Turkish J. Ineq., 9 (2) (2025), Pages 47 –69 .

Turkish Journal of
INEQUALITIES

Available online at www.tjinequality.com

APPLICATIONS OF INEQUALITIES INVOLVING k-FRACTIONAL CONFORMABLE INTEGRALS VIA JENSEN-MERCER INEQUALITIES

MURTI YASHOBANT KUMAR[1], MADAN MOHAN SOREN[1], AND HÜSEYIN BUDAK[2]

Abstract. This paper is devoted to the development of new inequalities of the Hermite-Hadamard, trapezoidal, and midpoint types through the use of generalized k-fractional integrals. The methodology relies on established mathematical tools, including the Jensen-Mercer inequality, the power mean inequality, and Hölder inequality. Fundamental identities involving generalized k-fractional integrals and convex functions form the basis of the main results. The paper also explores connections between these results and earlier research on classical Riemann-Liouville fractional integrals, fractional conformable integrals, and generalized k-fractional integrals. Several examples, supported by graphical representations, are presented to illustrate and confirm the validity and applicability of the derived inequalities.

1. Introduction and Definitions

Fractional calculus, which extends the classical definitions of differentiation and integration to arbitrary (non-integer) orders, has witnessed significant growth in recent years due to its broad applicability in modeling real-world phenomena with memory and hereditary properties. Unlike traditional integer-order derivatives, fractional-order operators are inherently nonlocal, as they incorporate the entire history of a function's behavior. This distinctive feature makes them particularly suitable for modeling complex dynamical systems in diverse areas such as physics [16], control theory, viscoelasticity, signal processing, fluid mechanics and biological systems [2,3,27].

Within the growing field, fractional integral inequalities play a central role in the theoretical study of solutions to fractional differential equations. Among them, the Hermite-Hadamard inequality, which provides bounds for the integral mean of convex functions, has received considerable attention. It is a fundamental tool in both mathematical analysis and

Key words and phrases. Fractional integrals, Fractional inequalities, Jensen-Mercer inequality, Riemann-Liouville fractional integrals.

2010 *Mathematics Subject Classification.* Primary: 26A33. Secondary: 26D07, 26D10, 26D15.

Received: 14/06/2025 *Accepted*: 10/11/2025.

Cite this article as: M.Y. Kumar, M.M. Soren, H. Budak, Applications of inequalities involving k-fractional conformable integrals via Jensen-Mercer inequalities, Turkish Journal of Inequalities, 9(2) (2025), 47-69.

numerical integration. For a convex function $f : [a, b] \to \mathbf{R}$ the classical Hermite-Hadamard inequality is expressed as:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}.$$

This inequality has inspired numerous extensions and generalizations, particularly in the setting of fractional integrals and generalized convexity. A particularly powerful framework for such generalizations is provided by generalized k-fractional integral operators. These operators not only extend the well-known Riemann-Liouville, Hadamard, and Katugampola fractional integrals, but also offer a more flexible structure for modelling and analysing integral inequalities (see [7, 8, 12, 17–19, 22, 29–31, 34, 36, 38]). The additional parameters and kernel functions embedded in the k-generalized operators yield a richer class of integral expressions, thereby enabling more refined bounds and deeper analytical results.

Recent research has investigated the interplay between convexity and fractional integrals. For instance, Sarikaya et al. [33] applied Riemann-Liouville fractional integrals to derive Hermite-Hadamard-type inequalities for convex functions. Building on this foundation, Set et al. [37] and others (see, e.g., [5,6,9,11,23,24,32]) introduced further refinements and new inequalities involving generalized convexity and diverse classes of fractional integrals.

The present paper advances this line of inquiry by establishing new Hermite-Hadamard, trapezoidal, and midpoint-type inequalities using generalized k-fractional integrals in conjunction with classical convex functions. The convexity assumption is essential here, as it provides the structural foundation for formulating and proving these inequalities. Our approach is supported by several classical analytical tools, including:

- the Jensen-Mercer inequality, which governs averages and convex combinations of functions,
- Hölder's inequality, which is fundamental for estimating integral bounds under weighted norms, and
- the power-mean inequality, which relates different mean values in the context of convex functions.

Through this combination of fractional calculus, convex analysis, and integral inequality techniques, we derive several novel results. These not only generalize known inequalities, but also unify them within a broader operator framework.

Furthermore, we provide illustrative examples and graphical visualizations to validate the theoretical results. These examples demonstrate the behavior of the derived inequalities under various convex functions and parameter choices, thereby highlighting their applicability in practical mathematical modeling.

The contributions of this paper are twofold.

Theoretical: We establish new Hermite-Hadamard, trapezoidal, and midpoint-type inequalities via the generalized k-fractional integrals, which encapsulating several existing results as special cases.

Practical: We present explicit examples and graphical plots to showcase the effectiveness and applicability of the proposed results.

Definition 1.1. A function $g : [a, b] \subset \mathbf{R} \to \mathbf{R}$, is said to be convex if

$$g(tx + (1-t)y) \leq tg(x) + (1-t)g(y)$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$. We say that f is concave if $(-g)$ is convex.

Definition 1.2. ([26],[39]) For $0 < y, z < \infty$ and $y, z \in \mathbf{R}$, the Gamma function, Beta function, and incomplete Beta function are given by

$$\Gamma(y) = \int_0^\infty t^{y-1} e^{-t} dt, \;\; \mathcal{B}(y, z) := \int_0^1 t^{y-1}(1-t)^{z-1} dt, \;\; \mathcal{B}(y, z, b) := \int_0^b t^{y-1}(1-t)^{z-1} dt.$$

The k-Gamma function and k-Beta function are defined as

$$\Gamma_k(y) = \int_0^\infty t^{y-1} e^{-t^k/k} dt, \;\; \mathcal{B}_k(y, z) := \frac{1}{k} \int_0^1 t^{y/k-1}(1-t)^{z/k-1} dt.$$

They satisfy the relations

$$\Gamma(y) = \lim_{k \to 1} \Gamma_k(y), \;\; \Gamma_k(y+k) = y\Gamma_k(y), \;\; \mathcal{B}_k(y, z) = \frac{1}{k}\mathcal{B}\left(\frac{y}{k}, \frac{z}{k}\right) = \frac{\Gamma_k(y)\Gamma_k(z)}{\Gamma_k(y+z)}.$$

Definition 1.3. [26] Let $f \in \mathcal{L}_1[a, b]$ with $a < b$, where $a, b \in \mathbf{R}$. The Riemann-Liouville fractional integrals $\Im^\beta_{a+} f$ and $\Im^\beta_{b-} f$ of order $\beta > 0$ are defined by

$$\Im^\beta_{a+} f(y) = \frac{1}{\Gamma(\beta)} \int_a^y (y-t)^{\beta-1} f(t) dt, \quad y > a,$$

and

$$\Im^\beta_{b-} f(y) = \frac{1}{\Gamma(\beta)} \int_y^b (t-y)^{\beta-1} f(t) dt, \quad y < b,$$

respectively, where $\Gamma(\beta)$ is the Gamma function. These are referred to as the left-sided and right-sided fractional integrals.

Conformable derivatives, often called local fractional derivatives, allow differentiation of arbitrary order, unlike traditional nonlocal fractional derivatives. These operators, along with modified conformable derivatives [4], are important for generating generalized non-local fractional derivatives with singular kernels [1, 20, 21]. In particular, the generalized k-fractional conformable integral [21] extends classical fractional operators and enables broader analytical developments. Feng et al. [14] subsequently advanced this theory, characterizing generalized k-fractional conformable integrals, evaluating their effectiveness relative to established fractional tools.

Definition 1.4. [21] Let $\beta > 0$, $\alpha \in (0, 1]$ and $f \in \mathcal{L}_1[a, b]$. The fractional conformable integrals ${}^\beta\Im^\alpha_{a+} f$ and ${}^\beta\Im^\alpha_{b-} f$ are defined by

$${}^\beta\Im^\alpha_{a+} f(y) = \frac{1}{\Gamma(\beta)} \int_a^y \left(\frac{(y-a)^\alpha - (t-a)^\alpha}{\alpha}\right)^{\beta-1} \frac{f(t)}{(t-a)^{1-\alpha}} dt, \quad y > a, \tag{1.1}$$

and

$${}^\beta\Im^\alpha_{b-} f(y) = \frac{1}{\Gamma(\beta)} \int_y^b \left(\frac{(b-y)^\alpha - (b-t)^\alpha}{\alpha}\right)^{\beta-1} \frac{f(t)}{(b-t)^{1-\alpha}} dt, \quad y < b. \tag{1.2}$$

Definition 1.5. [14] Let $\beta > 0$, $\alpha \in (0,1]$ and $f \in \mathcal{L}_1[a,b]$. The generalized left and right k-fractional conformable integrals (k-FCI) of order β are respectively defined as

$$ {}^{\beta}_{k}\Im^{\alpha}_{a+} f(y) = \frac{1}{k\Gamma_k(\beta)} \int_a^y \left(\frac{(y-a)^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta/k-1} \frac{f(t)}{(t-a)^{1-\alpha}} dt, \quad y > a, \tag{1.3} $$

and

$$ {}^{\beta}_{k}\Im^{\alpha}_{b-} f(y) = \frac{1}{k\Gamma_k(\beta)} \int_y^b \left(\frac{(b-y)^\alpha - (b-t)^\alpha}{\alpha} \right)^{\beta/k-1} \frac{f(t)}{(b-t)^{1-\alpha}} dt, \quad y < b, \tag{1.4} $$

where $k > 0$.

If we assign $k = 1$, then the generalized k-fractional conformable integrals in (1.3) and (1.4) reduce to the fractional conformable integrals in (1.1) and (1.2), respectively.

Theorem 1.1. *If the function f is convex over $[a,b]$, then we have the following inequality*

$$ f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(t)dt \le \frac{f(a)+f(b)}{2}. $$

This inequality is known as the Hermite-Hadamard inequality in the literature.

Theorem 1.2. [13] *If $f : [a,b] \to \mathbf{R}$ is a convex function, then*

$$ f\left(a + b - \sum_{i=1}^n r_i x_i\right) \le f(a) + f(b) - \sum_{i=1}^n r_i f(x_i) \tag{1.5} $$

for all $x_i \in [a,b], r_i \in [0,1], i = 1, ..., n$ and $\sum_{i=1}^n r_i = 1$. This inequality is known as the Jensen–Mercer inequality in the literature.

For further results related to the Jensen-Mercer inequality, see [25, 28].

Let $\frac{1}{\lambda} + \frac{1}{\mu} = 1$ with $p, q > 1$. Then Hölder's inequality for integrals states that

$$ \int_a^b |f(t)g(t)|dt \le \left(\int_a^b |f(t)|^\lambda dt \right)^{\frac{1}{\lambda}} \left(\int_a^b |g(t)|^\mu dt \right)^{\frac{1}{\mu}}. $$

The aim of this study is to establish new inequalities for generalized k-fractional conformable integrals. In particular, Hermite-Hadamard, trapezoidal, and midpoint-type inequalities will be derived using the Jensen-Mercer inequality. These results not only offer new insights into generalized k-fractional conformable integrals, but also contribute novel mathematical tools for applications in fractional calculus.

2. Main Results

In this section, we recall some known results and subsequently utilize them to derive new and interesting findings.

