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EXTENSIONS OF BULLEN-TYPE INEQUALITIES FOR DIFFERENT CLASSES OF CONVEX FUNCTIONS

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Abstract. In this study, we obtain several novel extensions of Bullen-type integral inequalities for differentiable functions whose second derivatives satisfy various convexity conditions. Using a fundamental identity related to Bullen-type inequalities, along with the power-mean inequality, the Hölder inequality, the Hölder-İşcan inequality, and the Young inequality, we obtain new estimates for functions whose second derivatives, in absolute value, belong to the $s-$convex, quasi$-$convex, and $\eta-$convex function classes. The obtained results generalize and improve upon several known inequalities available in the literature. Furthermore, several results and special cases are presented, showing that a set of previously formulated inequalities can be obtained as specific examples of our findings.

1. Introduction

Convexity theory plays a fundamental role in modern mathematical analysis and optimization theory due to its extensive applications in inequalities, numerical analysis, economics, engineering, and applied sciences. Let's begin by recalling the classic definition of convexity.

Definition 1.1. [3] Let I be a convex set on $\mathbb{R}$. The function $F : I \to \mathbb{R}$ is called convex on I if it satisfies the following inequality:

$$F(\tau\mu + (1-\tau)\nu) \leq \tau F(\mu) + (1-\tau)F(\nu)$$

for all $(\mu, \nu) \in I$ and $\tau \in [0,1]$. The function F is called concave if $(-F)$ is convex.

In [2], Bullen proved the following inequality which is known as Bullen's inequality for convex function:

Let $F : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$\frac{1}{\nu-\mu}\int_{\mu}^{\nu} F(\tau)d\tau \leq \frac{1}{2}\left[F\left(\frac{\mu+\nu}{2}\right) + \frac{F(\mu)+F(\nu)}{2}\right].$$

Key words and phrases. Bullen type inequalities, $s-$convex function, quasi$-$convex function, $\eta-$convex function, Hölder inequality, Hölder-İscan inequality, Power-mean inequality, Young inequaliy.

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In particular, convex functions provide a powerful framework for establishing integral inequalities of Hermite–Hadamard, Bullen, Ostrowski, Simpson, and Jensen types. Because of their geometric structure and analytical properties, convexity notions have been generalized in various directions in order to obtain more refined estimates and wider classes of inequalities. Among these generalizations, $s-$convexity, quasi-convexity, and η-convexity have attracted considerable attention in recent years. These concepts extend the classical convexity structure and enable the derivation of new integral inequalities and approximation results for broader families of functions.

There are many different classes of convexity in the literature. The three types of convexity that will be used in this article are as follows, respectivelty.

Definition 1.2. [1] A function $\digamma : \mathbb{R}^+ \to \mathbb{R}$, is said to be $s-$convex in the second sense class as follows

$$\digamma(\tau\mu + (1-\tau)\nu) \leq \tau^s \digamma(\mu) + (1-\tau)^s \digamma(\nu) \tag{1.1}$$

for all $\tau \in [0,1]$, $\mu, \nu \geq 0$ and for some fixed $s \in (0,1]$.

Definition 1.3. [5] The function $\digamma : I \to \mathbb{R}$ is called a quasi-convex if the following inequality

$$\digamma(\tau\mu + (1-\tau)\nu) \leq \max\{\digamma(\mu), \digamma(\nu)\}$$

holds for all $\mu, \nu \in I$ and $\tau \in [0,1]$.

Definition 1.4. [4] The function $\digamma : I \to \mathbb{R}$ is said to be $\eta-$convex (or convex with respect to η) if the inequality

$$\digamma(\tau\mu + (1-\tau)\nu) \leq \digamma(\nu) + \tau\eta(\digamma(\mu), \digamma(\nu))$$

hold for all $\mu, \nu \in I$, $\tau \in [0,1]$, and η is defined by $\eta : \digamma(I) \times \digamma(I) \to \mathbb{R}$.

It is well known that every convex function is quasi-convex; however, the converse is not necessarily true. Moreover, $s-$convexity and $\eta-$convexity provide useful extensions of classical convexity and have been widely employed in the study of integral inequalities and fractional integral operators. Motivated by these developments, in this paper we establish several new inequalities associated with Lemma 2.1 by using power-mean, Hölder, Hölder–İşcan and Young inequalities under different generalized convexity assumptions.

To present our main results, the following well-known definition will also be needed throughout the paper.

Definition 1.5. [8, Section 1.1] The Euler Beta function is defined by

$$\mathcal{B}(\alpha, \beta) = \int_0^1 \tau^{\alpha-1}(1-\tau)^{\beta-1} d\tau, \qquad \alpha, \beta > 0.$$

2. Main Results

In [6], Sarıkaya established the following identity for differentiable function:

Lemma 2.1. *Let $F : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, the interior of the interval, where $\mu, \nu \in I^\circ$ with $\mu < \nu$, and suppose that $F'' \in L[\mu, \nu]$. Then the following identity holds:*

$$\frac{1}{2}\int_\mu^\nu \kappa(\varkappa,\tau)F''(\tau)d\tau = F(\varkappa) + \frac{F(\mu)+F(\nu)}{2} - \left[\frac{1}{\varkappa-\mu}\int_\mu^\varkappa F(\tau)d\tau + \frac{1}{\nu-\varkappa}\int_\varkappa^\nu F(\tau)d\tau\right] \tag{2.1}$$

where

$$\kappa(\varkappa,\tau) = \begin{cases} \frac{1}{\varkappa-\mu}(\varkappa-\tau)(\tau-\mu) & \text{for} \quad \mu \le \tau \le \varkappa, \\ \\ \frac{1}{\nu-\varkappa}(\tau-\varkappa)(\nu-\tau) & \text{for} \quad \varkappa \le \tau \le \nu. \end{cases}$$

Theorem 2.1. *Under the assumptions of Lemma 2.1, if $|F''|^q$ is $s-$convex in the second sense on $[\mu, \nu]$ for some $q \ge 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|F(\varkappa) + \frac{F(\mu)+F(\nu)}{2} - \left[\frac{1}{\varkappa-\mu}\int_\mu^\varkappa F(\tau)d\tau + \frac{1}{\nu-\varkappa}\int_\varkappa^\nu F(\tau)d\tau\right]\right| \\
&\le \frac{1}{2}\left(\frac{(\varkappa-\mu)^2+(\nu-\varkappa)^2}{6}\right)^{1-\frac{1}{q}} \\
&\quad\times\left[\frac{(\varkappa-\mu)^2}{(s+2)(s+3)}\left(|F''(\varkappa)|^q+|F''(\mu)|^q\right) + \frac{(\nu-\varkappa)^2}{(s+2)(s+3)}\left(|F''(\nu)|^q+|F''(\varkappa)|^q\right)\right]^{\frac{1}{q}}.
\end{aligned}$$

