac{1}{(3n+1)(3n+1)!} - \frac{3}{(3(n+1))!}] z^{3n+1} \\
= & \sum_{n=0}^{\infty} [\frac{1}{(3n+1)} - \frac{1}{(3n+2)(n+1))}] \frac{z^{3n+1}}{(3n+1)!} < 0.
\end{aligned}$$

Also, we have

$$\begin{aligned}
& \frac{5z}{6} + \frac{H_{3\ 0}(z)-1}{z^2} - Hi_{3\ 1}(z) \\
= & \frac{5z}{6} + \sum_{n=1}^{\infty} \frac{z^{3n-2}}{(3n)!} - \sum_{n=0}^{\infty} \frac{z^{3n+1}}{(3n+1)(3n+1)!} \\
= & \frac{5z}{6} + \frac{z}{6} - z + \sum_{n=2}^{\infty} \frac{z^{3n-2}}{(3n)!} - \sum_{n=1}^{\infty} \frac{z^{3n+1}}{(3n+1)(3n+1)!} \\
= & \sum_{n=2}^{\infty} \frac{z^{3n-2}}{(3n)!} - \sum_{n=2}^{\infty} \frac{z^{3n-2}}{(3n-2)(3n-2)!} \\
= & \sum_{n=2}^{\infty} [\frac{1}{(3n-1)(3n)} - \frac{1}{(3n-2)}] \frac{z^{3n+1}}{(3n-2)!} < 0.
\end{aligned}$$

Thus, the proof is completed. $\square$

Example 2.7. The following identities about the derivations of Hi are valid:

$$\begin{aligned}
Hi_{4\ i}(z) &= \int_0^1 \frac{H_{4\ i}(zt)}{t^i} dt, i = 1, 2, 3, \\
Hi_{4\ 1}^{(1)}(z) &= \int_0^1 H_{4\ 0}(zt) dt, \\
Hi_{4\ 1}^{(2)}(z) &= \int_0^1 t\ H_{4\ 2}(zt) dt, \\
Hi_{4\ 1}^{(3)}(z) &= \int_0^1 t^2 H_{4\ 1}(zt) dt.
\end{aligned}$$

Also, we have

$$
\begin{aligned}
Hi^{(1)}_{3\ 1}(z) &= \int_0^1 H_{3\ 0}(zt)dt = \frac{H_{3\ 1}(z)}{z},\\
Hi^{(2)}_{3\ 1}(z) &= \int_0^1 t\ H_{3\ 2}(zt)dt = \frac{H_{3\ 0}(z)}{z} - \frac{1}{z}\int_0^1 H_{3\ 0}(zt)dt\\
&= \frac{H_{3\ 0}(z)}{z} - \frac{H_{3\ 1}(z)}{z^2},\\
Hi^{(3)}_{3\ 1}(z) &= \int_0^1 t^2 H_{3\ 1}(zt)dt\\
&= \frac{H_{3\ 2}(z)}{z} - \frac{2H_{3\ 0}(z)}{z^2} - \frac{H_{3\ 1}(z)}{z^3}.
\end{aligned}
$$

3. Counterexample

In this section, we plan two examples to show that the following theorem of paper in [6] is not always true.

Theorem 3.1. [6]*Let* $z > 0$ *and* $\lambda \in (0, 1)$. *Then the inequality*

$$Shi(\lambda z) > \lambda\ Shi(z), \tag{3.1}$$

holds. If $\lambda > 1$, *then*

$$Shi(\lambda z) < \lambda Shi(z), \tag{3.2}$$

where, $Shi(z) = \int_0^z \frac{\sinh(t)}{t}$.

Example 3.1. If we choose $z = 1$ and $\lambda = 5$, then, we have

$$Shi(5) - 5\ Shi(1) = \int_0^5 \frac{\sinh(t)}{t}dt - 5\int_0^1 \frac{\sinh(t)}{t} \cong 14.807.$$

That is,

$$Shi(5) > 5\ Shi(1).$$

This inequality contradicts the inequality in (3.2).

Example 3.2. If we choose $z = 2000$ and $\lambda = 0.001$, then we have,

$$Shi(2) - 0.001\ Shi(2000) = \int_0^2 \frac{\sinh(t)}{t}dt - 0.001\int_0^{2000} \frac{\sinh(t)}{t} \cong -9.707806 \times 10^{861}.$$

That is,

$$Shi((0.001) \times 2000) < 0.001\ Shi(2000).$$

This inequality is a contradiction with the inequality in (3.1).

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