Theorem 2.1. [25] *If $f : [a,b] \to \mathbf{R}$ is convex, then*

$$ f\left(\frac{a+b}{2}\right) \le \frac{\alpha^{\beta/k}\Gamma_k(\beta+k)}{2(b-a)^{\frac{\alpha\beta}{k}}} \left[{}^{\beta}_{k}\Im^{\alpha}_{a+} f(b) + {}^{\beta}_{k}\Im^{\alpha}_{b-} f(a) \right] \le \frac{f(a)+f(b)}{2} \tag{2.1} $$

for $\beta > 0, k > 0$ and $\alpha \in (0,1]$.

Lemma 2.1. [28] *Let $f : [a,b] \to \mathbf{R}$ be differentiable on (a,b) with $f' \in \mathcal{L}_1[a,b]$. Then*

$$\frac{f(a)+f(b)}{2} - \frac{\alpha^{\beta/k}\Gamma_k(\beta+k)}{2(b-a)^{\frac{\alpha\beta}{k}}}\left[{}^{\beta}_{k}\Im^{\alpha}_{a+}f(b) + {}^{\beta}_{k}\Im^{\alpha}_{b-}f(a)\right]$$
$$= \frac{(b-a)\alpha^{\beta/k}}{2}\int_0^1\left[\left(\frac{1-t^\alpha}{\alpha}\right)^{\frac{\beta}{k}} - \left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}\right] f'(ta+(1-t)b)dt \quad (2.2)$$

for $\beta > 0, k > 0$ and $\alpha \in (0,1]$.

Remark 2.1. By choosing $k = 1$ in (2.2), we obtain the equality for fractional conformable integrals proved by Set et al. in [35].

Lemma 2.2. *Let $f : [a,b] \to \mathbf{R}$ be differentiable on (a,b) with $f' \in \mathcal{L}_1[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right) - \frac{\alpha^{\beta/k}\Gamma_k(\beta+k)}{2(b-a)^{\frac{\alpha\beta}{k}}}\left[{}^{\beta}_{k}\Im^{\alpha}_{a+}f(b) + {}^{\beta}_{k}\Im^{\alpha}_{b-}f(a)\right] = \frac{(b-a)\alpha^{\beta/k}}{2}(B_1 - B_2 - B_3 + B_4), \quad (2.3)$$

where

$$B_1 = \int_0^{1/2}\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\frac{\beta}{k}} f'(tb+(1-t)a)dt,$$
$$B_2 = \int_0^{1/2}\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\frac{\beta}{k}} f'(ta+(1-t)b)dt,$$
$$B_3 = \int_{1/2}^{1}\left[\frac{1}{\alpha^{\beta/k}} - \left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}\right] f'(tb+(1-t)a)dt,$$
$$B_4 = \int_{1/2}^{1}\left[\frac{1}{\alpha^{\beta/k}} - \left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}\right] f'(ta+(1-t)b)dt.$$

Proof. Using integration by parts, we obtain

$$\begin{aligned}
B_1 &= \int_0^{1/2}\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\frac{\beta}{k}} f'(tb+(1-t)a)dt \\
&= \left(\frac{1}{\alpha}\right)^{\beta/k}\left[(1-(1-t)^\alpha)^{\beta/k}\frac{f(tb+(1-t)a)}{b-a}\Big|_0^{1/2}\right. \\
&\left. -\frac{\alpha\beta}{k}\int_0^{1/2}(1-(1-t)^\alpha)^{\frac{\beta}{k}-1}(1-t)^{\alpha-1}\frac{f(tb+(1-t)a)}{b-a}dt\right] \\
&= \left(\frac{1}{\alpha}\right)^{\beta/k}\frac{1}{(b-a)}\left[(1-(1/2)^\alpha)^{\frac{\beta}{k}} f\left(\frac{a+b}{2}\right)\right. \\
&\left. -\frac{\alpha\beta}{k}\int_0^{1/2}(1-(1-t)^\alpha)^{\frac{\beta}{k}-1}(1-t)^{\alpha-1}f(tb+(1-t)a)dt\right].
\end{aligned}$$

Similarly, we get

$$B_2 = \left(\frac{1}{\alpha}\right)^{\beta/k}\frac{1}{(a-b)}\left[(1-(1/2)^\alpha)^{\frac{\beta}{k}} f\left(\frac{a+b}{2}\right)\right.$$

$$-\frac{\alpha\beta}{k}\int_0^{1/2}(1-(1-t)^\alpha)^{\frac{\beta}{k}-1}(1-t)^{\alpha-1}f(ta+(1-t)b)dt\Bigg],$$

$$B_3=\left(\frac{1}{\alpha}\right)^{\beta/k}\frac{1}{(b-a)}\Bigg[(1-(1/2)^\alpha)^{\frac{\beta}{k}}f\left(\frac{a+b}{2}\right)-f\left(\frac{a+b}{2}\right)$$
$$+\frac{\alpha\beta}{k}\int_{1/2}^{1}(1-(1-t)^\alpha)^{\frac{\beta}{k}-1}(1-t)^{\alpha-1}f(tb+(1-t)a)dt\Bigg],$$

$$B_4=\left(\frac{1}{\alpha}\right)^{\beta/k}\frac{1}{(a-b)}\Bigg[(1-(1/2)^\alpha)^{\frac{\beta}{k}}f\left(\frac{a+b}{2}\right)-f\left(\frac{a+b}{2}\right)$$
$$+\frac{\alpha\beta}{k}\int_{1/2}^{1}(1-(1-t)^\alpha)^{\frac{\beta}{k}-1}(1-t)^{\alpha-1}f(tb+(1-t)a)dt\Bigg].$$

Now, substituting the calculated integrals, we obtain

$$B_1-B_2-B_3+B_4=\left(\frac{1}{\alpha}\right)^{\beta/k}\frac{1}{(b-a)}\Bigg[2f\left(\frac{a+b}{2}\right)$$
$$-\frac{\beta}{k}\alpha^{\frac{\beta}{k}}\int_0^1\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}(1-t)^{\alpha-1}f(tb+(1-t)a)dt$$
$$-\frac{\beta}{k}\alpha^{\frac{\beta}{k}}\int_0^1\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}(1-t)^{\alpha-1}f(ta+(1-t)b)dt\Bigg]$$
$$=\frac{2}{b-a}\left(\frac{1}{\alpha}\right)^{\beta/k}\left[f\left(\frac{a+b}{2}\right)-\frac{\alpha^{\beta/k}\Gamma_k(\beta+k)}{2(b-a)^{\frac{\alpha\beta}{k}}}\left[{}_k^\beta\Im_{a+}^\alpha f(b)+{}_k^\beta\Im_{b-}^\alpha f(a)\right]\right]. \tag{2.4}$$

Multiplying both sides of (2.4) by $\frac{(b-a)\alpha^{\beta/k}}{2}$, gives the desired equality (2.3). ☐

Remark 2.2. By choosing $k=1$ in Lemma 2.2, we obtain the equality for fractional conformable integrals proved by Hezenci and Budak in [15].

Theorem 2.2. *If $f:[a,b]\to\mathbf{R}$ is convex function, then*

$$f\left(a+b-\frac{y+z}{2}\right)\leq f(a)+f(b)-\frac{\alpha^{\beta/k}\Gamma_k(\beta+k)}{2(z-y)^{\frac{\alpha\beta}{k}}}\left[{}_k^\beta\Im_{y+}^\alpha f(z)+{}_k^\beta\Im_{z-}^\alpha f(y)\right]$$
$$\leq f(a)+f(b)-f\left(\frac{y+z}{2}\right) \tag{2.5}$$

for all $y,z\in[a,b]$ with $y<z$, for $\beta>0, k>0$ and $\alpha\in(0,1]$.

Proof. Since f is convex on $[a,b]$, the Jensen-Mercer inequality gives

$$f\left(a+b-\frac{x_1+y_1}{2}\right)\leq f(a)+f(b)-\frac{f(x_1)+f(y_1)}{2} \tag{2.6}$$

for all $x_1,y_1\in[a,b]$.

Now, set $x_1=ty+(1-t)z$ and $y_1=(1-t)y+tz$ with $t\in[0,1]$. Substituting into (2.6) yields

$$f\left(a+b-\frac{y+z}{2}\right)\leq f(a)+f(b)-\frac{f(ty+(1-t)z)+f((1-t)y+tz)}{2}. \tag{2.7}$$

Multiplying both sides of (2.7) by $(\frac{1-t^\alpha}{\alpha})^{\frac{\beta}{k}-1}t^{\alpha-1}$ and integrating over $[0,1]$, we obtain

$$f\left(a+b-\frac{y+z}{2}\right)\int_0^1\left(\frac{1-t^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}t^{\alpha-1}dt \le \{f(a)+f(b)\}\int_0^1\left(\frac{1-t^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}t^{\alpha-1}dt$$
$$-\frac{1}{2}\int_0^1\{f(ty+(1-t)z)+f((1-t)y+tz)\}\left(\frac{1-t^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}t^{\alpha-1}dt.$$

Using the evaluation technique from Theorem 2.1, this simplifies to

$$f\left(a+b-\frac{y+z}{2}\right)\left(\frac{k}{\beta}\right)\left(\frac{1}{\alpha}\right)^{\frac{\beta}{k}} \le$$
$$\{f(a)+f(b)\}\left(\frac{k}{\beta}\right)\left(\frac{1}{\alpha}\right)^{\frac{\beta}{k}}-\frac{k\Gamma_k(\beta)}{2(z-y)^{\frac{\alpha\beta}{k}}}\left[{}_k^\beta\Im_{y+}^\alpha f(z)+{}_k^\beta\Im_{z-}^\alpha f(y)\right]$$
$$\implies f\left(a+b-\frac{y+z}{2}\right)\le\{f(a)+f(b)\}-\frac{\alpha^{\beta/k}\Gamma_k(\beta+k)}{2(z-y)^{\frac{\alpha\beta}{k}}}\left[{}_k^\beta\Im_{y+}^\alpha f(z)+{}_k^\beta\Im_{z-}^\alpha f(y)\right], \quad (2.8)$$

which is the first inequality of (2.5).

To prove the second inequality, observe that convexity of f implies

$$f\left(\frac{y+z}{2}\right)=f\left(\frac{ty+(1-t)z+(1-t)y+tz}{2}\right)$$
$$\le\frac{f(ty+(1-t)z)+f((1-t)y+tz)}{2},\quad t\in[0,1]. \quad (2.9)$$

Multiplying both sides of (2.9) by $(\frac{1-t^\alpha}{\alpha})^{\frac{\beta}{k}-1}t^{\alpha-1}$ and integrating over $[0,1]$ gives

$$f\left(\frac{y+z}{2}\right)\int_0^1\left(\frac{1-t^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}t^{\alpha-1}dt$$
$$\le\frac{1}{2}\int_0^1\{f(ty+(1-t)z)+f((1-t)y+tz)\}\left(\frac{1-t^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}t^{\alpha-1}dt,$$

this reduces to

$$f\left(\frac{y+z}{2}\right)\le\frac{\alpha^{\beta/k}\Gamma_k(\beta+k)}{2(z-y)^{\frac{\alpha\beta}{k}}}\left[{}_k^\beta\Im_{y+}^\alpha f(b)+{}_k^\beta\Im_{z-}^\alpha f(a)\right]. \quad (2.10)$$

Rearranging (2.10), we get

$$-\frac{\alpha^{\beta/k}\Gamma_k(\beta+k)}{2(z-y)^{\frac{\alpha\beta}{k}}}\left[{}_k^\beta\Im_{y+}^\alpha f(b)+{}_k^\beta\Im_{z-}^\alpha f(a)\right]\le -f\left(\frac{y+z}{2}\right).$$

Adding $f(a)+f(b)$ to both sides yields

$$f(a)+f(b)-\frac{\alpha^{\beta/k}\Gamma_k(\beta+k)}{2(z-y)^{\frac{\alpha\beta}{k}}}\left[{}_k^\beta\Im_{y+}^\alpha f(z)+{}_k^\beta\Im_{z-}^\alpha f(y)\right]\le f(a)+f(b)-f\left(\frac{y+z}{2}\right), \quad (2.11)$$

which is exactly the second inequality in (2.5).