Proof. Taking the absolute value of identity (2.1) and applying power-mean inequality, together with the the $s-$convexity in the second sense of $|F''|^q$ we obtain

$$\begin{aligned}
&\left|F(\varkappa) + \frac{F(\mu)+F(\nu)}{2} - \left[\frac{1}{\varkappa-\mu}\int_\mu^\varkappa F(\tau)d\tau + \frac{1}{\nu-\varkappa}\int_\varkappa^\nu F(\tau)d\tau\right]\right| \\
&\le \frac{1}{2}\left(\frac{1}{\varkappa-\mu}\int_\mu^\varkappa(\varkappa-\tau)(\tau-\mu)d\tau + \frac{1}{\nu-\varkappa}\int_\varkappa^\nu(\tau-\varkappa)(\nu-\tau)d\tau\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\frac{1}{\varkappa-\mu}\int_\mu^\varkappa(\varkappa-\tau)(\tau-\mu)|F''(\tau)|^q d\tau + \frac{1}{\nu-\varkappa}\int_\varkappa^\nu(\tau-\varkappa)(\nu-\tau)|F''(\tau)|^q d\tau\right)^{\frac{1}{q}} \\
&\le \frac{1}{2}\left(\frac{(\varkappa-\mu)^2+(\nu-\varkappa)^2}{6}\right)^{1-\frac{1}{q}} \\
&\quad\times\left[\frac{1}{\varkappa-\mu}\int_\mu^\varkappa(\varkappa-\tau)(\tau-\mu)\left(\left(\frac{\tau-\mu}{\varkappa-\mu}\right)^s|F''(\varkappa)|^q + \left(\frac{\varkappa-\tau}{\varkappa-\mu}\right)^s|F''(\mu)|^q\right)d\tau\right. \\
&\quad\left.+\frac{1}{\nu-\varkappa}\int_\varkappa^\nu(\tau-\varkappa)(\nu-\tau)\left(\left(\frac{\tau-\varkappa}{\nu-\varkappa}\right)^s|F''(\nu)|^q + \left(\frac{\nu-\tau}{\nu-\varkappa}\right)^s|F''(\varkappa)|^q\right)d\tau\right]^{\frac{1}{q}} \\
&= \frac{1}{2}\left(\frac{(\varkappa-\mu)^2+(\nu-\varkappa)^2}{6}\right)^{1-\frac{1}{q}} \\
&\quad\times\left[\frac{(\varkappa-\mu)^2}{(s+2)(s+3)}\left(|F''(\varkappa)|^q+|F''(\mu)|^q\right) + \frac{(\nu-\varkappa)^2}{(s+2)(s+3)}\left(|F''(\nu)|^q+|F''(\varkappa)|^q\right)\right]^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. □

Corollary 2.1. *If we choose $q = 1$ in Theorem 2.1, then we obtain*

$$\left|F(\varkappa) + \frac{F(\mu)+F(\nu)}{2} - \left[\frac{1}{\varkappa-\mu}\int_{\mu}^{\varkappa} F(\tau)d\tau + \frac{1}{\nu-\varkappa}\int_{\varkappa}^{\nu} F(\tau)d\tau\right]\right|$$
$$\leq \frac{[(\varkappa-\mu)^2 + (\nu-\varkappa)^2]|F''(\varkappa)| + (\varkappa-\mu)^2|F''(\mu)| + (\nu-\varkappa)^2|F''(\nu)|}{2(s+2)(s+3)}.$$

Remark 2.1. If we choose $s = 1$ in Corollary 2.1, then we obtain [6, Theorem 2.3].

Corollary 2.2. *If we choose $\varkappa = \frac{\mu+\nu}{2}$ in Corollary 2.1, then we obtain*

$$\left|\frac{1}{2}\left[F\left(\frac{\mu+\nu}{2}\right) + \frac{F(\mu)+F(\nu)}{2}\right] - \frac{1}{\nu-\mu}\int_{\mu}^{\nu} F(\tau)d\tau\right|$$
$$\leq \frac{(\nu-\mu)^2}{4(s+2)(s+3)}\left(\frac{|F''(\mu)| + |F''(\nu)|}{2}\right).$$

Remark 2.2. If we choose $s = 1$ in Corollary 2.2, then we obtain [7, Theorem 4].

Theorem 2.2. *Under the assumptions of Lemma 2.1, if $|F''|^q$ is $s-$convex in the second sense on $[\mu,\nu]$ for some $q > 1$, then the following inequality holds:*

$$\left|F(\varkappa) + \frac{F(\mu)+F(\nu)}{2} - \left[\frac{1}{\varkappa-\mu}\int_{\mu}^{\varkappa} F(\tau)d\tau + \frac{1}{\nu-\varkappa}\int_{\varkappa}^{\nu} F(\tau)d\tau\right]\right| \tag{2.2}$$
$$\leq \frac{1}{2}[(\varkappa-\mu)^{p+1} + (\nu-\varkappa)^{p+1}]^{\frac{1}{p}}\mathcal{B}^{\frac{1}{p}}(p+1,p+1)(\nu-\mu)^{\frac{1}{q}}\left(\frac{|F''(\mu)|^q + |F''(\nu)|^q}{s+1}\right)^{\frac{1}{q}},$$

for all $\varkappa \in (\mu,\nu)$ and $\frac{1}{p} + \frac{1}{q}$.

Proof. Taking the absolute value of identity (2.1) and applying Hölder's integral inequality, together with the the $s-$convexity in the second sense of $|f''|^q$ we obtain

$$\left|F(\varkappa) + \frac{F(\mu)+F(\nu)}{2} - \left[\frac{1}{\varkappa-\mu}\int_{\mu}^{\varkappa} F(\tau)d\tau + \frac{1}{\nu-\varkappa}\int_{\varkappa}^{\nu} F(\tau)d\tau\right]\right|$$
$$\leq \frac{1}{2}\left(\frac{1}{(\varkappa-\mu)^p}\int_{\mu}^{\varkappa}(\varkappa-\tau)^p(\tau-\mu)^p d\tau + \frac{1}{(\nu-\varkappa)^p}\int_{\varkappa}^{\nu}(\tau-\varkappa)^p(\nu-\tau)^p d\tau\right)^{\frac{1}{p}}$$
$$\times\left(\int_{\mu}^{\nu}\left[\left(\frac{\tau-\mu}{\nu-\mu}\right)^s|F''(\nu)|^q + \left(\frac{\nu-\tau}{\nu-\mu}\right)^s|F''(\mu)|^q\right]d\tau\right)^{\frac{1}{q}}$$
$$= \frac{1}{2}[(\varkappa-\mu)^{p+1} + (\nu-\varkappa)^{p+1}]^{\frac{1}{p}}\mathcal{B}^{\frac{1}{p}}(p+1,p+1)(\nu-\mu)^{\frac{1}{q}}\left(\frac{|F''(\mu)|^q + |F''(\nu)|^q}{s+1}\right)^{\frac{1}{q}}.$$

This completes the proof. □

Remark 2.3. If we choose $s = 1$ in Theorem 2.2, then we obtain [6, Theorem 2.5].