Combining (2.8) and (2.11) completes the proof. □

Remark 2.3. By choosing $k=1$ in Theorem 2.2, we recover the Hermite-Hadamard-Mercer inequality for fractional conformable integrals established by Butt et al. in [10].

Example 2.1. Consider the convex function $f : [1,6] \to \mathbf{R}$ defined by $f(x) = x^2$. For $y = 3$ and $z = 5$, equation (2.5) gives

$$f\left(a+b-\frac{y+z}{2}\right) = f(7-4) = f(3) = 9$$

and

$$f(a)+f(b)-f\left(\frac{y+z}{2}\right) = 37 - f(4) = 21.$$

These values correspond to the left-hand and right-hand sides of (2.5), respectively.

From (1.3) and (1.4), we compute

$$\begin{aligned}
{}^{\beta}_{k}\Im^{\alpha}_{y+}f(z) = {}^{\beta}_{k}\Im^{\alpha}_{3+}f(5) &= \frac{1}{k\Gamma_k(\beta)}\int_3^5\left(\frac{(2)^\alpha-(t-3)^\alpha}{\alpha}\right)^{\beta/k-1}\frac{t^2}{(t-3)^{1-\alpha}}dt\\
&= \left(\frac{2^\alpha}{\alpha}\right)^{\beta/k-1}\frac{1}{k\Gamma_k(\beta)}\int_3^5\left(1-(\frac{t-3}{2})^\alpha\right)^{\beta/k-1}(t-3)^{\alpha-1}[(t-3)+3]^2dt\\
&= \left(\frac{2^\alpha}{\alpha}\right)^{\beta/k-1}\frac{1}{k\Gamma_k(\beta)}\int_3^5\left(1-(\frac{t-3}{2})^\alpha\right)^{\beta/k-1}[(t-3)^{\alpha+1}+6(t-3)^\alpha+9(t-3)^{\alpha-1}]dt\\
&= \left(\frac{2^\alpha}{\alpha}\right)^{\beta/k}\frac{1}{k\Gamma_k(\beta)}\int_0^1 t^{\beta/k-1}[4(1-t)^{\frac{2}{\alpha}}+12(1-t)^{\frac{1}{\alpha}}+9]dt\\
&= \left(\frac{2^\alpha}{\alpha}\right)^{\beta/k}\frac{1}{k\Gamma_k(\beta)}\left[4\mathfrak{B}\left(\frac{\beta}{k},1+\frac{2}{\alpha}\right)+12\mathfrak{B}\left(\frac{\beta}{k},1+\frac{1}{\alpha}\right)+9\frac{k}{\beta}\right].
\end{aligned}$$

Similarly,

$${}^{\beta}_{k}\Im^{\alpha}_{z-}f(y) = {}^{\beta}_{k}\Im^{\alpha}_{5-}f(3) = \left(\frac{2^\alpha}{\alpha}\right)^{\beta/k}\frac{1}{k\Gamma_k(\beta)}\left[4\mathfrak{B}\left(\frac{\beta}{k},1+\frac{2}{\alpha}\right)-20\mathfrak{B}\left(\frac{\beta}{k},1+\frac{1}{\alpha}\right)+25\frac{k}{\beta}\right].$$

Thus, the middle term of (2.5) becomes

$$\begin{aligned}
f(a)+f(b)-\frac{\alpha^{\beta/k}\Gamma_k(\beta+k)}{2(z-y)^{\frac{\alpha\beta}{k}}}\left[{}^{\beta}_{k}\Im^{\alpha}_{y+}f(z)+{}^{\beta}_{k}\Im^{\alpha}_{z-}f(y)\right]&\\
= 37-\frac{\beta}{k}\left[4\mathfrak{B}\left(\frac{\beta}{k},1+\frac{2}{\alpha}\right)-4\mathfrak{B}\left(\frac{\beta}{k},1+\frac{1}{\alpha}\right)+17\frac{k}{\beta}\right].&
\end{aligned}$$

Hence, inequality (2.5) reduces to

$$9 \le 37-\frac{\beta}{k}\left[4\mathfrak{B}\left(\frac{\beta}{k},1+\frac{2}{\alpha}\right)-4\mathfrak{B}\left(\frac{\beta}{k},1+\frac{1}{\alpha}\right)+17\frac{k}{\beta}\right] \le 21. \tag{2.12}$$

The validity of inequality (2.12) is illustrated in Figure 1.

3. Trapezoidal-Type Inequalities with Jensen–Mercer Inequality

In this section, we develop new inequalities for generalized k-fractional conformable integrals by establishing a key identity based on Lemma 2.1. This identity is then combined with Hölder's and power-mean inequalities, and the Jensen–Mercer inequality is applied to derive trapezoidal-type inequalities.

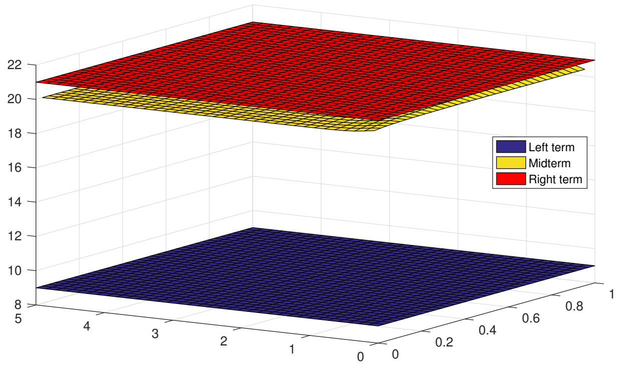

FIGURE 1. The graph of the inequalities of (2.12) is evaluated by MATLAB software for $\alpha \in (0,1]$, $\beta \in (0,5]$ and $k = 0.5$.

Lemma 3.1. *Let $f : [a,b] \to \mathbf{R}$ be differentiable on (a,b) with $f' \in \mathcal{L}_1[a,b]$. Then*

$$\begin{aligned}&\frac{f(a+b-y)+f(a+b-z)}{2}\\ &\quad-\frac{\alpha^{\beta/k}\Gamma_k(\beta+k)}{2(z-y)^{\frac{\alpha\beta}{k}}}\left[{}^{\beta}_{k}\Im^{\alpha}_{(a+b-z)+}f(a+b-y)+{}^{\beta}_{k}\Im^{\alpha}_{(a+b-y)-}f(a+b-z)\right]\\ &=\frac{(z-y)\alpha^{\beta/k}}{2}\int_0^1\left[\left(\frac{1-t^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}-\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\right]f'(a+b-(tz+(1-t)y))dt \quad (3.1)\end{aligned}$$

for $y,z \in [a,b]$ with $y < z$, $\beta > 0, k > 0$ and $\alpha \in (0,1]$.

Proof. Replacing a with $a+b-z$ and b with $a+b-y$ in Lemma 2.1 immediately yields the desired result. □

Lemma 3.2. *For $\beta > 0, k > 0$ and $\alpha \in (0,1]$, the following equalities hold:*

$$\tau_1=\int_0^{\frac{1}{2}}(1-t^{\alpha})^{\beta/k}dt=\int_{\frac{1}{2}}^1(1-(1-t)^{\alpha})^{\beta/k}dt=\frac{1}{\alpha}\mathcal{B}\left(\frac{1}{\alpha},\frac{\beta}{k}+1;(\frac{1}{2})^{\alpha}\right),$$

$$\begin{aligned}\tau_2&=\int_0^{\frac{1}{2}}(1-(1-t)^{\alpha})^{\beta/k}dt\\ &=\int_{\frac{1}{2}}^1(1-t^{\alpha})^{\beta/k}dt=\frac{1}{\alpha}\left[\mathcal{B}\left(\frac{1}{\alpha},\frac{\beta}{k}+1\right)-\mathcal{B}\left(\frac{1}{\alpha},\frac{\beta}{k}+1;\left(\frac{1}{2}\right)^{\alpha}\right)\right],\end{aligned}$$

$$\tau_3=\int_0^{\frac{1}{2}}t(1-t^{\alpha})^{\beta/k}dt=\int_{\frac{1}{2}}^1(1-t)(1-(1-t)^{\alpha})^{\beta/k}dt=\frac{1}{\alpha}\mathcal{B}\left(\frac{2}{\alpha},\frac{\beta}{k}+1;\left(\frac{1}{2}\right)^{\alpha}\right),$$

$$\begin{aligned}\tau_4&=\int_0^{\frac{1}{2}}(1-t)(1-(1-t)^{\alpha})^{\beta/k}dt\\ &=\int_{\frac{1}{2}}^1 t(1-t^{\alpha})^{\beta/k}dt=\frac{1}{\alpha}\left[\mathcal{B}\left(\frac{2}{\alpha},\frac{\beta}{k}+1\right)-\mathcal{B}\left(\frac{2}{\alpha},\frac{\beta}{k}+1;\left(\frac{1}{2}\right)^{\alpha}\right)\right],\end{aligned}$$

$$\tau_5 = \int_{\frac{1}{2}}^{1} t(1-(1-t)^{\alpha})^{\beta/k}dt = \int_0^{\frac{1}{2}} (1-t)(1-t^{\alpha})^{\beta/k}dt = \tau_1 - \tau_3,$$
$$\tau_6 = \int_{\frac{1}{2}}^{1} (1-t)(1-t^{\alpha})^{\beta/k}dt = \int_0^{\frac{1}{2}} t(1-(1-t)^{\alpha})^{\beta/k}dt = \tau_2 - \tau_4.$$

Proof. By substituting $t^{\alpha} = Z$, we obtain

$$\tau_1 = \frac{1}{\alpha}\int_0^{(\frac{1}{2})^{\alpha}} Z^{\frac{1}{\alpha}-1}(1-Z)^{\frac{\beta}{k}}dZ = \frac{1}{\alpha}\mathcal{B}\left(\frac{1}{\alpha}, \frac{\beta}{k}+1; \left(\frac{1}{2}\right)^{\alpha}\right).$$