Theorem 2.3. *Under the assumptions of Lemma 2.1, if $|F''|^q$ is $s-$convex on $[\mu,\nu]$, for some $q > 1$, then the following inequality holds:*

$$\left|F(\varkappa) + \frac{F(\mu)+F(\nu)}{2} - \left[\frac{1}{\varkappa-\mu}\int_{\mu}^{\varkappa} F(\tau)d\tau + \frac{1}{\nu-\varkappa}\int_{\varkappa}^{\nu} F(\tau)d\tau\right]\right|$$

$$\leq \frac{1}{2(\nu-\mu)}[\mathcal{A}_1{}^{\frac{1}{p}}\mathcal{B}_1{}^{\frac{1}{q}} + \mathcal{A}_2{}^{\frac{1}{p}}\mathcal{B}_2{}^{\frac{1}{q}}],$$

where

$$\mathcal{A}_1 = \frac{1}{2}\mathcal{B}(p+1,p+1)[((\nu-\mu)+(\nu-\varkappa))(\varkappa-\mu)^{p+1} + (\nu-\varkappa)^{p+2}],$$
$$\mathcal{A}_2 = \frac{1}{2}\mathcal{B}(p+1,p+1)[(\varkappa-\mu)^{p+2} + (\nu+\varkappa-2\mu)(\nu-\varkappa)^{p+1}],$$
$$\mathcal{B}_1 = (\nu-\mu)^2\left(\frac{|F''(\mu)|^q}{s+2} + \frac{|F''(\nu)|^q}{(s+1)(s+2)}\right),$$
$$\mathcal{B}_2 = (\nu-\mu)^2\left(\frac{|F''(\nu)|^q}{s+2} + \frac{|F''(\mu)|^q}{(s+1)(s+2)}\right).$$

Proof. Taking the absolute value of identity (2.1) and applying Hölder-İscan integral inequality, together with the $s-$convexity in the second sense of $|F''|^q$ we obtain

$$\begin{aligned}
&\left|F(\varkappa) + \frac{F(\mu)+F(\nu)}{2} - \left[\frac{1}{\varkappa-\mu}\int_\mu^\varkappa F(\tau)d\tau + \frac{1}{\nu-\varkappa}\int_\varkappa^\nu F(\tau)d\tau\right]\right| \\
&\leq \frac{1}{2(\nu-\mu)}\left(\int_\mu^\nu (\nu-\tau)|\kappa(\varkappa,\tau)|^p d\tau\right)^{\frac{1}{p}}\left(\int_\mu^\nu (\nu-\tau)|F''(\tau)|^q d\tau\right)^{\frac{1}{q}} \\
&\quad + \frac{1}{2(\nu-\mu)}\left(\int_\mu^\nu (\tau-\mu)|\kappa(\varkappa,\tau)|^p d\tau\right)^{\frac{1}{p}}\left(\int_\mu^\nu (\tau-\mu)|F''(\tau)|^q d\tau\right)^{\frac{1}{q}} \\
&\leq \frac{1}{2(\nu-\mu)}\left(\int_\mu^\varkappa (\nu-\tau)\left(\frac{(\varkappa-\tau)(\tau-\mu)}{\varkappa-\mu}\right)^p d\tau + \int_\varkappa^\nu (\nu-\tau)\left(\frac{(\tau-\varkappa)(\nu-\tau)}{\nu-\varkappa}\right)^p d\tau\right)^{\frac{1}{p}} \\
&\quad \times \left(\int_\mu^\nu (\nu-\tau)\left[\left(\frac{\tau-\mu}{\nu-\mu}\right)^s |F''(\nu)|^q + \left(\frac{\nu-\tau}{\nu-\mu}\right)^s |F''(\mu)|^q\right] d\tau\right)^{\frac{1}{q}} \\
&\quad + \frac{1}{2(\nu-\mu)}\left(\int_\mu^\varkappa (\tau-\mu)\left(\frac{(\varkappa-\tau)(\tau-\mu)}{\varkappa-\mu}\right)^p d\tau + \int_\varkappa^\nu (\tau-\mu)\left(\frac{(\tau-\varkappa)(\nu-\tau)}{\nu-\varkappa}\right)^p d\tau\right)^{\frac{1}{p}} \\
&\quad \times \left(\int_\mu^\nu (\tau-\mu)\left[\left(\frac{\tau-\mu}{\nu-\mu}\right)^s |F''(\nu)|^q + \left(\frac{\nu-\tau}{\nu-\mu}\right)^s |F''(\mu)|^q\right] d\tau\right)^{\frac{1}{q}} \\
&= \frac{1}{2(\nu-\mu)}\left\{\left(\left(\nu - \frac{\mu+\varkappa}{2}\right)(\varkappa-\mu)^{p+1}\mathcal{B}(p+1,p+1) + \frac{1}{2}(\nu-\varkappa)^{p+2}\mathcal{B}(p+1,p+1)\right)^{\frac{1}{p}}\right. \\
&\quad \left.\times (\nu-\mu)^2\left(\frac{|F''(\mu)|^q}{s+2} + \frac{|F''(\nu)|^q}{(s+1)(s+2)}\right)^{\frac{1}{q}}\right\} \\
&\quad + \frac{1}{2(\nu-\mu)}\left\{\left(\frac{1}{2}(\varkappa-\mu)^{p+2}\mathcal{B}(p+1,p+1) + \left(\frac{\nu+\varkappa}{2} - \mu\right)(\nu-\varkappa)^{p+1}\mathcal{B}(p+1,p+1)\right)^{\frac{1}{p}}\right. \\
&\quad \left.\times (\nu-\mu)^2\left(\frac{|F''(\nu)|^q}{s+2} + \frac{|F''(\mu)|^q}{(s+1)(s+2)}\right)^{\frac{1}{q}}\right\} \\
&= \frac{1}{2(\nu-\mu)}\left\{\left(\frac{1}{2}\mathcal{B}(p+1,p+1)[((\nu-\mu)+(\nu-\varkappa))(\varkappa-\mu)^{p+1} + (\nu-\varkappa)^{p+2}]\right)^{\frac{1}{p}}\right.
\end{aligned}$$

$$\times (\nu-\mu)^2\left(\frac{|F''(\mu)|^q}{s+2}+\frac{|F''(\nu)|^q}{(s+1)(s+2)}\right)^{\frac{1}{q}}\Bigg\}$$
$$+\frac{1}{2(\nu-\mu)}\Bigg\{\left(\frac{1}{2}\mathcal{B}(p+1,p+1)[(\varkappa-\mu)^{p+2}+(\nu+\varkappa-2\mu)(\nu-\varkappa)^{p+1}]\right)^{\frac{1}{p}}$$
$$\times (\nu-\mu)^2\left(\frac{|F''(\nu)|^q}{s+2}+\frac{|F''(\mu)|^q}{(s+1)(s+2)}\right)^{\frac{1}{q}}\Bigg\}.$$

This completes the proof. □

Corollary 2.3. *If we choose $s=1$ in Theorem 2.3, then we obtain*

$$\left|F(\varkappa)+\frac{F(\mu)+F(\nu)}{2}-\left[\frac{1}{\varkappa-\mu}\int_\mu^\varkappa F(\tau)d\tau+\frac{1}{\nu-\varkappa}\int_\varkappa^\nu F(\tau)d\tau\right]\right|$$
$$\le\frac{1}{2(\nu-\mu)}\Bigg\{\left(\frac{1}{2}\mathcal{B}(p+1,p+1)[((\nu-\mu)+(\nu-\varkappa))(\varkappa-\mu)^{p+1}+(\nu-\varkappa)^{p+2}]\right)^{\frac{1}{p}}$$
$$\times(\nu-\mu)^2\left(\frac{2|F''(\mu)|^q+|F''(\nu)|^q}{6}\right)^{\frac{1}{q}}\Bigg\}$$
$$+\frac{1}{2(\nu-\mu)}\Bigg\{\left(\frac{1}{2}\mathcal{B}(p+1,p+1)[(\varkappa-\mu)^{p+2}+(\nu+\varkappa-2\mu)(\nu-\varkappa)^{p+1}]\right)^{\frac{1}{p}}$$
$$\times(\nu-\mu)^2\left(\frac{2|F''(\nu)|^q+|F''(\mu)|^q}{6}\right)^{\frac{1}{q}}\Bigg\}.$$

Theorem 2.4. *Under the assumptions of Lemma 2.1, if $|F''|^q$ is $s-$convex on $[\mu,\nu]$, for some $q>1$, then the following inequality holds:*

$$\left|F(\varkappa)+\frac{F(\mu)+F(\nu)}{2}-\left[\frac{1}{\varkappa-\mu}\int_\mu^\varkappa F(\tau)d\tau+\frac{1}{\nu-\varkappa}\int_\varkappa^\nu F(\tau)d\tau\right]\right| \tag{2.3}$$
$$\le\frac{1}{2p}[(\varkappa-\mu)^{p+1}+(\nu-\varkappa)^{p+1}]\mathcal{B}(p+1,p+1)+\frac{\nu-\mu}{2q}\left(\frac{|F''(\mu)|^q+|F''(\nu)|^q}{s+1}\right),$$

for all $\varkappa\in(\mu,\nu)$.

Proof. By using the identity that is given in Lemma 2.1, we have

$$\left|F(\varkappa)+\frac{F(\mu)+F(\nu)}{2}-\left[\frac{1}{\varkappa-\mu}\int_\mu^\varkappa F(\tau)d\tau+\frac{1}{\nu-\varkappa}\int_\varkappa^\nu F(\tau)d\tau\right]\right|$$
$$\le\frac{1}{2(\varkappa-\mu)}\int_\mu^\varkappa(\varkappa-\tau)(\tau-\mu)|F''(\tau)|d\tau+\frac{1}{2(\nu-\varkappa)}\int_\varkappa^\nu(\tau-\varkappa)(\nu-\tau)|F''(\tau)|d\tau.$$

By using the Young inequality as $xy\le\frac{1}{p}x^p+\frac{1}{q}y^q$,

$$\left|F(\varkappa)+\frac{F(\mu)+F(\nu)}{2}-\left[\frac{1}{\varkappa-\mu}\int_\mu^\varkappa F(\tau)d\tau+\frac{1}{\nu-\varkappa}\int_\varkappa^\nu F(\tau)d\tau\right]\right|$$
$$\le\frac{1}{2}\left(\frac{1}{p}\int_\mu^\nu|\kappa(\varkappa,\tau)|^pd\tau+\frac{1}{q}\int_\mu^\nu|F''(\tau)|^qd\tau\right)$$
$$=\frac{1}{2p}\left(\frac{1}{(\varkappa-\mu)^p}\int_\mu^\varkappa(\varkappa-\tau)^p(\tau-\mu)^pd\tau+\frac{1}{(\nu-\varkappa)^p}\int_\varkappa^\nu(\tau-\varkappa)^p(\nu-\tau)^pd\tau\right)$$

$$+\frac{1}{2q}\int_{\mu}^{\nu}|F''(\tau)|^q d\tau.$$

By using $s-$convexity in the second sense of $|F''(\tau)|^q$ and by a simple computation, we have the desired result. □

Corollary 2.4. *If we choose $s=1$ in Theorem 2.4, then we obtain*

$$\left|F(\varkappa)+\frac{F(\mu)+F(\nu)}{2}-\left[\frac{1}{\varkappa-\mu}\int_{\mu}^{\varkappa}F(\tau)d\tau+\frac{1}{\nu-\varkappa}\int_{\varkappa}^{\nu}F(\tau)d\tau\right]\right| \tag{2.4}$$
$$\leq\frac{1}{2p}[(\varkappa-\mu)^{p+1}+(\nu-\varkappa)^{p+1}]\mathcal{B}(p+1,p+1)+\frac{\nu-\mu}{4q}\left(|F''(\mu)|^q+|F''(\nu)|^q\right).$$

Theorem 2.5. *Under the assumptions of Lemma 2.1, if $|F''|$ is quasi$-$convex on $[\mu,\nu]$ for some $q\geq 1$, then the following inequalty holds:*

$$\left|F(\varkappa)+\frac{F(\mu)+F(\nu)}{2}-\left[\frac{1}{\varkappa-\mu}\int_{\mu}^{\varkappa}F(\tau)d\tau+\frac{1}{\nu-\varkappa}\int_{\varkappa}^{\nu}F(\tau)d\tau\right]\right|$$
$$\leq\frac{1}{2}\left(\frac{(\varkappa-\mu)^2+(\nu-\varkappa)^2}{6}\right)^{1-\frac{1}{q}}$$
$$\times\left(\frac{(\varkappa-\mu)^2\max\{|F''(\mu)|^q,|F''(\varkappa)|^q\}}{6}+\frac{(\nu-\varkappa)^2\max\{\ |F''(\varkappa)|^q,|F''(\nu)|^q\}}{6}\right)^{\frac{1}{q}}.$$

Proof. Since $|F''|$ is quasi$-$convex on $[\mu,\nu]$, we have

$$|F''(\tau)|\leq\max\{|F''(\mu)|,|F''(\varkappa)|\}\quad\text{for}\quad\tau\in[\mu,\varkappa]$$

and

$$|F''(\tau)|\leq\max\{|F''(\varkappa)|,|F''(\nu)|\},\quad\text{for}\quad\tau\in[\varkappa,\nu].$$