The remaining equalities can be derived in a similar manner. □

Theorem 3.1. *Let $f : [a,b] \to \mathbf{R}$ be differentiable on (a,b) and suppose that $|f'|$ is convex on $[a,b]$. Then, for $y, z \in [a,b]$ with $y < z$, $\beta > 0$, $k > 0$, and $\alpha \in (0,1]$, we have*

$$\Bigg|\frac{f(a+b-y)+f(a+b-z)}{2} - \frac{\alpha^{\beta/k}\Gamma_k(\beta+k)}{2(z-y)^{\frac{\alpha\beta}{k}}}\left[{}^{\beta}_{k}\Im^{\alpha}_{(a+b-z)+}f(a+b-y) + {}^{\beta}_{k}\Im^{\alpha}_{(a+b-y)-}f(a+b-z)\right]\Bigg| \le \frac{(z-y)}{\alpha}\left[|f'(a)|+|f'(b)| - \frac{|f'(y)|+|f'(z)|}{2}\right]\left[2\mathcal{B}\left(\frac{1}{\alpha}, \frac{\beta}{k}+1; \left(\frac{1}{2}\right)^{\alpha}\right) - \mathcal{B}\left(\frac{1}{\alpha}, \frac{\beta}{k}+1\right)\right]. \tag{3.2}$$

Proof. From Lemma 3.1, we can write

$$\Bigg|\frac{f(a+b-y)+f(a+b-z)}{2} - \frac{\alpha^{\beta/k}\Gamma_k(\beta+k)}{2(z-y)^{\frac{\alpha\beta}{k}}}\left[{}^{\beta}_{k}\Im^{\alpha}_{(a+b-z)+}f(a+b-y) + {}^{\beta}_{k}\Im^{\alpha}_{(a+b-y)-}f(a+b-z)\right]\Bigg|$$
$$\le \frac{(z-y)\alpha^{\beta/k}}{2}\int_0^1 \left|\left(\frac{1-t^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}} - \left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\right| |f'(a+b-(tz+(1-t)y))|dt.$$

Since $|f'|$ is convex on $[a,b]$, applying the Jensen–Mercer inequality together with Lemma 3.2, we obtain

$$\Bigg|\frac{f(a+b-y)+f(a+b-z)}{2} - \frac{\alpha^{\beta/k}\Gamma_k(\beta+k)}{2(z-y)^{\frac{\alpha\beta}{k}}}\left[{}^{\beta}_{k}\Im^{\alpha}_{(a+b-z)+}f(a+b-y) + {}^{\beta}_{k}\Im^{\alpha}_{(a+b-y)-}f(a+b-z)\right]\Bigg|$$
$$\le \frac{(z-y)\alpha^{\beta/k}}{2}\int_0^1 \left|\left(\frac{1-t^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}} - \left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\right|$$
$$\times [|f'(a)|+|f'(b)| - t|f'(z)| - (1-t)|f'(y)|]dt$$
$$= \frac{(z-y)\alpha^{\beta/k}}{2}[\{|f'(a)|+|f'(b)|\}\theta_1 - |f'(z)|\theta_2 - |f'(y)|\theta_3]. \tag{3.3}$$

Here,

$$\begin{aligned}\theta_1 &= \int_0^1 \left| \left(\frac{1-t^\alpha}{\alpha}\right)^{\frac{\beta}{k}} - \left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\frac{\beta}{k}} \right| dt \\ &= \left(\frac{1}{\alpha}\right)^{\beta/k} \left[\int_0^{\frac{1}{2}} \{(1-t^\alpha)^{\beta/k} - (1-(1-t)^\alpha)^{\beta/k}\} dt \right. \\ &\qquad \left. + \int_{\frac{1}{2}}^1 \{(1-(1-t)^\alpha)^{\beta/k} - (1-t^\alpha)^{\beta/k}\} dt\right] \\ &= 2\left(\frac{1}{\alpha}\right)^{\beta/k} \int_0^{\frac{1}{2}} \{(1-t^\alpha)^{\beta/k} - (1-(1-t)^\alpha)^{\beta/k}\} dt = 2\left(\frac{1}{\alpha}\right)^{\frac{\beta}{k}} [\tau_1 - \tau_2].\end{aligned}$$

Thus,

$$\theta_1 = 2\left(\frac{1}{\alpha}\right)^{\frac{\beta}{k}+1} \left[2\mathcal{B}\left(\frac{1}{\alpha}, \frac{\beta}{k}+1; \left(\frac{1}{2}\right)^\alpha\right) - \mathcal{B}\left(\frac{1}{\alpha}, \frac{\beta}{k}+1\right)\right]. \tag{3.4}$$

Similarly,

$$\begin{aligned}\theta_2 &= \int_0^1 t \left| \left(\frac{1-t^\alpha}{\alpha}\right)^{\frac{\beta}{k}} - \left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\frac{\beta}{k}} \right| dt \\ &= \left(\frac{1}{\alpha}\right)^{\frac{\beta}{k}+1} \left[2\mathcal{B}\left(\frac{1}{\alpha}, \frac{\beta}{k}+1; \left(\frac{1}{2}\right)^\alpha\right) - \mathcal{B}\left(\frac{1}{\alpha}, \frac{\beta}{k}+1\right)\right] \qquad (3.5)\end{aligned}$$

and

$$\begin{aligned}\theta_3 &= \int_0^1 (1-t) \left| \left(\frac{1-t^\alpha}{\alpha}\right)^{\frac{\beta}{k}} - \left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\frac{\beta}{k}} \right| dt \\ &= \left(\frac{1}{\alpha}\right)^{\frac{\beta}{k}+1} \left[2\mathcal{B}\left(\frac{1}{\alpha}, \frac{\beta}{k}+1; \left(\frac{1}{2}\right)^\alpha\right) - \mathcal{B}\left(\frac{1}{\alpha}, \frac{\beta}{k}+1\right)\right]. \qquad (3.6)\end{aligned}$$

Substituting the equalities (3.4), (3.5), and (3.6) into (3.3) yields the desired inequality (3.2). □

Remark 3.1. By setting $k = 1$ in Theorem 3.1, we obtain the following inequalities for fractional conformable integrals:

$$\begin{aligned}&\left| \frac{f(a+b-y)+f(a+b-z)}{2} \right. \\ &\qquad \left. - \frac{\alpha^\beta \Gamma(\beta+1)}{2(z-y)^{\alpha\beta}} \left[{}^\beta\Im^\alpha_{(a+b-z)+} f(a+b-y) + {}^\beta\Im^\alpha_{(a+b-y)-} f(a+b-z)\right] \right| \\ &\le \frac{(z-y)}{\alpha}\left[|f'(a)|+|f'(b)| - \frac{|f'(y)|+|f'(z)|}{2}\right]\left[2\mathcal{B}\left(\frac{1}{\alpha}, \beta+1; (\frac{1}{2})^\alpha\right) - \mathcal{B}\left(\frac{1}{\alpha}, \beta+1\right)\right],\end{aligned}$$

which was proved by Hyder et al. in [19].

Remark 3.2. By taking $k = 1$ and $\alpha = 1$ in Theorem3.1, we obtain the inequality

$$\left| \frac{f(a+b-y)+f(a+b-z)}{2} - \frac{\Gamma(\beta+1)}{2(z-y)^\beta} \left[\Im^\beta_{(a+b-z)+} f(a+b-y) + \Im^\beta_{(a+b-y)-} f(a+b-z)\right] \right|$$

$$\leq \frac{z-y}{\beta+1}(1-\frac{1}{2^{\beta}})\left[|f'(a)|+|f'(b)|-\frac{|f'(z)|-|f'(y)|}{2}\right]$$

which was established by Ogulmus and Sarikaya in [30].

Remark 3.3. By choosing $k=1$, $y=a$, and $z=b$ in Theorem 3.1, we arrive at the inequality

$$\left|\frac{f(a)+f(b)}{2}-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathfrak{I}^{\alpha}_{a+}f(b)+{}^{\beta}\mathfrak{I}^{\alpha}_{b-}f(a)\right]\right|$$
$$\leq \frac{(b-a)}{\alpha}\left[|f'(a)|+|f'(b)|\right]\times\left[2\mathcal{B}\left(\frac{1}{\alpha},\beta+1;\left(\frac{1}{2}\right)^{\alpha}\right)-\mathcal{B}\left(\frac{1}{\alpha},\beta+1\right)\right]$$

which was demonstrated by Set et al. in [35].

Example 3.1. Consider the function $f:[1,6]\to\mathbf{R}$ defined by $f(x)=x^2$. Clearly, $|f'|$ is convex. For $y=3$ and $z=5$, we calculate the k-fractional conformable integrals in (3.2) as follows:

$$\begin{aligned}{}^{\beta}_{k}\mathfrak{I}^{\alpha}_{(a+b-z)+}f(a+b-y)&={}^{\beta}_{k}\mathfrak{I}^{\alpha}_{(2)+}f(4)\\&=\left(\frac{2^{\alpha}}{\alpha}\right)^{\beta/k}\frac{1}{k\Gamma_k(\beta)}\left[4\mathcal{B}\left(\frac{\beta}{k},1+\frac{2}{\alpha}\right)+8\mathcal{B}\left(\frac{\beta}{k},1+\frac{1}{\alpha}\right)+4\frac{k}{\beta}\right]\end{aligned}$$

and

$$\begin{aligned}{}^{\beta}_{k}\mathfrak{I}^{\alpha}_{(a+b-y)-}f(a+b-z)&={}^{\beta}_{k}\mathfrak{I}^{\alpha}_{(4)-}f(2)\\&=\left(\frac{2^{\alpha}}{\alpha}\right)^{\beta/k}\frac{1}{k\Gamma_k(\beta)}\left[8\mathcal{B}\left(\frac{\beta}{k},1+\frac{2}{\alpha}\right)-8\mathcal{B}\left(\frac{\beta}{k},1+\frac{1}{\alpha}\right)+16\frac{k}{\beta}\right].\end{aligned}$$

Thus, the left-hand side of (3.2) becomes

$$\begin{aligned}&\left|\frac{f(a+b-y)+f(a+b-z)}{2}\right.\\&\quad\left.-\frac{\alpha^{\beta/k}\Gamma_k(\beta+k)}{2(z-y)^{\frac{\alpha\beta}{k}}}\left[{}^{\beta}_{k}\mathfrak{I}^{\alpha}_{(a+b-z)+}f(a+b-y)+{}^{\beta}_{k}\mathfrak{I}^{\alpha}_{(a+b-y)-}f(a+b-z)\right]\right|\\&=\left|10-\frac{\beta}{k}\left[4\mathcal{B}\left(\frac{\beta}{k},1+\frac{2}{\alpha}\right)-4\mathcal{B}\left(\frac{\beta}{k},1+\frac{1}{\alpha}\right)+10\frac{k}{\beta}\right]\right|.\end{aligned}$$

On the other hand, the right-hand side of (3.2) is computed as

$$\begin{aligned}&\frac{(z-y)}{\alpha}\left[|f'(a)|+|f'(b)|-\frac{|f'(y)|+|f'(z)|}{2}\right]\left[2\mathcal{B}\left(\frac{1}{\alpha},\frac{\beta}{k}+1;\left(\frac{1}{2}\right)^{\alpha}\right)-\mathcal{B}\left(\frac{1}{\alpha},\frac{\beta}{k}+1\right)\right]\\&\qquad=\frac{12}{\alpha}\left[2\mathcal{B}\left(\frac{1}{\alpha},\frac{\beta}{k}+1;\left(\frac{1}{2}\right)^{\alpha}\right)-\mathcal{B}\left(\frac{1}{\alpha},\frac{\beta}{k}+1\right)\right].\end{aligned}$$

Therefore, the inequality obtained is

$$\begin{aligned}&\left|10-\frac{\beta}{k}\left[4\mathcal{B}\left(\frac{\beta}{k},1+\frac{2}{\alpha}\right)-4\mathcal{B}\left(\frac{\beta}{k},1+\frac{1}{\alpha}\right)+10\frac{k}{\beta}\right]\right|\\&\leq\frac{12}{\alpha}\left[2\mathcal{B}\left(\frac{1}{\alpha},\frac{\beta}{k}+1;\left(\frac{1}{2}\right)^{\alpha}\right)-\mathcal{B}\left(\frac{1}{\alpha},\frac{\beta}{k}+1\right)\right].\end{aligned}\tag{3.7}$$

The validity of inequality (3.7) is illustrated in Figure 2.