Taking the absolute value of identity (2.1) and applying power-mean inequality, together with the the quasi$-$convexity of $|F''|^q$ we obtain

$$\left|F(\varkappa)+\frac{F(\mu)+F(\nu)}{2}-\left[\frac{1}{\varkappa-\mu}\int_{\mu}^{\varkappa}F(\tau)d\tau+\frac{1}{\nu-\varkappa}\int_{\varkappa}^{\nu}F(\tau)d\tau\right]\right|$$
$$\leq\frac{1}{2}\left(\frac{1}{\varkappa-\mu}\int_{\mu}^{\varkappa}(\varkappa-\tau)(\tau-\mu)d\tau+\frac{1}{\nu-\varkappa}\int_{\varkappa}^{\nu}(\tau-\varkappa)(\nu-\tau)d\tau\right)^{1-\frac{1}{q}}$$
$$\times\left(\frac{1}{\varkappa-\mu}\int_{\mu}^{\varkappa}(\varkappa-\tau)(\tau-\mu)|F''(\tau)|^q d\tau+\frac{1}{\nu-\varkappa}\int_{\varkappa}^{\nu}(\tau-\varkappa)(\nu-\tau)|F''(\tau)|^q d\tau\right)^{\frac{1}{q}}$$
$$\leq\frac{1}{2}\left(\frac{(\varkappa-\mu)^2+(\nu-\varkappa)^2}{6}\right)^{1-\frac{1}{q}}$$
$$\times\left(\frac{\max\{|F''(\mu)|^q,|F''(\varkappa)|^q\}}{\varkappa-\mu}\int_{\mu}^{\varkappa}(\varkappa-\tau)(\tau-\mu)d\tau\right.$$
$$\left.+\frac{\max\{\ |F''(\varkappa)|^q,|F''(\nu)|^q\}}{\nu-\varkappa}\int_{\varkappa}^{\nu}(\tau-\varkappa)(\nu-\tau)d\tau\right)^{\frac{1}{q}}$$

$$= \frac{1}{2}\left(\frac{(\varkappa-\mu)^2+(\nu-\varkappa)^2}{6}\right)^{1-\frac{1}{q}}$$
$$\times\left(\frac{(\varkappa-\mu)^2\max\{|F''(\mu)|^q,|F''(\varkappa)|^q\}}{6}+\frac{(\nu-\varkappa)^2\max\{\ |F''(\varkappa)|^q,|F''(\nu)|^q\}}{6}\right)^{\frac{1}{q}}.$$

This completes the proof. □

Theorem 2.6. *Under the assumptions of Lemma 2.1, if $|F''|^q$ is quasi−convex on $[\mu,\nu]$, for some $q>1$, then the following inequality holds:*

$$\left|F(\varkappa)+\frac{F(\mu)+F(\nu)}{2}-\left[\frac{1}{\varkappa-\mu}\int_\mu^\varkappa F(\tau)d\tau+\frac{1}{\nu-\varkappa}\int_\varkappa^\nu F(\tau)d\tau\right]\right|$$
$$\leq\frac{1}{2}[(\varkappa-\mu)^{p+1}+(\nu-\varkappa)^{p+1}]^{\frac{1}{p}}\mathcal{B}^{\frac{1}{p}}(p+1,p+1)(\nu-\mu)^{\frac{1}{q}}\left(\max\{|F''(\mu)|^q,|F''(\nu)|^q\}\right)^{\frac{1}{q}}$$

for all $\varkappa\in(\mu,\nu)$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Taking the absolute value of identity (2.1) and applying Hölder's integral inequality, together with the the quasi−convexity of $|f''|^q$ we obtain

$$\left|F(\varkappa)+\frac{F(\mu)+F(\nu)}{2}-\left[\frac{1}{\varkappa-\mu}\int_\mu^\varkappa F(\tau)d\tau+\frac{1}{\nu-\varkappa}\int_\varkappa^\nu F(\tau)d\tau\right]\right|$$
$$\leq\frac{1}{2}\left(\int_\mu^\nu|\kappa(\varkappa,\tau)|^p d\tau\right)^{\frac{1}{p}}\left(\int_\mu^\nu|F''(\tau)|^q d\tau\right)^{\frac{1}{q}},\qquad\frac{1}{p}+\frac{1}{q}=1.$$
$$\leq\frac{1}{2}\left(\frac{1}{(\varkappa-\mu)^p}\int_\mu^\varkappa(\varkappa-\tau)^p(\tau-\mu)^p d\tau+\frac{1}{(\nu-\varkappa)^p}\int_\varkappa^\nu(\tau-\varkappa)^p(\nu-\tau)^p d\tau\right)^{\frac{1}{p}}$$
$$\times\left(\int_\mu^\nu\max\{|F''(\mu)|^q,|F''(\nu)|^q\}d\tau\right)^{\frac{1}{q}}$$
$$=\frac{1}{2}[(\varkappa-\mu)^{p+1}+(\nu-\varkappa)^{p+1}]^{\frac{1}{p}}\mathcal{B}^{\frac{1}{p}}(p+1,p+1)(\nu-\mu)^{\frac{1}{q}}\left(\max\{|F''(\mu)|^q,|F''(\nu)|^q\}\right)^{\frac{1}{q}}.$$

This completes the proof. □

Theorem 2.7. *Under the assumptions of Lemma 2.1, if $|F''|^q$ is quasi−convex on $[\mu,\nu]$, for some $q>1$, then the following inequality holds:*

$$\left|F(\varkappa)+\frac{F(\mu)+F(\nu)}{2}-\left[\frac{1}{\varkappa-\mu}\int_\mu^\varkappa F(\tau)d\tau+\frac{1}{\nu-\varkappa}\int_\varkappa^\nu F(\tau)d\tau\right]\right|$$
$$\leq\frac{1}{2(\nu-\mu)}[\mathcal{A}_1{}^{\frac{1}{p}}+\mathcal{A}_2{}^{\frac{1}{p}}]\left(\frac{(\nu-\mu)^2}{2}\max\{|F''(\nu)|^q,|F''(\mu)|^q\}\right)^{\frac{1}{q}}$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are as in Theorem 2.3.

Proof. Taking the absolute value of identity (2.1) and applying Hölder-İscan integral inequality, together with the the quasi−convexity of $|F''|^q$ we obtain

$$\left|F(\varkappa)+\frac{F(\mu)+F(\nu)}{2}-\left[\frac{1}{\varkappa-\mu}\int_\mu^\varkappa F(\tau)d\tau+\frac{1}{\nu-\varkappa}\int_\varkappa^\nu F(\tau)d\tau\right]\right|$$