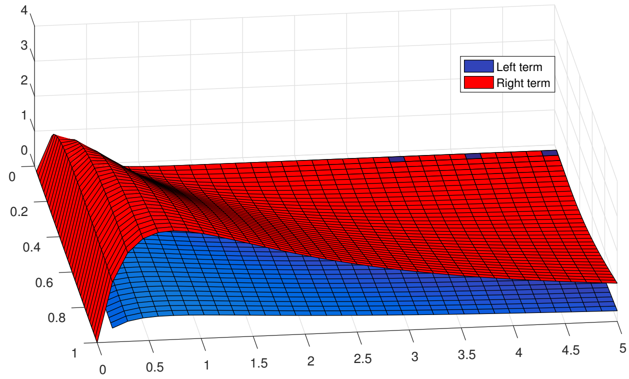

FIGURE 2. The graph of the inequalitt of (3.7) is evaluated by MATLAB software for $\alpha \in (0,1]$, $\beta \in (0,5]$ and $k = 0.5$.

Theorem 3.2. *Let $f : [a,b] \to \mathbf{R}$ be differentiable on (a,b) and assume that $|f'|^{\mu}$ is convex on $[a,b]$ for $\mu > 1$. Then,*

$$\Bigg|\frac{f(a+b-y)+f(a+b-z)}{2} - \frac{\alpha^{\beta/k}\Gamma_k(\beta+k)}{2(z-y)^{\frac{\alpha\beta}{k}}}\left[{}^{\beta}_{k}\Im^{\alpha}_{(a+b-z)+}f(a+b-y) + {}^{\beta}_{k}\Im^{\alpha}_{(a+b-y)-}f(a+b-z)\right]\Bigg| \leq \frac{(z-y)}{2}$$
$$\left(\int_0^1 \left|(1-t^{\alpha})^{\frac{\beta}{k}} - (1-(1-t)^{\alpha})^{\frac{\beta}{k}}\right|^{\lambda} dt\right)^{\frac{1}{\lambda}} \left[|f'(a)|^{\mu} + |f'(b)|^{\mu} - \frac{|f'(z)|^{\mu}+|f'(y)|^{\mu}}{2}\right]^{\frac{1}{\mu}}, \quad (3.8)$$

where $\frac{1}{\lambda} + \frac{1}{\mu} = 1$.

Proof. According to Lemma 3.1, we get

$$\Bigg|\frac{f(a+b-y)+f(a+b-z)}{2} - \frac{\alpha^{\beta/k}\Gamma_k(\beta+k)}{2(z-y)^{\frac{\alpha\beta}{k}}}\left[{}^{\beta}_{k}\Im^{\alpha}_{(a+b-z)+}f(a+b-y) + {}^{\beta}_{k}\Im^{\alpha}_{(a+b-y)-}f(a+b-z)\right]\Bigg|$$
$$\leq \frac{(z-y)\alpha^{\beta/k}}{2}\int_0^1 \left|\left[\left(\frac{1-t^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}} - \left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\right] f'(a+b-(tz+(1-t)y))dt\right|.$$

By Hölder's inequality, it follows that

$$\Bigg|\frac{f(a+b-y)+f(a+b-z)}{2} - \frac{\alpha^{\beta/k}\Gamma_k(\beta+k)}{2(z-y)^{\frac{\alpha\beta}{k}}}\left[{}^{\beta}_{k}\Im^{\alpha}_{(a+b-z)+}f(a+b-y) + {}^{\beta}_{k}\Im^{\alpha}_{(a+b-y)-}f(a+b-z)\right]\Bigg| \leq \frac{(z-y)\alpha^{\beta/k}}{2}$$
$$\left(\int_0^1 \left|\left(\frac{1-t^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}} - \left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\right|^{\lambda} dt\right)^{\frac{1}{\lambda}} \left(\int_0^1 |f'(a+b-(tz+(1-t)y))|^{\mu}dt\right)^{\frac{1}{\mu}}.$$

Since $|f'|^{\mu}$ is convex on $[a,b]$, we obtain

$$\begin{aligned}&\Bigg|\frac{f(a+b-y)+f(a+b-z)}{2}\\&\qquad-\frac{\alpha^{\beta/k}\Gamma_k(\beta+k)}{2(z-y)^{\frac{\alpha\beta}{k}}}\left[{}_k^{\beta}\Im^{\alpha}_{(a+b-z)+}f(a+b-y)+{}_k^{\beta}\Im^{\alpha}_{(a+b-y)-}f(a+b-z)\right]\Bigg|\\&\leq\frac{(z-y)}{2}\left(\int_0^1\left|(1-t^{\alpha})^{\frac{\beta}{k}}-(1-(1-t)^{\alpha})^{\frac{\beta}{k}}\right|^{\lambda}dt\right)^{\frac{1}{\lambda}}\\&\quad\times\left(\int_0^1\{|f'(a)|^{\mu}+|f'(b)|^{\mu}-t|f'(z)|^{\mu}-(1-t)|f'(y)|^{\mu}\}dt\right)^{\frac{1}{\mu}}\\&=\frac{(z-y)}{2}\left(\int_0^1\left|(1-t^{\alpha})^{\frac{\beta}{k}}-(1-(1-t)^{\alpha})^{\frac{\beta}{k}}\right|^{\lambda}dt\right)^{\frac{1}{\lambda}}\left[|f'(a)|^{\mu}+|f'(b)|^{\mu}-\frac{|f'(z)|^{\mu}+|f'(y)|^{\mu}}{2}\right]^{\frac{1}{\mu}}.\end{aligned}$$

This completes the proof. □

Remark 3.4. By setting $k=1$ in Theorem 3.2, we obtain the following inequality for fractional conformable integrals

$$\begin{aligned}&\Bigg|\frac{f(a+b-y)+f(a+b-z)}{2}\\&\qquad-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(z-y)^{\alpha\beta}}\left[{}^{\beta}\Im^{\alpha}_{(a+b-z)+}f(a+b-y)+{}^{\beta}\Im^{\alpha}_{(a+b-y)-}f(a+b-z)\right]\Bigg|\leq\\&\frac{(z-y)}{2}\left(\int_0^1\left|(1-t^{\alpha})^{\beta}-(1-(1-t)^{\alpha})^{\beta}\right|^{\lambda}dt\right)^{\frac{1}{\lambda}}\left[|f'(a)|^{\mu}+|f'(b)|^{\mu}-\frac{|f'(z)|^{\mu}+|f'(y)|^{\mu}}{2}\right]^{\frac{1}{\mu}}\end{aligned}$$

which was proved by Hyder et al. in [19, Theorem 6].

Remark 3.5. By taking $k=1$ and $\alpha=1$ in Theorem 3.2, we obtain the following inequality

$$\begin{aligned}&\Bigg|\frac{f(a+b-y)+f(a+b-z)}{2}\\&\qquad-\frac{\Gamma(\beta+1)}{2(z-y)^{\beta}}\left[\Im^{\beta}_{(a+b-z)+}f(a+b-y)+\Im^{\beta}_{(a+b-y)-}f(a+b-z)\right]\Bigg|\\&\leq\frac{(z-y)}{2}\left(\int_0^1\left|(1-t)^{\beta}-t^{\beta}\right|^{\lambda}dt\right)^{\frac{1}{\lambda}}\times\left[|f'(a)|^{\mu}+|f'(b)|^{\mu}-\frac{|f'(z)|^{\mu}+|f'(y)|^{\mu}}{2}\right]^{\frac{1}{\mu}},\end{aligned}$$

which was given in [19, Remark 4].

Remark 3.6. By choosing $k=1$, $y=a$, and $z=b$ in Theorem 3.2, we obtain the following inequality

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}\Im^{\alpha}_{a+}f(b)+{}^{\beta}\Im^{\alpha}_{b-}f(a)\right]\right|\\&\leq\frac{(b-a)}{2}\left(\int_0^1\left|(1-t^{\alpha})^{\frac{\beta}{k}}-(1-(1-t)^{\alpha})^{\frac{\beta}{k}}\right|^{\lambda}dt\right)^{\frac{1}{\lambda}}\left[\frac{|f'(a)|^{\mu}+|f'(b)|^{\mu}}{2}\right]^{\frac{1}{\mu}},\end{aligned}$$

which corresponds to the Riemann-Liouville fractional integrals.

4. Midpoint-Type Inequalities with Jensen–Mercer Inequality

In this section, we establish an equality based on Lemma 2.2 and, by applying the Jensen–Mercer inequality, derive several midpoint-type inequalities.

Lemma 4.1. *Let $f:[a,b]\to\mathbf{R}$ be differentiable on (a,b) and suppose that $f'\in\mathcal{L}_1[a,b]$. Then*

$$f\left(a+b-\frac{y+z}{2}\right)-\frac{\alpha^{\beta/k}\Gamma_k(\beta+k)}{2(z-y)^{\frac{\alpha\beta}{k}}}\left[{}^{\beta}_{k}\mathfrak{I}^{\alpha}_{(a+b-z)+}f(a+b-y)+{}^{\beta}_{k}\mathfrak{I}^{\alpha}_{(a+b-y)-}f(a+b-z)\right]$$
$$=\frac{(z-y)\alpha^{\beta/k}}{2}(\gamma_1-\gamma_2-\gamma_3+\gamma_4). \tag{4.1}$$

Here,

$$\gamma_1=\int_0^{1/2}\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}f'(a+b-(ty+(1-t)z))dt,$$
$$\gamma_2=\int_0^{1/2}\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}f'(a+b-(tz+(1-t)y))dt,$$
$$\gamma_3=\int_{1/2}^{1}\left[\frac{1}{\alpha^{\beta/k}}-\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\right]f'(a+b-(ty+(1-t)z))dt,$$
$$\gamma_4=\int_{1/2}^{1}\left[\frac{1}{\alpha^{\beta/k}}-\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\right]f'(a+b-(tz+(1-t)y))dt.$$

Proof. By replacing a with $a+b-z$ and b with $a+b-y$ in Lemma 2.2, we obtain the desired result. □

Theorem 4.1. *Let $f:[a,b]\to\mathbf{R}$ be differentiable on (a,b) and assume that $|f'|$ is convex on $[a,b]$. Then, for $y,z\in[a,b]$ with $y<z$, $\beta>0$, $k>0$, and $\alpha\in(0,1]$, we have*

$$\Bigg|f\left(a+b-\frac{y+z}{2}\right)-\frac{\alpha^{\beta/k}\Gamma_k(\beta+k)}{2(z-y)^{\frac{\alpha\beta}{k}}}\left[{}^{\beta}_{k}\mathfrak{I}^{\alpha}_{(a+b-z)+}f(a+b-y)+{}^{\beta}_{k}\mathfrak{I}^{\alpha}_{(a+b-y)-}f(a+b-z)\right]\Bigg|\le(z-y)$$
$$\left[|f'(a)|+|f'(b)|-\frac{|f'(y)|+|f'(z)|}{2}\right]\left[\frac{1}{2}+\frac{1}{\alpha}\left[\mathcal{B}\left(\frac{1}{\alpha},\frac{\beta}{k}+1\right)-2\mathcal{B}\left(\frac{1}{\alpha},\frac{\beta}{k}+1;\left(\frac{1}{2}\right)^{\alpha}\right)\right]\right]. \tag{4.2}$$

Proof. From Lemma 4.1, we have

$$\left|f\left(a+b-\frac{y+z}{2}\right)-\frac{\alpha^{\beta/k}\Gamma_k(\beta+k)}{2(z-y)^{\frac{\alpha\beta}{k}}}\left[{}^{\beta}_{k}\mathfrak{I}^{\alpha}_{(a+b-z)+}f(a+b-y)+{}^{\beta}_{k}\mathfrak{I}^{\alpha}_{(a+b-y)-}f(a+b-z)\right]\right|$$
$$\le\frac{(z-y)\alpha^{\beta/k}}{2}\{U_1+U_2+U_3+U_4\}. \tag{4.3}$$

Applying the Jensen-Mercer inequality and Lemma 3.2, we obtain

$$\begin{aligned}U_1 &= \int_0^{1/2}\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}|f'(a+b-(ty+(1-t)z))|dt\\ &\le \int_0^{1/2}\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}[|f'(a)|+|f'(b)|-t|f'(z)|-(1-t)|f'(y)|]dt\\ &= \left(\frac{1}{\alpha}\right)^{\frac{\beta}{k}}\left[(|f'(a)|+|f'(b)|)\tau_2-|f'(z)|\tau_4-|f'(y)|\tau_6\right].\end{aligned}$$

Similarly,

$$\begin{aligned}U_2 &= \int_0^{1/2}\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}|f'(a+b-(tz+(1-t)y))|dt\\ &\le \left(\frac{1}{\alpha}\right)^{\frac{\beta}{k}}\left[(|f'(a)|+|f'(b)|)\tau_2-|f'(z)|\tau_6-|f'(y)|\tau_4\right],\\ U_3 &= \int_{1/2}^{1}\left[\frac{1}{\alpha^{\beta/k}}-\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}\right]|f'(a+b-(ty+(1-t)z))|dt\\ &\le \left(\frac{1}{\alpha}\right)^{\frac{\beta}{k}}\left[(|f'(a)|+|f'(b)|)\left(\frac{1}{2}-\tau_1\right)-|f'(z)|\left(\frac{1}{8}-\tau_3\right)-|f'(y)|\left(\frac{3}{8}-\tau_5\right)\right],\\ U_4 &= \int_{1/2}^{1}\left[\frac{1}{\alpha^{\beta/k}}-\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}\right]|f'(a+b-(tz+(1-t)y))|dt\\ &\le \left(\frac{1}{\alpha}\right)^{\frac{\beta}{k}}\left[(|f'(a)|+|f'(b)|)\left(\frac{1}{2}-\tau_1\right)-|f'(y)|\left(\frac{1}{8}-\tau_3\right)-|f'(z)|\left(\frac{3}{8}-\tau_5\right)\right].\end{aligned}$$

Substituting U_1, U_2, U_3, U_4 into (4.3), and simplifying, yields

$$\begin{aligned}&\left|f\left(a+b-\frac{y+z}{2}\right)-\frac{\alpha^{\beta/k}\Gamma_k(\beta+k)}{2(z-y)^{\frac{\alpha\beta}{k}}}\left[{}_k^\beta\Im^\alpha_{(a+b-z)+}f(a+b-y)+{}_k^\beta\Im^\alpha_{(a+b-y)-}f(a+b-z)\right]\right|\\ &\le \frac{(z-y)}{2}\Big[(|f'(a)|+|f'(b)|)\tau_2-|f'(z)|\tau_4-|f'(y)|\tau_6+(|f'(a)|+|f'(b)|)\tau_2-|f'(z)|\tau_6\\ &-|f'(y)|\tau_4+(|f'(a)|+|f'(b)|)\left(\frac{1}{2}-\tau_1\right)-|f'(z)|\left(\frac{1}{8}-\tau_3\right)-|f'(y)|\left(\frac{3}{8}-\tau_5\right)\\ &+(|f'(a)|+|f'(b)|)\left(\frac{1}{2}-\tau_1\right)-|f'(y)|\left(\frac{1}{8}-\tau_3\right)-|f'(z)|\left(\frac{3}{8}-\tau_5\right)\Big]\\ &=\frac{(z-y)}{2}\left[(|f'(a)|+|f'(b)|)(1-2\tau_1+2\tau_2)-(|f'(y)|+|f'(z)|)(\frac{1}{2}+(\tau_4+\tau_6)-(\tau_3+\tau_5))\right]\\ &=\frac{(z-y)}{2}\left[(|f'(a)|+|f'(b)|)(1-2\tau_1+2\tau_2)-(|f'(y)|+|f'(z)|)(\frac{1}{2}+\tau_2-\tau_1)\right]\\ &=(z-y)(\frac{1}{2}-\tau_1+\tau_2)\left[|f'(a)|+|f'(b)|-\frac{|f'(y)|+|f'(z)|}{2}\right]\le(z-y)\\ &\quad\left[|f'(a)|+|f'(b)|-\frac{|f'(y)|+|f'(z)|}{2}\right]\left[\frac{1}{2}+\frac{1}{\alpha}\left[\mathcal{B}\left(\frac{1}{\alpha},\frac{\beta}{k}+1\right)-2\mathcal{B}\left(\frac{1}{\alpha},\frac{\beta}{k}+1;\left(\frac{1}{2}\right)^\alpha\right)\right]\right].\end{aligned}$$

This completes the proof. $\square$

Remark 4.1. By setting $k = 1$ in Theorem 4.1, we obtain the following inequality for fractional conformable integrals

$$\left| f\left(a+b-\frac{y+z}{2}\right) - \frac{\alpha^{\beta}\Gamma(\beta+1)}{2(z-y)^{\alpha\beta}}\left[{}^{\beta}\Im^{\alpha}_{(a+b-z)+}f(a+b-y) + {}^{\beta}\Im^{\alpha}_{(a+b-y)-}f(a+b-z)\right]\right| \leq (z-y)$$
$$\left[|f'(a)|+|f'(b)|-\frac{|f'(y)|+|f'(z)|}{2}\right]\left[\frac{1}{2}+\frac{1}{\alpha}\left[\mathcal{B}\left(\frac{1}{\alpha},\beta+1\right)-2\mathcal{B}\left(\frac{1}{\alpha},\beta+1;\left(\frac{1}{2}\right)^{\alpha}\right)\right]\right],$$

which was demonstrated by Hyder et al. in [19, Theorem 7].

Remark 4.2. By taking $k = 1$, $y = a$, and $z = b$ in Theorem 4.1, we obtain the following inequality

$$\left| f\left(\frac{a+b}{2}\right) - \frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}\Im^{\alpha}_{a+}f(b) + {}^{\beta}\Im^{\alpha}_{b-}f(a)\right]\right|$$
$$\leq \frac{(b-a)}{2}\left[|f'(a)|+|f'(b)|\right] \times \left[\mathcal{B}\left(\frac{1}{\alpha},\beta+1\right)-2\mathcal{B}\left(\frac{1}{\alpha},\beta+1;\left(\frac{1}{2}\right)^{\alpha}\right)\right],$$

which was proved by Hezenci and Budak in [15].

Example 4.1. Consider the function $f : [1, 6] \to \mathbf{R}$ defined by $f(x) = x^2$. Clearly, $|f'|$ is convex on $[1, 6]$. Let $y = 3$ and $z = 5$. Then the left-hand side of (4.2) becomes

$$\left| f\left(a+b-\frac{y+z}{2}\right) - \frac{\alpha^{\beta/k}\Gamma_k(\beta+k)}{2(z-y)^{\frac{\alpha\beta}{k}}}\left[{}^{\beta}_{k}\Im^{\alpha}_{(a+b-z)+}f(a+b-y) + {}^{\beta}_{k}\Im^{\alpha}_{(a+b-y)-}f(a+b-z)\right]\right|$$
$$= \left|9 - 2\frac{\beta}{k}\left[2\mathcal{B}\left(\frac{\beta}{k},1+\frac{2}{\alpha}\right)-2\mathcal{B}\left(\frac{\beta}{k},1+\frac{1}{\alpha}\right)+5\frac{k}{\beta}\right]\right|.$$

The right-hand side of (4.2) is computed as

$$\left[|f'(a)|+|f'(b)|-\frac{|f'(y)|+|f'(z)|}{2}\right]\left[\frac{1}{2}+\frac{1}{\alpha}\left[\mathcal{B}\left(\frac{1}{\alpha},\frac{\beta}{k}+1\right)-2\mathcal{B}\left(\frac{1}{\alpha},\frac{\beta}{k}+1;\left(\frac{1}{2}\right)^{\alpha}\right)\right]\right]$$
$$\times (z-y) = 12\left[\frac{1}{2}+\frac{1}{\alpha}\left[\mathcal{B}\left(\frac{1}{\alpha},\frac{\beta}{k}+1\right)-2\mathcal{B}\left(\frac{1}{\alpha},\frac{\beta}{k}+1;\left(\frac{1}{2}\right)^{\alpha}\right)\right]\right].$$

Therefore, the inequality takes the form

$$\left|9 - 2\frac{\beta}{k}\left[2\mathcal{B}\left(\frac{\beta}{k},1+\frac{2}{\alpha}\right)-2\mathcal{B}\left(\frac{\beta}{k},1+\frac{1}{\alpha}\right)+5\frac{k}{\beta}\right]\right|$$
$$\leq 12\left[\frac{1}{2}+\frac{1}{\alpha}\left[\mathcal{B}\left(\frac{1}{\alpha},\frac{\beta}{k}+1\right)-2\mathcal{B}\left(\frac{1}{\alpha},\frac{\beta}{k}+1;\left(\frac{1}{2}\right)^{\alpha}\right)\right]\right]. \tag{4.4}$$

The validity of the inequality (4.4) is illustrated in Figure 3.