$$
\begin{aligned}
&\leq \frac{1}{2(\nu-\mu)}\left(\int_{\mu}^{\nu}(\nu-\tau)|\kappa(\varkappa,\tau)|^{p}d\tau\right)^{\frac{1}{p}}\left(\int_{\mu}^{\nu}(\nu-\tau)|F''(\tau)|^{q}d\tau\right)^{\frac{1}{q}}\\
&\quad+\frac{1}{2(\nu-\mu)}\left(\int_{\mu}^{\nu}(\tau-\mu)|\kappa(\varkappa,\tau)|^{p}d\tau\right)^{\frac{1}{p}}\left(\int_{\mu}^{\nu}(\tau-\mu)|F''(\tau)|^{q}d\tau\right)^{\frac{1}{q}}\\
&\leq \frac{1}{2(\nu-\mu)}\left(\int_{\mu}^{\varkappa}(\nu-\tau)\left(\frac{(\varkappa-\tau)(\tau-\mu)}{\varkappa-\mu}\right)^{p}d\tau+\int_{\varkappa}^{\nu}(\nu-\tau)\left(\frac{(\tau-\varkappa)(\nu-\tau)}{\nu-\varkappa}\right)^{p}d\tau\right)^{\frac{1}{p}}\\
&\quad\times\left(\max\{|F''(\nu)|^{q},|F''(\mu)|^{q}\}\int_{\mu}^{\nu}(\nu-\tau)d\tau\right)^{\frac{1}{q}}\\
&\quad+\frac{1}{2(\nu-\mu)}\left(\int_{\mu}^{\varkappa}(\tau-\mu)\left(\frac{(\varkappa-\tau)(\tau-\mu)}{\varkappa-\mu}\right)^{p}d\tau+\int_{\varkappa}^{\nu}(\tau-\mu)\left(\frac{(\tau-\varkappa)(\nu-\tau)}{\nu-\varkappa}\right)^{p}d\tau\right)^{\frac{1}{p}}\\
&\quad\times\left(\max\{F''(\nu)|^{q},F''(\mu)|^{q}\}\int_{\mu}^{\nu}(\tau-\mu)d\tau\right)^{\frac{1}{q}}\\
&=\frac{1}{2(\nu-\mu)}\left\{\left(\frac{1}{2}\mathcal{B}(p+1,p+1)[((\nu-\mu)+(\nu-\varkappa))(\varkappa-\mu)^{p+1}+(\nu-\varkappa)^{p+2}]\right)^{\frac{1}{p}}\right.\\
&\quad\left.\times\left(\frac{\max\{|F''(\nu)|^{q},|F''(\mu)|^{q}\}(\nu-\mu)^{2}}{2}\right)^{\frac{1}{q}}\right\}\\
&\quad+\frac{1}{2(\nu-\mu)}\left\{\left(\frac{1}{2}\mathcal{B}(p+1,p+1)[(\varkappa-\mu)^{p+2}+(\nu+\varkappa-2\mu)(\nu-\varkappa)^{p+1}]\right)^{\frac{1}{p}}\right.\\
&\quad\left.\times\left(\frac{\max\{|F''(\nu)|^{q},|F''(\mu)|^{q}\}(\nu-\mu)^{2}}{2}\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

This completes the proof. □

Theorem 2.8. *Under the assumptions of Lemma 2.1, if $|F''|^{q}$ is quasi−convex on $[\mu,\nu]$, for some $q>1$, then the following inequality holds:*

$$
\begin{aligned}
&\left|F(\varkappa)+\frac{F(\mu)+F(\nu)}{2}-\left[\frac{1}{\varkappa-\mu}\int_{\mu}^{\varkappa}F(\tau)d\tau+\frac{1}{\nu-\varkappa}\int_{\varkappa}^{\nu}F(\tau)d\tau\right]\right|\\
&\leq\frac{1}{2p}[(\varkappa-\mu)^{p+1}+(\nu-\varkappa)^{p+1}]\mathcal{B}(p+1,p+1)+\frac{1}{2q}(\nu-\mu)^{\frac{1}{q}}\left(\max\{|F''(\mu)|^{q},|F''(\nu)|^{q}\}\right)^{\frac{1}{q}},
\end{aligned}
$$

for all $\varkappa\in(\mu,\nu)$.

Proof. By using the identity that is given in Lemma 2.1, we have

$$
\begin{aligned}
&\left|F(\varkappa)+\frac{F(\mu)+F(\nu)}{2}-\left[\frac{1}{\varkappa-\mu}\int_{\mu}^{\varkappa}F(\tau)d\tau+\frac{1}{\nu-\varkappa}\int_{\varkappa}^{\nu}F(\tau)d\tau\right]\right|\\
&\leq\frac{1}{2(\varkappa-\mu)}\int_{\mu}^{\varkappa}(\varkappa-\tau)(\tau-\mu)|F''(\tau)|d\tau+\frac{1}{2(\nu-\varkappa)}\int_{\varkappa}^{\nu}(\tau-\varkappa)(\nu-\tau)|F''(\tau)|d\tau.
\end{aligned}
$$

By using the Young inequality as $xy\leq\frac{1}{p}x^{p}+\frac{1}{q}y^{q}$,

$$
\left|F(\varkappa)+\frac{F(\mu)+F(\nu)}{2}-\left[\frac{1}{\varkappa-\mu}\int_{\mu}^{\varkappa}F(\tau)d\tau+\frac{1}{\nu-\varkappa}\int_{\varkappa}^{\nu}F(\tau)d\tau\right]\right|
$$

$$\leq \frac{1}{2}\left(\frac{1}{p}\int_{\mu}^{\nu}|\kappa(\varkappa,\tau)|^{p}d\tau+\frac{1}{q}\int_{\mu}^{\nu}|F''(\tau)|^{q}d\tau\right)$$
$$=\frac{1}{2p}\left(\frac{1}{(\varkappa-\mu)^{p}}\int_{\mu}^{\varkappa}(\varkappa-\tau)^{p}(\tau-\mu)^{p}d\tau+\frac{1}{(\nu-\varkappa)^{p}}\int_{\varkappa}^{\nu}(\tau-\varkappa)^{p}(\nu-\tau)^{p}d\tau\right)$$
$$+\frac{1}{2q}\int_{\mu}^{\nu}|F''(\tau)|^{q}d\tau.$$

By using quasi−convexity of $|F''(\tau)|^{q}$ and by a simple computation, we have the desired result. □

Theorem 2.9. *Under the assumptions of Lemma 2.1, if $|F''|$ is $\eta-$convex on $[\mu,\nu]$, then the following inequalty holds:*

$$\left|F(\varkappa)+\frac{F(\mu)+F(\nu)}{2}-\left[\frac{1}{\varkappa-\mu}\int_{\mu}^{\varkappa}F(\tau)d\tau+\frac{1}{\nu-\varkappa}\int_{\varkappa}^{\nu}F(\tau)d\tau\right]\right| \tag{2.5}$$
$$\leq\frac{|F''(\varkappa)|(\varkappa-\mu)^{2}}{12}+\frac{|F''(\mu)|(\nu-\varkappa)^{2}}{12}$$
$$+\frac{(\varkappa-\mu)^{2}}{24}\eta(|F''(\varkappa)|,|F''(\mu)|)+\frac{(\nu-\varkappa)^{2}}{24}\eta(|F''(\nu)|,|F''(\varkappa)|)$$

for all $\varkappa\in(\mu,\nu)$.

Proof. Since $|F''|$ is $\eta-$convex on $[\mu,\nu]$, we have

$$|F''(\tau)|\leq|F''(\mu)|+\left(\frac{\tau-\mu}{\varkappa-\mu}\right)\eta(|F''(\varkappa)|,|F''(\mu)|),\quad \text{for}\quad \tau\in[\mu,\varkappa]$$

and

$$|F''(\tau)|\leq|F''(\varkappa)|+\left(\frac{\tau-\varkappa}{\nu-\varkappa}\right)\eta(|F''(\nu)|,|F''(\varkappa)|),\quad \text{for}\quad \tau\in[\varkappa,\nu].$$

Taking the absolute value of both sides in equation (2.1), and applying the $\eta-$convexity estimates, we obtain

$$\left|F(\varkappa)+\frac{F(\mu)+F(\nu)}{2}-\left[\frac{1}{\varkappa-\mu}\int_{\mu}^{\varkappa}F(\tau)d\tau+\frac{1}{\nu-\varkappa}\int_{\varkappa}^{\nu}F(\tau)d\tau\right]\right|$$
$$\leq\frac{1}{2(\varkappa-\mu)}\int_{\mu}^{\varkappa}(\varkappa-\tau)(\tau-\mu)|F''(\tau)|d\tau+\frac{1}{2(\nu-\varkappa)}\int_{\varkappa}^{\nu}(\tau-\varkappa)(\nu-\tau)|F''(\tau)|d\tau$$
$$\leq\frac{1}{2(\varkappa-\mu)}\int_{\mu}^{\varkappa}(\varkappa-\tau)(\tau-\mu)\left[|F''(\mu)|+\left(\frac{\tau-\mu}{\varkappa-\mu}\right)\eta(|F''(\varkappa)|,|F''(\mu)|)\right]d\tau$$
$$+\frac{1}{2(\nu-\varkappa)}\int_{\varkappa}^{\nu}(\tau-\varkappa)(\nu-\tau)\left(|F''(\varkappa)|+\left(\frac{\tau-\varkappa}{\nu-\varkappa}\right)\eta(|F''(\nu)|,|F''(\varkappa)|)\right)d\tau.$$

Evaluating the integrals and simplifying yields:

$$\left|F(\varkappa)+\frac{F(\mu)+F(\nu)}{2}-\left[\frac{1}{\varkappa-\mu}\int_{\mu}^{\varkappa}F(\tau)d\tau+\frac{1}{\nu-\varkappa}\int_{\varkappa}^{\nu}F(\tau)d\tau\right]\right|$$
$$\leq\frac{|F''(\varkappa)|(\varkappa-\mu)^{2}}{12}+\frac{|F''(\mu)|(\nu-\varkappa)^{2}}{12}$$
$$+\frac{(\varkappa-\mu)^{2}}{24}\eta(|F''(\varkappa)|,|F''(\mu)|)+\frac{(\nu-\varkappa)^{2}}{24}\eta(|F''(\nu)|,|F''(\varkappa)|).$$

□

Remark 2.4. If we choose $\eta(F(x),F(y)) = F(x) - F(y)$ in Theorem 2.9, then we obtain [6, Theorem 2.3].

Theorem 2.10. *Under the assumptions of Lemma 2.1, if $|F''|^q$ is $\eta-$convex on $[\mu,\nu]$, for some $q>1$, then the following inequality holds:*

$$\left|F(\varkappa)+\frac{F(\mu)+F(\nu)}{2}-\left[\frac{1}{\varkappa-\mu}\int_{\mu}^{\varkappa}F(\tau)d\tau+\frac{1}{\nu-\varkappa}\int_{\varkappa}^{\nu}F(\tau)d\tau\right]\right| \tag{2.6}$$
$$\leq \frac{1}{2}[(\varkappa-\mu)^{p+1}+(\nu-\varkappa)^{p+1}]^{\frac{1}{p}}\mathcal{B}^{\frac{1}{p}}(p+1,p+1)(\nu-\mu)^{\frac{1}{q}}\left(|F''(\mu)|+\frac{1}{2}\eta(|F''(\nu)|,|F''(\mu)\right)^{\frac{1}{q}}$$

for all $\varkappa\in(\mu,\nu)$.

Proof. Taking the absolute value of identity (2.1) and applying Hölder's integral inequality, together with the the $\eta-$convexity of $|F''|^q$ we obtain

$$\left|F(\varkappa)+\frac{F(\mu)+F(\nu)}{2}-\left[\frac{1}{\varkappa-\mu}\int_{\mu}^{\varkappa}F(\tau)d\tau+\frac{1}{\nu-\varkappa}\int_{\varkappa}^{\nu}F(\tau)d\tau\right]\right|$$
$$\leq \frac{1}{2}\left(\int_{\mu}^{\nu}|\kappa(\varkappa,\tau)|^p d\tau\right)^{\frac{1}{p}}\left(\int_{\mu}^{\nu}|F''(\tau)|^q d\tau\right)^{\frac{1}{q}}, \qquad \frac{1}{p}+\frac{1}{q}=1.$$
$$\leq \frac{1}{2}\left(\frac{1}{(\varkappa-\mu)^p}\int_{\mu}^{\varkappa}(\varkappa-\tau)^p(\tau-\mu)^p d\tau+\frac{1}{(\nu-\varkappa)^p}\int_{\varkappa}^{\nu}(\tau-\varkappa)^p(\nu-\tau)^p d\tau\right)^{\frac{1}{p}}$$
$$\times\left(\int_{\mu}^{\nu}\left[|F''(\mu)|+\left(\frac{\tau-\mu}{\nu-\mu}\right)\eta(|F''(\nu)|,|F''(\mu)|)\right]d\tau\right)^{\frac{1}{q}}$$
$$=\frac{1}{2}[(\varkappa-\mu)^{p+1}+(\nu-\varkappa)^{p+1}]^{\frac{1}{p}}\mathcal{B}^{\frac{1}{p}}(p+1,p+1)(\nu-\mu)^{\frac{1}{q}}\left(|F''(\mu)|+\frac{1}{2}\eta(|F''(\nu)|,|F''(\mu)|)\right)^{\frac{1}{q}}.$$

This completes the proof. □

Remark 2.5. If we choose $\eta(F(x),F(y)) = F(x) - F(y)$ in Theorem 2.10, then we obtain [6, Theorem 2.5].

Theorem 2.11. *Under the assumptions of Lemma 2.1, if $|F''|^q$ is $\eta-$convex on $[\mu,\nu]$, for some $q>1$, then the following inequality holds:*

$$\left|F(\varkappa)+\frac{F(\mu)+F(\nu)}{2}-\left[\frac{1}{\varkappa-\mu}\int_{\mu}^{\varkappa}F(\tau)d\tau+\frac{1}{\nu-\varkappa}\int_{\varkappa}^{\nu}F(\tau)d\tau\right]\right|$$
$$\leq \frac{1}{2(\nu-\mu)}[\mathcal{A}_1{}^{\frac{1}{p}}\mathcal{C}_1{}^{\frac{1}{q}}+\mathcal{A}_2{}^{\frac{1}{p}}\mathcal{C}_2{}^{\frac{1}{q}}],$$

where

$$\mathcal{C}_1=\frac{(\nu-\mu)^2}{2}|F''(\mu)|^q+\frac{(\nu-\mu)^2}{6}\eta(|F''(\nu)|^q,|F''(\mu)|^q),$$
$$\mathcal{C}_2=\frac{(\nu-\mu)^2}{2}|F''(\mu)|^q+\frac{(\nu-\mu)^2}{3}\eta(|F''(\nu)|^q,|F''(\mu)|^q$$

and $\mathcal{A}_1$ and $\mathcal{A}_2$ are as in Theorem 2.3.