Theorem 4.2. *Let $f : [a, b] \to \mathbf{R}$ be differentiable on (a, b) and assume that $|f'|^{\mu}$ is convex on $[a, b]$ for $\mu > 1$, with $\frac{1}{\lambda} + \frac{1}{\mu} = 1$. Then*

$$\left| f\left(a+b-\frac{y+z}{2}\right) - \frac{\alpha^{\beta/k}\Gamma_k(\beta+k)}{2(z-y)^{\frac{\alpha\beta}{k}}}\left[{}^{\beta}_{k}\Im^{\alpha}_{(a+b-z)+}f(a+b-y) + {}^{\beta}_{k}\Im^{\alpha}_{(a+b-y)-}f(a+b-z)\right]\right|$$

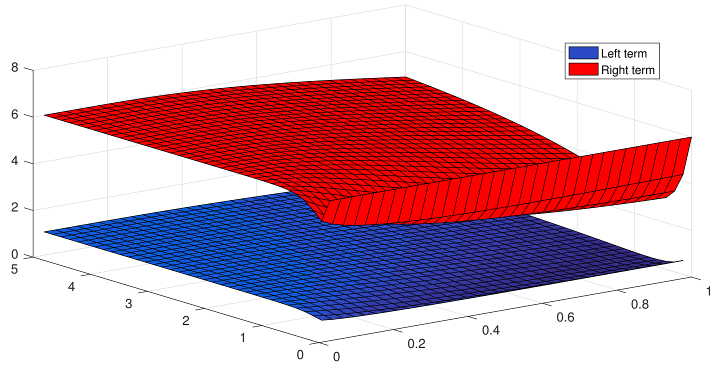

FIGURE 3. The graph of the inequality of (4.4) is evaluated by MATLAB software for $\alpha \in (0,1]$, $\beta \in (0,5]$ and $k = 0.5$.

$$\leq \frac{(z-y)}{2^{1+\frac{1}{\mu}}} \left[\left(\frac{1}{\alpha} \left[\mathcal{B}\left(\frac{1}{\alpha}, \frac{\lambda\beta}{k} + 1 \right) - \mathcal{B}\left(\frac{1}{\alpha}, \frac{\lambda\beta}{k} + 1; \left(\frac{1}{2}\right)^{\alpha} \right) \right] \right)^{\frac{1}{\lambda}} \right.$$
$$\left. + \left(\int_{\frac{1}{2}}^{1} [1-(1-(1-t)^{\alpha})^{\beta/k}]^{\lambda} dt \right)^{\frac{1}{\lambda}} \right] \left[\left(|f'(a)|^{\mu} + |f'(b)|^{\mu} - \frac{|f'(z)|^{\mu} + 3|f'(y)|^{\mu}}{4} \right)^{\frac{1}{\mu}} \right.$$
$$\left. + \left(|f'(a)|^{\mu} + |f'(b)|^{\mu} - \frac{3|f'(z)|^{\mu} + |f'(y)|^{\mu}}{4} \right)^{\frac{1}{\mu}} \right], \tag{4.5}$$

for $y, z \in [a,b]$ *with* $y < z$, $\beta > 0$, $k > 0$, *and* $\alpha \in (0,1]$.

Proof. From Lemma 4.1, we have

$$\left| f\left(a+b-\frac{y+z}{2}\right) - \frac{\alpha^{\beta/k}\Gamma_k(\beta+k)}{2(z-y)^{\frac{\alpha\beta}{k}}} \left[{}_k^{\beta}\Im^{\alpha}_{(a+b-z)+} f(a+b-y) + {}_k^{\beta}\Im^{\alpha}_{(a+b-y)-} f(a+b-z) \right] \right|$$
$$\leq \frac{(z-y)\alpha^{\beta/k}}{2}(V_1+V_2+V_3+V_4). \tag{4.6}$$

Applying the Jensen-Mercer inequality and Hölder's inequality, we obtain

$$V_1 = \int_0^{1/2} \left(\frac{1-(1-t)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k}} |f'(a+b-(ty+(1-t)z))| dt$$
$$\leq \left(\int_0^{1/2} \left(\frac{1-(1-t)^{\alpha}}{\alpha} \right)^{\frac{\lambda\beta}{k}} \right)^{\frac{1}{\lambda}} \times \left(\int_0^{1/2} |f'(a+b-(ty+(1-t)z))|^{\mu} dt \right)^{\frac{1}{\mu}}$$
$$\leq \left(\frac{1}{\alpha}\right)^{\frac{\beta}{k}} \left(\frac{1}{\alpha} \left[\mathcal{B}\left(\frac{1}{\alpha}, \frac{\lambda\beta}{k} + 1 \right) - \mathcal{B}\left(\frac{1}{\alpha}, \frac{\lambda\beta}{k} + 1; \left(\frac{1}{2}\right)^{\alpha} \right) \right] \right)^{\frac{1}{\lambda}}$$
$$\times \left(\int_0^{1/2} \{ |f'(a)|^{\mu} + |f'(b)|^{\mu} - t|f'(y)|^{\mu} - (1-t)|f'(z)|^{\mu} \} dt \right)^{\frac{1}{\mu}}$$

$$
\begin{aligned}
&= \left(\frac{1}{\alpha}\right)^{\frac{\beta}{k}} \left(\frac{1}{\alpha}\left[\mathcal{B}\left(\frac{1}{\alpha}, \frac{\lambda\beta}{k}+1\right) - \mathcal{B}\left(\frac{1}{\alpha}, \frac{\lambda\beta}{k}+1; \left(\frac{1}{2}\right)^{\alpha}\right)\right]\right)^{\frac{1}{\lambda}} \\
&\quad \times \left(\frac{|f'(a)|^{\mu}+|f'(b)|^{\mu}}{2} - \frac{3|f'(z)|^{\mu}+|f'(y)|^{\mu}}{8}\right)^{\frac{1}{\mu}}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
V_2 &= \int_0^{1/2} \left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}} |f'(a+b-(tz+(1-t)y))|dt \\
&\leq \left(\frac{1}{\alpha}\right)^{\frac{\beta}{k}} \left(\frac{1}{\alpha}\left[\mathcal{B}\left(\frac{1}{\alpha}, \frac{\lambda\beta}{k}+1\right) - \mathcal{B}\left(\frac{1}{\alpha}, \frac{\lambda\beta}{k}+1; \left(\frac{1}{2}\right)^{\alpha}\right)\right]\right)^{\frac{1}{\lambda}} \\
&\quad \times \left(\frac{|f'(a)|^{\mu}+|f'(b)|^{\mu}}{2} - \frac{|f'(z)|^{\mu}+3|f'(y)|^{\mu}}{8}\right)^{\frac{1}{\mu}}, \\
V_3 &= \int_{1/2}^{1} \left[\frac{1}{\alpha^{\beta/k}} - \left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\right] |f'(a+b-(ty+(1-t)z))|dt \\
&\leq \frac{\left(\int_{\frac{1}{2}}^{1}[1-(1-(1-t)^{\alpha})^{\beta/k}]^{\lambda}dt\right)^{\frac{1}{\lambda}}}{(\alpha)^{\frac{\beta}{k}}} \left(\frac{|f'(a)|^{\mu}+|f'(b)|^{\mu}}{2} - \frac{|f'(z)|^{\mu}+3|f'(y)|^{\mu}}{8}\right)^{\frac{1}{\mu}}, \\
V_4 &= \int_{1/2}^{1} \left[\frac{1}{\alpha^{\beta/k}} - \left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\right] |f'(a+b-(tz+(1-t)y))|dt \\
&\leq \frac{\left(\int_{\frac{1}{2}}^{1}[1-(1-(1-t)^{\alpha})^{\beta/k}]^{\lambda}dt\right)^{\frac{1}{\lambda}}}{(\alpha)^{\frac{\beta}{k}}} \left(\frac{|f'(a)|^{\mu}+|f'(b)|^{\mu}}{2} - \frac{3|f'(z)|^{\mu}+|f'(y)|^{\mu}}{8}\right)^{\frac{1}{\mu}}.
\end{aligned}
$$

Substituting V_1, V_2, V_3, and V_4 in (4.6), we obtain

$$
\begin{aligned}
&\left|f(a+b-\frac{y+z}{2}) - \frac{\alpha^{\beta/k}\Gamma_k(\beta+k)}{2(z-y)^{\frac{\alpha\beta}{k}}}\left[{}^{\beta}_{k}\Im^{\alpha}_{(a+b-z)+}f(a+b-y) +^{\beta}_{k} \Im^{\alpha}_{(a+b-y)-}f(a+b-z)\right]\right| \\
&\leq \frac{(z-y)}{2}\left[\left(\frac{1}{\alpha}\left[\mathcal{B}\left(\frac{1}{\alpha}, \frac{\lambda\beta}{k}+1\right) - \mathcal{B}\left(\frac{1}{\alpha}, \frac{\lambda\beta}{k}+1; \left(\frac{1}{2}\right)^{\alpha}\right)\right]\right)^{\frac{1}{\lambda}}\right. \\
&\quad \left. + \left(\int_{\frac{1}{2}}^{1}[1-(1-(1-t)^{\alpha})^{\beta/k}]^{\lambda}dt\right)^{\frac{1}{\lambda}}\right]\left[\left(\frac{|f'(a)|^{\mu}+|f'(b)|^{\mu}}{2} - \frac{3|f'(z)|^{\mu}+|f'(y)|^{\mu}}{8}\right)^{\frac{1}{\mu}}\right. \\
&\quad \left. + \left(\frac{|f'(a)|^{\mu}+|f'(b)|^{\mu}}{2} - \frac{|f'(z)|^{\mu}+3|f'(y)|^{\mu}}{8}\right)^{\frac{1}{\mu}}\right].
\end{aligned}
$$

This complete the proof. □

Remark 4.3. By setting $k = 1$ in Theorem 4.2, we obtain the following inequality

$$
\left|f(a+b-\frac{y+z}{2}) - \frac{\alpha^{\beta}\Gamma(\beta+1)}{2(z-y)^{\alpha\beta}}\left[{}^{\beta}\Im^{\alpha}_{(a+b-z)+}f(a+b-y) +^{\beta} \Im^{\alpha}_{(a+b-y)-}f(a+b-z)\right]\right|
$$

$$\leq \frac{(z-y)}{2^{1+\frac{1}{\mu}}}\left[\left(\frac{1}{\alpha}\left[\mathcal{B}\left(\frac{1}{\alpha},\lambda\beta+1\right)-\mathcal{B}\left(\frac{1}{\alpha},\lambda\beta+1;\left(\frac{1}{2}\right)^{\alpha}\right)\right]\right)^{\frac{1}{\lambda}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^{1}[1-(1-(1-t)^{\alpha})^{\beta}]^{\lambda}dt\right)^{\frac{1}{\lambda}}\right]\left[\left(|f'(a)|^{\mu}+|f'(b)|^{\mu}-\frac{|f'(z)|^{\mu}+3|f'(y)|^{\mu}}{4}\right)^{\frac{1}{\mu}}\right.$$
$$\left.+\left(|f'(a)|^{\mu}+|f'(b)|^{\mu}-\frac{3|f'(z)|^{\mu}+|f'(y)|^{\mu}}{4}\right)^{\frac{1}{\mu}}\right],$$

which was proved by Hyder et al. in [19, Theorem 8].

Remark 4.4. By taking $k=1$ and $\alpha=1$ in Theorem 4.2, we obtain the following inequality

$$\left|f(a+b-\frac{y+z}{2})-\frac{\Gamma(\beta+1)}{2(z-y)^{\beta}}\left[\Im^{\beta}_{(a+b-z)+}f(a+b-y)+\Im^{\beta}_{(a+b-y)-}f(a+b-z)\right]\right|$$
$$\leq \frac{(z-y)}{2^{1+\frac{1}{\mu}}}\left[\left(\frac{1}{(\lambda\beta+1)2^{(\lambda\beta+1)}}\right)^{\frac{1}{\lambda}}+\left(\int_{\frac{1}{2}}^{1}[1-t^{\beta}]^{\lambda}dt\right)^{\frac{1}{\lambda}}\right]$$
$$\times\left[\left(|f'(a)|^{\mu}+|f'(b)|^{\mu}-\frac{|f'(z)|^{\mu}+3|f'(y)|^{\mu}}{4}\right)^{\frac{1}{\mu}}\right.$$
$$\left.+\left(|f'(a)|^{\mu}+|f'(b)|^{\mu}-\frac{3|f'(z)|^{\mu}+|f'(y)|^{\mu}}{4}\right)^{\frac{1}{\mu}}\right],$$

which was given in [19, Remark 8].