Proof. Taking the absolute value of identity (2.1) and applying Hölder-İşcan integral inequality, together with the the $\eta-$convexity of $|F''|^q$ we obtain

$$\begin{aligned}
&\left|F(\varkappa)+\frac{F(\mu)+F(\nu)}{2}-\left[\frac{1}{\varkappa-\mu}\int_\mu^\varkappa F(\tau)d\tau+\frac{1}{\nu-\varkappa}\int_\varkappa^\nu F(\tau)d\tau\right]\right| \\
&\leq \frac{1}{2(\nu-\mu)}\left(\int_\mu^\nu(\nu-\tau)|\kappa(\varkappa,\tau)|^p d\tau\right)^{\frac{1}{p}}\left(\int_\mu^\nu(\nu-\tau)|F''(\tau)|^q d\tau\right)^{\frac{1}{q}} \\
&\quad+\frac{1}{2(\nu-\mu)}\left(\int_\mu^\nu(\tau-\mu)|\kappa(\varkappa,\tau)|^p d\tau\right)^{\frac{1}{p}}\left(\int_\mu^\nu(\tau-\mu)|F''(\tau)|^q d\tau\right)^{\frac{1}{q}} \\
&\leq \frac{1}{2(\nu-\mu)}\left(\int_\mu^\varkappa(\nu-\tau)\left(\frac{(\varkappa-\tau)(\tau-\mu)}{\varkappa-\mu}\right)^p d\tau+\int_\varkappa^\nu(\nu-\tau)\left(\frac{(\tau-\varkappa)(\nu-\tau)}{\nu-\varkappa}\right)^p d\tau\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_\mu^\nu(\nu-\tau)\left[|F''(\mu)|^q+\left(\frac{\tau-\mu}{\nu-\mu}\right)\eta(|F''(\nu)|^q,|F''(\mu)|^q)\right]d\tau\right)^{\frac{1}{q}} \\
&\quad+\frac{1}{2(\nu-\mu)}\left(\int_\mu^\varkappa(\tau-\mu)\left(\frac{(\varkappa-\tau)(\tau-\mu)}{\varkappa-\mu}\right)^p d\tau+\int_\varkappa^\nu(\tau-\mu)\left(\frac{(\tau-\varkappa)(\nu-\tau)}{\nu-\varkappa}\right)^p d\tau\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_\mu^\nu(\tau-\mu)\left[|F''(\mu)|^q+\left(\frac{\tau-\mu}{\nu-\mu}\right)\eta(|F''(\nu)|^q,|F''(\mu)|^q)\right]d\tau\right)^{\frac{1}{q}} \\
&=\frac{1}{2(\nu-\mu)}\left\{\left(\frac{1}{2}\mathcal{B}(p+1,p+1)[((\nu-\mu)+(\nu-\varkappa))(\varkappa-\mu)^{p+1}+(\nu-\varkappa)^{p+2}]\right)^{\frac{1}{p}}\right. \\
&\quad\left.\times\left(\frac{(\nu-\mu)^2}{2}|F''(\mu)|^q+\frac{(\nu-\mu)^2}{6}\eta(|F''(\nu)|^q,|F''(\mu)|^q)\right)^{\frac{1}{q}}\right\} \\
&\quad+\frac{1}{2(\nu-\mu)}\left\{\left(\frac{1}{2}\mathcal{B}(p+1,p+1)[(\varkappa-\mu)^{p+2}+(\nu+\varkappa-2\mu)(\nu-\varkappa)^{p+1}]\right)^{\frac{1}{p}}\right. \\
&\quad\left.\times\left(\frac{(\nu-\mu)^2}{2}|F''(\mu)|^q+\frac{(\nu-\mu)^2}{3}\eta(|F''(\nu)|^q,|F''(\mu)|^q)\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

This completes the proof. □

Theorem 2.12. *Under the assumptions of Lemma 2.1, if $|F''|^q$ is $s-$convex on $[\mu,\nu]$, for some $q>1$, then the following inequality holds:*

$$\begin{aligned}
&\left|F(\varkappa)+\frac{F(\mu)+F(\nu)}{2}-\left[\frac{1}{\varkappa-\mu}\int_\mu^\varkappa F(\tau)d\tau+\frac{1}{\nu-\varkappa}\int_\varkappa^\nu F(\tau)d\tau\right]\right| \qquad (2.7)\\
&\leq\frac{1}{2p}[(\varkappa-\mu)^{p+1}+(\nu-\varkappa)^{p+1}]\mathcal{B}(p+1,p+1)+\frac{\nu-\mu}{2q}\left(|F''(\mu)|^q+\frac{1}{2}\eta(|F''(\nu)|^q,|F''(\mu)|^q)\right),
\end{aligned}$$

for all $\varkappa\in(\mu,\nu)$.

Proof. By using the identity that is given in Lemma 2.1, we have

$$\begin{aligned}
&\left|F(\varkappa)+\frac{F(\mu)+F(\nu)}{2}-\left[\frac{1}{\varkappa-\mu}\int_\mu^\varkappa F(\tau)d\tau+\frac{1}{\nu-\varkappa}\int_\varkappa^\nu F(\tau)d\tau\right]\right| \\
&\leq\frac{1}{2(\varkappa-\mu)}\int_\mu^\varkappa(\varkappa-\tau)(\tau-\mu)|F''(\tau)|d\tau+\frac{1}{2(\nu-\varkappa)}\int_\varkappa^\nu(\tau-\varkappa)(\nu-\tau)|F''(\tau)|d\tau.
\end{aligned}$$

By using the Young inequality as $xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q$,

$$\begin{aligned}
&\left|F(\varkappa) + \frac{F(\mu)+F(\nu)}{2} - \left[\frac{1}{\varkappa-\mu}\int_{\mu}^{\varkappa} F(\tau)d\tau + \frac{1}{\nu-\varkappa}\int_{\varkappa}^{\nu} F(\tau)d\tau\right]\right| \\
&\leq \frac{1}{2}\left(\frac{1}{p}\int_{\mu}^{\nu} |\kappa(\varkappa,\tau)|^p d\tau + \frac{1}{q}\int_{\mu}^{\nu} |F''(\tau)|^q d\tau\right) \\
&= \frac{1}{2p}\left(\frac{1}{(\varkappa-\mu)^p}\int_{\mu}^{\varkappa} (\varkappa-\tau)^p(\tau-\mu)^p d\tau + \frac{1}{(\nu-\varkappa)^p}\int_{\varkappa}^{\nu} (\tau-\varkappa)^p(\nu-\tau)^p d\tau\right) \\
&\quad + \frac{1}{2q}\int_{\mu}^{\nu} |F''(\tau)|^q d\tau.
\end{aligned}$$

By using $\eta-$convexity of $|F''(\tau)|^q$ and by a simple computation, we have the desired result. □

Remark 2.6. If we choose $\eta(F(x), F(y)) = F(x) - F(y)$, then the inequality (2.7) becomes inequality (2.4).

3. Conclusion

In this work, we obtained a series of new Bullen-type integral inequalities for differentiable functions by utilizing a fundamental integral identity involving the second derivative. Under the assumptions that the absolute value of the second derivative belongs to the classes of $s-$convex, quasi$-$convex, and $\eta-$convex functions, several upper bounds were established by means of power-mean, Hölder, Hölder–İşcan, and Young inequalities. The derived inequalities extend and unify a number of existing results in the literature, while the corresponding corollaries demonstrate that previously known inequalities are recovered as special cases. The techniques developed in this paper may be useful for obtaining further estimates related to numerical integration formulas and other classes of generalized convex functions. Future research may focus on extending these results to fractional integral operators and other generalized convexity structures.

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