Remark 4.5. By choosing $k=1$, $y=a$, and $z=b$ in Theorem 4.2, we obtain the following inequality

$$\left|\frac{f(a+b)}{2}-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}\Im^{\alpha}_{a+}f(b)+{}^{\beta}\Im^{\alpha}_{b-}f(a)\right]\right|\leq\frac{(b-a)}{2^{1+\frac{1}{\mu}}}$$
$$\left[\left(\frac{1}{\alpha}\left[\mathcal{B}\left(\frac{1}{\alpha},\lambda\beta+1\right)-\mathcal{B}\left(\frac{1}{\alpha},\lambda\beta+1;\left(\frac{1}{2}\right)^{\alpha}\right)\right]\right)^{\frac{1}{\lambda}}+\left(\int_{\frac{1}{2}}^{1}[1-(1-(1-t)^{\alpha})^{\beta}]^{\lambda}dt\right)^{\frac{1}{\lambda}}\right]$$
$$\times\left[\left(\frac{|f'(a)|^{\mu}+3|f'(b)|^{\mu}}{4}\right)^{\frac{1}{\mu}}+\left(\frac{3|f'(a)|^{\mu}+|f'(b)|^{\mu}}{4}\right)^{\frac{1}{\mu}}\right],$$

which was demonstrated by Hezenci and Budak in [15].

5. Discussion

This study establishes novel inequalities using generalized k-fractional conformable integrals, extending previous results for Riemann-Liouville and fractional conformable integrals. The derived Hermite-Hadamard, trapezoidal, and midpoint-type inequalities provide deeper insights into the behavior of convex functions under these operators. The practical significance lies in their potential applications in mathematical analysis, applied sciences, and engineering problems involving fractional calculus. Future work may explore extensions to higher-order fractional operators, multi-dimensional domains, or other classes of convex and non-convex functions, as well as potential applications in differential equations, optimization, and modeling of real-world phenomena.

6. Conclusion

This study investigates inequalities using generalized k-fractional conformable integrals. It utilizes the Jensen-Mercer inequality to establish new results. The study focuses on Hermite-Hadamard, trapezoidal, and midpoint-type inequalities. It explores the connections between these inequalities and convex functions. Equalities with generalized k-fractional conformable integrals are established to derive the new inequalities. The results build upon prior work on Riemann-Liouville fractional integrals, fractional conformable integrals and generalized k-fractional conformable integrals. The study provides a comprehensive analysis of the inequalities. Examples with graphical representations are used to validate the findings. The results have potential applications in various fields. The study contributes to the development of fractional calculus. It provides new insights into the properties of convex functions. The inequalities established in the study are novel and original. The research is significant, as it fills a gap in the existing literature. The study's findings are accurate and reliable, as demonstrated by the examples.

Acknowledgements. The authors would like to thank the reviewers for their comments and suggestions, which have improved the final version of the paper.

References

[1] T. Abdeljawad, F. Jarad and J. Alzabut, *Fractional proportional differences with memory,* Eur. Phys. J. Spec. Top., **226** (2017), 3333–3354.

[2] R. Agarwal, S. D. Purohit and Kritika, *A mathematical fractional model with nonsingular kernel for thrombin receptor activation in calcium signalling,* Math. Method Appl. Sci., **42**(18) (2019), 7160–7171.

[3] R. Agarwal, M. P. Yadav, D. Baleanu and S. D. Purohit, *Existence and uniqueness of miscible flow equation through porous media with a non singular fractional derivative,* AIMS Mathematics, **5**(2) (2020), 1062–1073.

[4] D. R. Anderson, D. J. Ulness, *Newly defined conformable derivatives,* Adv. Dyn. Syst. Appl., **10** (2015), 109–137.

[5] I. A. Baloch, Y. -M. Chu, *Petrović-type inequalities for harmonic h-convex functions,* J. Funct. Spaces, **2020**, Article ID: 3075390, 1–7.

[6] M. A. Barakat, A. -A. Hyder and D. Rizk, *New fractional results for Langevin equations through extensive fractional operators,* AIMS Mathematics, **8**(3) (2023), 6119–6135.

[7] H. Budak, T. Tunç and M. Z. Sarıkaya, *Fractional Hermite-Hadamard-type inequalities for interval-valued functions,* Proc. Amer. Math. Soc., **148**(2) (2020), 705–718.

[8] H. Budak, S. K. Yıldırım, M. Z. Sarıkaya and H. Yıldırım, *Some parameterized Simpson, midpoint and trapezoid-type inequalities for generalized fractional integrals,* J. Inequal. Appl., **2022** (2022), Article ID: 40.

[9] S. I. Butt, M. Umar, S. Rashid, A. O. Akdemir and Y. -M. Chu, *New Hermite–Jensen–Mercer-type inequalities via k-fractional integrals,* Adv. Differ. Equ., **2020** (2020), 635.

[10] S. I. Butt, A. O. Akdemir, J. Nasir and F. Jarad, *Some Hermite-Jensen-Mercer like inequalities for convex functions through a certain generalized fractional integrals and related results,* Miskolc Math. Notes, 2020, **21**(2) (2020), 689–715.

[11] I. Demir, T. Tunc, *New midpoint-type inequalities in the context of the proportional Caputo-hybrid operator,* J. Inequal. Appl., **2024** (2024), Article ID: 2.
[12] S. S. Dragomir, *Hermite–Hadamard type inequalities for generalized Riemann–Liouville fractional integrals of h-convex functions,* Math. Method Appl. Sci., **44**(3) (2021), 2364–2380.
[13] S. S. Dragomir, C. E. M. Pearce, *Selected Topics on Hermite-Hadamard inequalities and applications,* RGMIA Monographs, Victoria University: Melbourne, VC, Canada, 2000.
[14] Q. Feng, S. Habib, S. Mubeen and M. N. Naeem, *Generalized k-fractional comformable integrals and related inequalities,* AIMS Mathematics, **4**(3) (2019), 343–358.
[15] H. Hezenci, H. Budak, *A remark on midpoint-type inequalities for conformable fractional integrals,* Miskolc Math. Notes, (2025), in press.
[16] R. Hilfer (Editor), *Applications of fractional calculus in physics,* World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 2000.
[17] A. A. Hyder, M. A. Barakat and A. Fathallah, *Enlarged integral inequalities through recent fractional generalized operators,* J. Inequal. Appl., **2022** (2022), Article ID: 95.
[18] A. A. Hyder, M. A. Barakat and A. H. Soliman, *A new class of fractional inequalities through the convexity concept and enlarged Riemann–Liouville integrals,* J. Inequal. Appl., **2023** (2023), Article ID: 137.
[19] A. A. Hyder, A. A. Almoneef and H. Budak, *Improvement in some inequalities via Jensen–Mercer inequality and fractional extended Riemann–Liouville integrals,* Axioms, **12**(9) (2023), 886.
[20] F. Jarad, T. Abdeljawad and J. Alzabut, *Generalized fractional derivatives generated by a class of local proportional derivatives,* Eur. Phys. J. Spec. Top., **226** (2017), 3457–3471.
[21] F. Jarad, E. Uğurlu, T. Abdeljawad and D. Baleanu, *On a new class of fractional operators,* Adv. Differ. Equ., **2017** (2017), 247.
[22] U. N. Katugampola„ *A new approach to generalized fractional derivatives,* Bull. Math. Anal. Appl., **6**(4) (2014), 1–15.
[23] M. A. Khan, Y. Khurshid, T.-S. Du and Y. -M. Chu, *Generalization of Hermite-Hadamard type inequalities via conformable fractional integrals,* J. Funct. Spaces, **2018**, Article ID: 5357463, 1–12.
[24] M. A. Khan, N. Mohammad, E.R. Nwaeze and Y. -M. Chu, *Quantum Hermite–Hadamard inequality by means of a Green function,* Adv. Differ. Equ., **2020** (2020), 99.
[25] M. Kian, M. S. Moslehian, *Refinements of the operator Jensen-Mercer Inequality,* Electron. J. Linear Algebra, **26** (2013), 742–753.
[26] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations,* North-Holland Mathematics Studies, 204; Elsevier Sci. B.V. : Amsterdam, The Netherlands, 2006.
[27] D. Kumar, J. Singh, S. D. Purohit and R. Swroop, *A hybrid analytical algorithm for nonlinear fractional wave-like equations,* Math. Model. Nat. Phenom., **14**(3) (2019), Article ID: 304.
[28] A. M. Mercer, *A variant of Jensens inequality,* J. Ineq. Pure Appl. Math., **4**(4) (2003), 73.
[29] P. O. Mohammed, I. Brevik, *A new version of the Hermite-Hadamard inequality for Riemann-Liouville fractional integrals,* Symmetry, **12**(4) (2020), 610.
[30] H. Ǒgülmüs, M. Z. Sarikaya, *Hermite-Hadamard-Mercer type inequalities for fractional integrals,* Filomat, **35**(7) (2021), 2425–2436.
[31] G. Rahman, K. S. Nisar and T. Abdeljawad, *Certain Hadamard proportional fractional integral inequalities,* Mathematics, **8**(4) (2020), 504.
[32] S. Rashid, T. Abdeljawad, F. Jarad and M. A. Noor, *Some estimates for generalized Riemann-Liouville fractional integrals of exponentially convex functions and their applications,* Mathematics, **7**(9) (2019), 807.
[33] M. Z. Sarikaya, E. Set, H. Yaldiz and N. Basak, *Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities,* Math. Comput. Model., **57**(9-10) (2013), 2403–2407.
[34] E. Set, B. Çelik, *On generalizations related to the left side of Fejér's inequality via fractional integral operator,* Miskolc Math. Notes, **18**(2) (2017), 1043–1057.
[35] E. Set, J. Choi and A. Gözpinar, *Hermite-Hadamard type inequalities involving nonlocal conformable fractional integrals,* Malays. J. Math. Sci., **15** (2021), 33–43.
[36] E. Set, A. Gözpinar and S. I. Butt, *A study on Hermite—Hadamard-type inequalities via new fractional conformable integrals,* Asian-Eur. J. Math., **14**(2) (2018), 21500169.

[37] E. Set, I. Ican, M. Z. Sarikaya and E.M. Ozdemir, *On new inequalities of Hermite–Hadamard–Fejér type for convex functions via fractional integrals,* Appl. Math. Comput., **259** (2015), 875–881.
[38] M. Sitthiwirattham, M. A. Vivas-Cortez, M. A. Ali, H. Budak and I. Avci, *A study of fractional Hermite–Hadamard–Mercer inequalities for differentiable functions,* Fractals, **32**(2) (2024), 2440016.
[39] W. S. Wang, *Some properties of k-gamma and k-beta functions,* ITM Web Conf., **7** (2016), 07003.

[1]Department of Mathematics,
Berhampur University,
Berhampur,760007,
Odisha, India
Email address: yashobant.9090@gmail.com
Email address: mms.math@buodisha.edu.in

[2]Department of Mathematics,
Faculty of Science and Arts,
Kocaeli University,
Kocaeli 41001, Türkiye
Email address: hsyn.budak@gmail.com